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Research Article

Updating Stiffness and Hysteretic Damping Matrices Using Measured Modal Data

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A new direct method for the finite element (FE) matrix updating problem in a hysteretic (or material) damping model based on measured incomplete vibration modal data is presented. With this method, the optimally approximated stiffness and hysteretic damping matrices can be easily constructed. The physical connectivity of the original model is preserved and the measured modal data are embedded in the updated model. The numerical results show that the proposed method works well.

1. Introduction

The dynamic behavior of a mechanical system with n degrees of freedom is modeled by the following set of second-order ordinary differential equations:

$$M_a\ddot{\mathbf{q}}(t) + (K_a + iH_a)\,\mathbf{q}(t) = \mathbf{0}, \tag{1}$$

where $M_a, K_a \in \mathbb{R}^{n\times n}$ and $H_a \in \mathbb{R}^{n\times n}$ are analytical mass, stiffness, and hysteretic damping matrices, respectively. $\mathbf{q}(t) \in \mathbb{R}^{n\times 1}$ is a displacement vector depending on time t. In general, M_a is symmetric and positive definite, and K_a and H_a are symmetric. Equation (1) is usually modeled by FE techniques and thus a FE model is a numerical model. Assuming a fundamental solution $\mathbf{q}(t) = \mathbf{x}e^{\alpha t}$, one obtains the structural eigenproblem

$$\lambda M_a\mathbf{x} + (K_a + iH_a)\,\mathbf{x} = \mathbf{0}, \tag{2}$$

where $\lambda = \alpha^2$.

The FE method is used in many applications in engineering practice such as structural response prediction, structural control, structural health monitoring, damage detection, and reliability and risk assessment [1–6]. However, the accuracy of the FE model may be adversely affected by the inaccuracies in the model often related to material properties, modeling of joints, boundary conditions, and damping and simplifications made. As a result, a significant discrepancy may exist between the modal properties calculated by the constructed FE model and those identified from the vibration measurements of the actual structure. Many investigations show that the differences between the numerical and experimental frequencies may exceed 10% [7, 8]. The problem of how to modify a numerical model from the dynamic measurements is known as model updating in structural dynamics. Generally speaking, FE model updating involves two major applications. The first aspect is the FE model tuning, in which the goal is to update a numerical model to characterize the behavior of a real structure; the second one is damage detection, where the discrepancies in the structural dynamic properties before and after damage are identified to help locate and quantify structural damage.

FE model updating has been an active area of research for the last four decades and a lot of approaches have been established for updating structural dynamic models [5, 6]. The direct matrix updating methods were first introduced by Baruch and Bar-Itzhack [9], Baruch [10], and Berman and Nagy [11]. An optimal solution was obtained using Lagrange multipliers for minimizing the changes in the matrices subject to the orthogonality properties of the modes, the eigenvalue equation, and the symmetry of the updated matrices. The matrix mixing methods were developed by Caesar [12] and Link et al. [13]. This approach utilized experimental modal data and analytical ones to construct the inverses of the mass and stiffness matrices. An alternative approach suggested by Wei [14, 15] was to update the mass and stiffness matrices simultaneously using a measured eigenvector matrix as the reference. The control-based eigenstructure assignment techniques for FE model updating were proposed by Zimmerman and Widengren [16] and Inman and Minas [17]. These early methods are simple and computationally efficient but do not generally respect the structural connectivity in the initial FE model. To deal with this difficulty, Kabe [18] presented a stiffness matrix adjustment technique using the constrained minimization theory. The adjusted stiffness matrix can predict the measured modal data accurately and the connectivity of the original stiffness matrix is preserved. Kammer [19] proposed the projector matrix (PM) method which uses the projector matrix theory and the Moore-Penrose inverse, resulting in a more computationally efficient solution. Halevi and Bucher [20] also established a direct matrix updating method by including the connectivity constraints.

It is observed that model updating problems for linear viscously damped elastic systems (or gyroscopic systems) have been considered by many authors [21–29]. However, problems for updating hysteretic damping models have not got enough attention in these years. It is known that the principal difference between viscous and hysteretic damping models is that, for the viscous system, the energy dissipation per cycle depends linearly upon the frequency of oscillation, while for the hysteretic case it is independent of frequency. Very few authors pay attention to hysteretic damping model updating problems because the free vibration response for a system with hysteretic damping is necessarily complex, whereas the modal data of viscously damped elastic systems are closed under complex conjugation. In [30], the authors provided an extended application of the constrained eigenstructure assignment method (CEAM), which was first introduced in [31], to finite element model updating. The existing formulation was modified to accommodate larger systems by developing a quadratic optimization procedure which is unconditionally stable.

The present paper proposes an efficient direct updating method for the FE model with hysteretic (or material) damping which preserves the connectivity of the original coefficient matrices. Assume that K_a and H_a are real-valued symmetric $(2r+1)$-diagonal matrices; that is, K_a and H_a are band matrices and the nonzero elements are $n(2r+1)-r(r+1)$. The problem of updating stiffness and hysteretic damping matrices simultaneously can be stated following the inverse eigenvalue problem and an associated optimal approximation problem, leading to Problems 1 and 2.

Problem 1. Let $\Gamma = \operatorname{diag}\{\gamma_1, \ldots, \gamma_p\} \in \mathbb{C}^{p\times p}$ and $Z = [\mathbf{z}_1, \ldots, \mathbf{z}_p] \in \mathbb{C}^{n\times p}$ represent the diagonal matrix of p measured eigenvalues and the matrix of corresponding p measured eigenvectors, where $p \ll n$. Find real-valued symmetric $(2r+1)$-diagonal matrices K and H such that

$$M_a Z\Gamma + (K + iH) Z = 0. \tag{3}$$

Problem 2. Find $(\widehat{K}, \widehat{H}) \in \mathcal{S}_{KH}$ such that

$$\begin{aligned} &\left\|\widehat{K} - K_a\right\|^2 + \left\|\widehat{H} - H_a\right\|^2 \\ &\quad = \min_{(K,H)\in\mathcal{S}_{KH}} \left(\left\|K - K_a\right\|^2 + \left\|H - H_a\right\|^2\right), \end{aligned} \tag{4}$$

where $\mathcal{S}_{KH}$ is the solution set of Problem 1.

This paper is divided into five sections. In Section 2, we give an explicit formula for the solution set $\mathcal{S}_{KH}$ of Problem 1 using the Kronecker product and vec operator. In Section 3, we show that the solution of Problem 2 is unique and present the expression of the unique solution $(\widehat{K}, \widehat{H})$ of this problem. Section 4 describes and discusses the numerical results which indicate the accuracy and efficiency of the proposed algorithm. Some concluding remarks will be drawn in Section 5.

In this paper, we use the following notations. $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ complex and real matrices, and $\mathbb{SR}^{n\times n}$ denotes the set of all symmetric matrices in $\mathbb{R}^{n\times n}$. $A^\top$ and A^+ denote the transpose and the Moore-Penrose inverse of a real matrix A, respectively. I_n represents the identity matrix of size n. In space $\mathbb{R}^{m\times n}$, we define an inner product as $(A, B) = \operatorname{trace}(B^\top A)$, for all $A, B \in \mathbb{R}^{m\times n}$; then, $\mathbb{R}^{m\times n}$ is a Hilbert space and the matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $B = [b_{ij}] \in \mathbb{R}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbb{R}^{mp\times nq}$. Also, $\operatorname{vec}(\cdot)$ represents the vec operator; that is, $\operatorname{vec}(A) = [\mathbf{a}_1^\top, \mathbf{a}_2^\top, \ldots, \mathbf{a}_n^\top]^\top$, for the matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, where $\mathbf{a}_j$, $j = 1, \ldots, n$, is the jth column vector of A. For convenience, the symbols E_A and F_A stand for the two orthogonal projectors: $E_A = I_m - AA^+$ and $F_A = I_n - A^+A$, where $A \in \mathbb{R}^{m\times n}$.

2. The Solution of Problem 1

To solve Problem 1, we need the following lemmas.

Lemma 3 (see [32]). *Assume that $L \in \mathbb{R}^{m\times q}$ and $\mathbf{b} \in \mathbb{R}^m$. Then, the equation of $L\mathbf{y} = \mathbf{b}$ has a solution $\mathbf{y} \in \mathbb{R}^q$ if and only if $LL^+\mathbf{b} = \mathbf{b}$. In this case, the set of all solutions of the equation is $\mathbf{y} = L^+\mathbf{b} + F_L\mathbf{z}$, where $\mathbf{z} \in \mathbb{R}^q$ is an arbitrary vector.*

Lemma 4 (see [33]). *Suppose that $D \in \mathbb{R}^{m\times n}$, $H \in \mathbb{R}^{n\times l}$, and $J \in \mathbb{R}^{l\times s}$. Then,*

$$\operatorname{vec}(DHJ) = \left(J^\top \otimes D\right) \operatorname{vec}(H). \tag{5}$$

Lemma 5 (see [34]). *Let $[A, B]$ be an arbitrary partitioned matrix, where $A \in \mathbb{R}^{m\times k}$, $B \in \mathbb{R}^{m\times l}$, and let $C = E_A B$, $D = A^+BF_C$. Then, the generalized inverse of the matrix $[A, B]$ is*

$$[A, B]^+ = \begin{bmatrix} \left(I_k + DD^\top\right)^{-1}\left(A^+ - A^+BC^+\right) \\ C^+ + D^\top\left(I_k + DD^\top\right)^{-1}\left(A^+ - A^+BC^+\right) \end{bmatrix}. \tag{6}$$

If we denote the set of all $n \times n$ real symmetric $(2r+1)$-diagonal matrices by S_0, then we can easily see that S_0 is a linear subspace of $\mathbb{SR}^{n\times n}$, and the dimension of S_0 is $N = (1/2)(2n-r)(r+1)$. Define

$$Y_{ij} = \begin{cases} \frac{\sqrt{2}}{2}\left(\mathbf{e}_i\mathbf{e}_j^\top + \mathbf{e}_j\mathbf{e}_i^\top\right), & i = 1,\ldots,n-1;\ j = i+1,\ldots,t_i, \\ \mathbf{e}_i\mathbf{e}_i^\top, & i = j = 1,\ldots,n, \end{cases} \tag{7}$$

where $t_i = \min\{i+r, n\}$ and $\mathbf{e}_j$, $j = 1,\ldots,n$, is the jth column vector of I_n. Clearly, $\{Y_{ij}\}$ forms an orthonormal basis of the subspace S_0; that is,

$$\left(Y_{ij}, Y_{kl}\right) = \begin{cases} 0, & i \neq k \text{ or } j \neq l, \\ 1, & i = k,\ j = l. \end{cases} \tag{8}$$

Now, if $K, H \in \mathbb{SR}^{n\times n}$ are $(2r+1)$-diagonal matrices, then K, H can be expressed as

$$\begin{aligned} K &= \sum_{i,j}\alpha_{ij}Y_{ij}, \\ H &= \sum_{i,j}\beta_{ij}Y_{ij}, \end{aligned} \tag{9}$$

where the real numbers α_{ij}, β_{ij}, $i = 1,\ldots,n$; $j = i,\ldots,t_i$, $t_i = \min\{i+r, n\}$, need to be determined.

Separating (3) into its real and imaginary parts yields

$$\begin{aligned} KZ_R - HZ_I &= M_a\left(Z_I\Gamma_I - Z_R\Gamma_R\right), \\ KZ_I + HZ_R &= -M_a\left(Z_R\Gamma_I + Z_I\Gamma_R\right), \end{aligned} \tag{10}$$

where $Z = Z_R + iZ_I (Z_R, Z_I \in \mathbb{R}^{n\times p})$, $\Gamma = \Gamma_R + i\Gamma_I (\Gamma_R, \Gamma_I \in \mathbb{R}^{p\times p})$. Substituting (9) into (10), we have

$$\begin{aligned} \sum_{i,j}\alpha_{ij}Y_{ij}Z_R - \sum_{i,j}\beta_{ij}Y_{ij}Z_I &= M_a\left(Z_I\Gamma_I - Z_R\Gamma_R\right), \\ \sum_{i,j}\alpha_{ij}Y_{ij}Z_I + \sum_{i,j}\beta_{ij}Y_{ij}Z_R &= -M_a\left(Z_R\Gamma_I + Z_I\Gamma_R\right). \end{aligned} \tag{11}$$

Let

$$\begin{aligned} \boldsymbol{\alpha} &= [\alpha_{11},\ldots,\alpha_{1,r+1},\ldots,\alpha_{n-r,n-r},\ldots,\alpha_{n-r,n},\ldots,\alpha_{n-1,n-1}, \\ &\quad \alpha_{n-1,n},\alpha_{n,n}]^\top, \\ \boldsymbol{\beta} &= [\beta_{11},\ldots,\beta_{1,r+1},\ldots,\beta_{n-r,n-r},\ldots,\beta_{n-r,n},\ldots,\beta_{n-1,n-1}, \\ &\quad \beta_{n-1,n},\beta_{n,n}]^\top, \\ G &= [\operatorname{vec}(Y_{11}),\ldots,\operatorname{vec}(Y_{1,r+1}),\ldots,\operatorname{vec}(Y_{n-r,n-r}),\ldots, \\ &\quad \operatorname{vec}(Y_{n-r,n}),\ldots,\operatorname{vec}(Y_{n-1,n-1}),\operatorname{vec}(Y_{n-1,n}), \\ &\quad \operatorname{vec}(Y_{n,n})] \in \mathbb{R}^{n^2\times N}, \end{aligned} \tag{12}$$

$$\begin{aligned} A &= \left(Z_R^\top \otimes I_n\right)G, \\ B &= \left(Z_I^\top \otimes I_n\right)G, \end{aligned} \tag{13}$$

$$\begin{aligned} \mathbf{d} &= \operatorname{vec}\left(M_a\left(Z_I\Gamma_I - Z_R\Gamma_R\right)\right), \\ \mathbf{f} &= \operatorname{vec}\left(-M_a\left(Z_R\Gamma_I + Z_I\Gamma_R\right)\right). \end{aligned} \tag{14}$$

By Lemma 4, (11) are equivalent to

$$\begin{aligned} J\boldsymbol{\xi} &= \mathbf{d}, \\ L\boldsymbol{\xi} &= \mathbf{f}, \end{aligned} \tag{15}$$

where $J = [A, -B]$, $L = [B, A]$, and $\boldsymbol{\xi} = [\boldsymbol{\alpha}^\top, \boldsymbol{\beta}^\top]^\top$. Applying Lemma 3, the first equation of (15) with unknown vector $\boldsymbol{\xi}$ has a solution if and only if

$$JJ^+\mathbf{d} = \mathbf{d}. \tag{16}$$

In this case, the general solution of the equation is

$$\boldsymbol{\xi} = J^+\mathbf{d} + F_J\mathbf{u}, \tag{17}$$

where $\mathbf{u} \in \mathbb{R}^{2N}$ is an arbitrary vector. Substituting (17) into the second equation of (15) yields

$$LF_J\mathbf{u} = \mathbf{f} - LJ^+\mathbf{d}. \tag{18}$$

Using Lemma 3 again, the equation of (18) with unknown vector $\mathbf{u}$ has a solution if and only if

$$E_T\left(\mathbf{f} - LJ^+\mathbf{d}\right) = \mathbf{0}, \tag{19}$$

where $T = LF_J$. When condition (19) is satisfied, the general solution of the equation of (18) is given by

$$\mathbf{u} = T^+\left(\mathbf{f} - LJ^+\mathbf{d}\right) + F_T\mathbf{v}, \tag{20}$$

where $\mathbf{v} \in \mathbb{R}^{2N}$ is an arbitrary vector. Substituting (20) into (17) results in

$$\boldsymbol{\xi} = J^+\mathbf{d} + F_JT^+\left(\mathbf{f} - LJ^+\mathbf{d}\right) + F_JF_T\mathbf{v}. \tag{21}$$

By Lemma 5, we have

$$J^+ = \begin{bmatrix} P \\ Q \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} C &= -E_AB, \\ D &= -A^+BF_C, \\ P &= \left(I_N + DD^\top\right)^{-1}\left(A^+ + A^+BC^+\right), \\ Q &= C^+ + D^\top\left(I_N + DD^\top\right)^{-1}\left(A^+ + A^+BC^+\right). \end{aligned} \tag{23}$$

Using (22), we have

$$T = LF_J = \left[A_1, B_1\right], \tag{24}$$

where

$$\begin{aligned} A_1 &= B - BPA - AQA, \\ B_1 &= BPB + A + AQB. \end{aligned} \tag{25}$$

Using Lemma 5 again, we can get

$$T^+ = \begin{bmatrix} U \\ V \end{bmatrix}, \tag{26}$$

where

$$
\begin{aligned}
C_1 &= E_{A_1} B_1, \\
D_1 &= A_1^{+} B_1 F_{C_1}, \\
U &= \left(I_N + D_1 D_1^{\top}\right)^{-1}\left(A_1^{+} - A_1^{+} B_1 C_1^{+}\right), \\
V &= C_1^{+} + D_1^{\top}\left(I_N + D_1 D_1^{\top}\right)^{-1}\left(A_1^{+} - A_1^{+} B_1 C_1^{+}\right).
\end{aligned} \tag{27}
$$

By (22), (24), and (26), (21) can be equivalently written as

$$
\begin{aligned}
\boldsymbol{\alpha} = {} & P\mathbf{d} + (U - PAU + PBV)(\mathbf{f} - BP\mathbf{d} - AQ\mathbf{d}) \\
& + \left(I_N - PA\right)\left(I_N - UA_1\right)\mathbf{s} - PBVA_1\mathbf{s} \\
& - \left(I_N - PA\right)UB_1\mathbf{t} + PB\left(I_N - VB_1\right)\mathbf{t},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\boldsymbol{\beta} = {} & Q\mathbf{d} + (V - QAU + QBV)(\mathbf{f} - BP\mathbf{d} - AQ\mathbf{d}) \\
& - QA\left(I_N - UA_1\right)\mathbf{s} - \left(I_N + QB\right)VA_1\mathbf{s} \\
& + QAUB_1\mathbf{t} + \left(I_N + QB\right)\left(I_N - VB_1\right)\mathbf{t},
\end{aligned} \tag{29}
$$

where $\mathbf{v} = [\mathbf{s}^{\top}, \mathbf{t}^{\top}]^{\top}$ and $\mathbf{s}, \mathbf{t} \in \mathbb{R}^N$ are arbitrary vectors.

To sum up the above discussion, we can get the following result.

Theorem 6. *Assume that $\Gamma = \operatorname{diag}\{\gamma_1, \ldots, \gamma_p\} \in \mathbb{C}^{p\times p}$ and $Z = [\mathbf{z}_1, \ldots, \mathbf{z}_p] \in \mathbb{C}^{n\times p}$ are the measured eigenvalue and eigenvector matrices, where $p \ll n$. Let $Z = Z_R + iZ_I$ $(Z_R, Z_I \in \mathbb{R}^{n\times p})$, $\Gamma = \Gamma_R + i\Gamma_I$ $(\Gamma_R, \Gamma_I \in \mathbb{R}^{p\times p})$, and G, A, B, $\mathbf{d}$, $\mathbf{f}$ be given by (12), (13), and (14), respectively. Also, let $N = (1/2)(2n - r)(r + 1)$, $J = [A, -B]$, $L = [B, A]$, and $T = LF_J = [A_1, B_1]$. J^{+} and T^{+} are given by (22) and (26), respectively. If conditions (16) and (19) are satisfied, then the solution set $\mathcal{S}_{KH}$ of Problem 1 is*

$$
\begin{aligned}
\mathcal{S}_{KH} = \{ & (K, H) \in \mathbb{SR}^{n\times n} \times \mathbb{SR}^{n\times n} \mid K \\
& = S\left(\boldsymbol{\alpha} \otimes I_n\right),\ H = S\left(\boldsymbol{\beta} \otimes I_n\right)\},
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
S = {} & [Y_{11}, \ldots, Y_{1,r+1}, \ldots, Y_{n-r,n-r}, \ldots, Y_{n-r,n}, \ldots, Y_{n-1,n-1}, \\
& Y_{n-1,n}, Y_{n,n}] \in \mathbb{R}^{n\times nN},
\end{aligned} \tag{31}
$$

where C, D, P, Q, A_1, B_1, C_1, D_1, U, V, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ are, respectively, given by (23), (25), (27), (28), and (29), with $\mathbf{s}, \mathbf{t} \in \mathbb{R}^N$ being arbitrary vectors.

3. The Solution of Problem 2

When the set $\mathcal{S}_{KH}$ is nonempty, it is easy to check from (30) that $\mathcal{S}_{KH}$ is a closed convex subset of $\mathbb{SR}^{n\times n}\times\mathbb{SR}^{n\times n}$. It follows from the best approximation theorem (see, e.g., [35]) that there exists a unique solution $(\widehat{K}, \widehat{H})$ in $\mathcal{S}_{KH}$ such that (4) holds.

Now, we set out to find the unique solution $(\widehat{K}, \widehat{H})$ in $\mathcal{S}_{KH}$. If K_a and H_a are real symmetric $(2r + 1)$-diagonal matrices, then K_a, H_a can be expressed as the linear combinations of the orthonormal basis $\{Y_{ij}\}$; namely,

$$
\begin{aligned}
K_a &= \sum_{i,j} \gamma_{ij} Y_{ij}, \\
H_a &= \sum_{i,j} \delta_{ij} Y_{ij},
\end{aligned} \tag{32}
$$

where γ_{ij}, δ_{ij}, $i = 1, \ldots, n$; $j = i, \ldots, t_i$, $t_i = \min\{i + r, n\}$, are uniquely determined by the elements of K_a and H_a. Let

$$
\begin{aligned}
\boldsymbol{\gamma} = {} & [\gamma_{11}, \ldots, \gamma_{1,r+1}, \ldots, \gamma_{n-r,n-r}, \ldots, \gamma_{n-r,n}, \ldots, \gamma_{n-1,n-1}, \\
& \gamma_{n-1,n}, \gamma_{n,n}]^{\top},
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\boldsymbol{\delta} = {} & [\delta_{11}, \ldots, \delta_{1,r+1}, \ldots, \delta_{n-r,n-r}, \ldots, \delta_{n-r,n}, \ldots, \delta_{n-1,n-1}, \\
& \delta_{n-1,n}, \delta_{n,n}]^{\top}.
\end{aligned} \tag{34}
$$

Then, for any $(K, H) \in \mathcal{S}_{KH}$ in (30), by (8), (9), (32), (33), and (34), we get

$$
\begin{aligned}
g &= \left\|K - K_a\right\|^2 + \left\|H - H_a\right\|^2 \\
&= \left\|\sum_{i,j}\left(\alpha_{ij} - \gamma_{ij}\right)Y_{ij}\right\|^2 + \left\|\sum_{i,j}\left(\beta_{ij} - \delta_{ij}\right)Y_{ij}\right\|^2 \\
&= \left(\sum_{i,j}\left(\alpha_{ij} - \gamma_{ij}\right)Y_{ij}, \sum_{i,j}\left(\alpha_{ij} - \gamma_{ij}\right)Y_{ij}\right) \\
&\quad + \left(\sum_{i,j}\left(\beta_{ij} - \delta_{ij}\right)Y_{ij}, \sum_{i,j}\left(\beta_{ij} - \delta_{ij}\right)Y_{ij}\right) \\
&= \sum_{i,j}\left(\alpha_{ij} - \gamma_{ij}\right)^2 + \sum_{i,j}\left(\beta_{ij} - \delta_{ij}\right)^2 \\
&= \left\|\boldsymbol{\alpha} - \boldsymbol{\gamma}\right\|^2 + \left\|\boldsymbol{\beta} - \boldsymbol{\delta}\right\|^2 = \left\|\begin{bmatrix}\boldsymbol{\alpha} \\ \boldsymbol{\beta}\end{bmatrix} - \begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\delta}\end{bmatrix}\right\|^2.
\end{aligned} \tag{35}
$$

Substituting (21) into the relation of g, we get

$$
g = \left\|J^{+}\mathbf{d} + F_J T^{+}\left(\mathbf{f} - LJ^{+}\mathbf{d}\right) + F_J F_T \mathbf{v} - \boldsymbol{\psi}\right\|^2, \tag{36}
$$

where $\boldsymbol{\psi} = [\boldsymbol{\gamma}^{\top}, \boldsymbol{\delta}^{\top}]^{\top}$. Notice that

$$
\begin{aligned}
F_T F_J &= \left[I_{2N} - \left(LF_J\right)^{+} LF_J\right] F_J = F_J - \left(LF_J\right)^{+} LF_J \\
&= F_J - T^{+}T,
\end{aligned} \tag{37}
$$

which implies that

$$
F_T F_J = F_J F_T. \tag{38}
$$

Therefore,

$$
\begin{aligned}
g = {} & \mathbf{v}^{\top} F_T F_J F_T \mathbf{v} - 2\mathbf{v}^{\top} F_T F_J \boldsymbol{\psi} - 2\boldsymbol{\psi}^{\top} J^{+}\mathbf{d} \\
& - 2\boldsymbol{\psi}^{\top} F_J T^{+}\left(\mathbf{f} - LJ^{+}\mathbf{d}\right) + \mathbf{d}^{\top}\left(JJ^{\top}\right)^{+}\mathbf{d} \\
& + \left(\mathbf{f} - LJ^{+}\mathbf{d}\right)^{\top}\left(T^{+}\right)^{\top} F_J T^{+}\left(\mathbf{f} - LJ^{+}\mathbf{d}\right) \\
& + \boldsymbol{\psi}^{\top}\boldsymbol{\psi},
\end{aligned} \tag{39}
$$

$$
\frac{\partial g}{\partial \mathbf{v}} = 2F_T F_J \mathbf{v} - 2F_T F_J \boldsymbol{\psi}.
$$

Clearly, the function $g = \|K-K_a\|^2+\|H-H_a\|^2 = \min$ attains the smallest value at $\partial g/\partial \mathbf{v} = 0$, which yields

$$F_T F_J \mathbf{v} = F_T F_J \boldsymbol{\psi}. \tag{40}$$

By substituting (40) into (28) and (29), we obtain

$$\begin{aligned}\widehat{\boldsymbol{\alpha}} &= P\mathbf{d} + (U - PAU + PBV)(\mathbf{f} - BP\mathbf{d} - AQ\mathbf{d}) \\ &\quad + (I_N - PA)(I_N - UA_1)\boldsymbol{\gamma} - PBVA_1\boldsymbol{\gamma} \\ &\quad - (I_N - PA)UB_1\boldsymbol{\delta} + PB(I_N - VB_1)\boldsymbol{\delta},\end{aligned} \tag{41}$$

$$\begin{aligned}\widehat{\boldsymbol{\beta}} &= Q\mathbf{d} + (V - QAU + QBV)(\mathbf{f} - BP\mathbf{d} - AQ\mathbf{d}) \\ &\quad - QA(I_N - UA_1)\boldsymbol{\gamma} - (I_N + QB)VA_1\boldsymbol{\gamma} \\ &\quad + QAUB_1\boldsymbol{\delta} + (I_N + QB)(I_N - VB_1)\boldsymbol{\delta}.\end{aligned} \tag{42}$$

By now, we proved the following theorem.

Theorem 7. *Suppose that the real symmetric $(2r+1)$-diagonal matrices K_a and H_a are given. If conditions (16) and (19) hold, then Problem 2 has a unique solution and the unique solution of Problem 2 can be expressed as*

$$\begin{aligned}\widehat{K} &= S(\widehat{\boldsymbol{\alpha}} \otimes I_n), \\ \widehat{H} &= S(\widehat{\boldsymbol{\beta}} \otimes I_n),\end{aligned} \tag{43}$$

where $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$ are given by (41) and (42), respectively.

4. A Numerical Example

Based on Theorems 6 and 7, we can state an algorithm for solving Problems 1 and 2 as follows.

Algorithm 8.

(1) Input M_a, K_a, H_a, Γ, Z.

(2) Form the orthonormal basis $\{Y_{ij}\}$ by (7).

(3) Separating Γ and Z into real and imaginary parts yields Γ_R, Γ_I, Z_R, and Z_I.

(4) Compute G, A, B, $\mathbf{d}$, $\mathbf{f}$ according to (12), (13), and (14), respectively.

(5) Compute $J = [A, -B]$, $L = [B, A]$.

(6) Compute C, D, P, Q by (23) and compute J^+ by (22).

(7) If condition (16) is not satisfied, then stop.

(8) Compute A_1, B_1 by (25) and form the matrix T by (24).

(9) Compute C_1, D_1, U, V by (27) and compute T^+ by (26).

(10) If condition (19) is not satisfied, then stop.

(11) Form vectors $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ by (33) and (34).

(12) Compute S, $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$ by (31), (41), and (42), respectively.

(13) Compute the unique solution $(\widehat{K}, \widehat{H})$ of Problem 2 by (43).

Example 1. Consider a five-DOF system with analytical mass, damping, and stiffness matrices as follows:

$$M_a = \operatorname{diag}\{1, 2, 5, 4, 3\},$$

$$H_a = \begin{bmatrix} 11.0 & -8.0 & 0 & 0 & 0 \\ -8.0 & 14.0 & -3.5 & 0 & 0 \\ 0 & -3.5 & 13.0 & -7.8 & 0 \\ 0 & 0 & -7.8 & 13.5 & -9.0 \\ 0 & 0 & 0 & -9.0 & 15.4 \end{bmatrix}, \tag{44}$$

$$K_a = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ -20 & 120 & -35 & 0 & 0 \\ 0 & -35 & 80 & -12 & 0 \\ 0 & 0 & -12 & 95 & -40 \\ 0 & 0 & 0 & -40 & 124 \end{bmatrix}.$$

That is, H_a, K_a are symmetric 3-diagonal matrices. The model used to simulate the consistent experimental data is given by $M = M_a$, $H = H_a$, and

$$K = \begin{bmatrix} 100 & -8 & 0 & 0 & 0 \\ -8 & 120 & -30 & 0 & 0 \\ 0 & -30 & 80 & -12 & 0 \\ 0 & 0 & -12 & 120 & -50 \\ 0 & 0 & 0 & -50 & 124 \end{bmatrix}. \tag{45}$$

Note that the differences between K_a and K are $(1,2)$, $(2,3)$, $(4,4)$, and $(4,5)$ elements. The eigensolution of the model is used to create the experimental modal data. Assume that the measured eigenvalue matrix $\Gamma = \operatorname{diag}\{\gamma_1, \gamma_2\}$ and corresponding eigenvector matrix $Z = [\mathbf{z}_1, \mathbf{z}_2]$ are given by

$$\begin{aligned}\Gamma &= \operatorname{diag}\{-13.553 - 1.3302i, -20.603 - 2.4171i\}, \\ Z &= \begin{bmatrix} 0.032335 + 0.023541i & -0.0090358 - 0.017342i \\ 0.30721 - 0.013746i & -0.12644 - 0.055369i \\ 0.94805 - 0.05195i & -0.3335 - 0.13811i \\ 0.3359 + 0.17504i & 0.85743 - 0.14257i \\ 0.19824 + 0.11415i & 0.69922 - 0.082153i \end{bmatrix}.\end{aligned} \tag{46}$$

All the tests are performed on an Intel Core 3.39 GHz PC with a main memory of 2.99 GB running MATLAB 6.5. It is easy to check that conditions (16) and (19) hold ($\|JJ^+\mathbf{d} - \mathbf{d}\| = 1.3277 \times 10^{-13}$, $\|E_T(\mathbf{f} - LJ^+\mathbf{d})\| = 0.10614$). According to Algorithm 8 with 0.1710 CPU time (in seconds), we solve

Table 1: Results for $\text{res}(\gamma_i, \mathbf{z}_i)$.

$(\gamma_i, \mathbf{z}_i)$	$(\gamma_1, \mathbf{z}_1)$	$(\gamma_2, \mathbf{z}_2)$
$\text{res}(\gamma_i, \mathbf{z}_i)$	0.039544	0.066018

the optimal approximation solution of Problem 2 as follows:

$$\widehat{K} = \begin{bmatrix} 100 & -8 & 0 & 0 & 0 \\ -8 & 120.04 & -29.995 & 0 & 0 \\ 0 & -29.995 & 79.986 & -12.002 & 0 \\ 0 & 0 & -12.002 & 120.04 & -50.049 \\ 0 & 0 & 0 & -50.049 & 123.99 \end{bmatrix},$$
$$\widehat{H} = \begin{bmatrix} 10.999 & -8 & 0 & 0 & 0 \\ -8 & 13.999 & -3.5013 & 0 & 0 \\ 0 & -3.5013 & 13.009 & -7.7795 & 0 \\ 0 & 0 & -7.7795 & 13.491 & -9.0188 \\ 0 & 0 & 0 & -9.0188 & 15.388 \end{bmatrix}. \tag{47}$$

We can see that Algorithm 8 works well and the updated matrices $\widehat{K}$ and $\widehat{H}$ are very close to the accurate stiffness matrix K and hysteretic damping matrix H_a with absolute errors

$$\begin{aligned} \left\|\widehat{H} - H_a\right\| &= 0.043152, \\ \left\|\widehat{K} - K\right\| &= 0.09293. \end{aligned} \tag{48}$$

Define

$$\text{res}\left(\gamma_i, \mathbf{z}_i\right) = \left\|\gamma_i M_a \mathbf{z}_i + \left(\widehat{K} + i\widehat{H}\right)\mathbf{z}_i\right\|. \tag{49}$$

By Algorithm 8, we can get the results in Table 1.

Therefore, the measured eigenvalues and eigenvectors are reproduced in the new model $M_a Z\Gamma + (\widehat{K} + i\widehat{H})Z = 0$, and the updated matrices $\widehat{K}, \widehat{H}$ are also symmetric 3-diagonal matrices, which means that the structural connectivity of the original model is preserved.

5. Concluding Remarks

Updating a structural FE model to match measured modal data has been an important task for engineers. In this paper, a new direct method for updating symmetric and banded mechanical systems has been introduced, which uses measured eigenvalues and eigenvectors and the additional constraint related to structural connectivity to adjust stiffness and hysteretic damping matrices simultaneously. The method was developed based on the constrained optimization theory. The error function was minimized such that the discrepancies between the estimated and actual stiffness and hysteretic damping matrices are minimal. The method can maintain the physical connectivity of the numerical model, and the adjusted model is able to reproduce the measured modal data accurately when conditions (16) and (19) are satisfied. A numerical example confirms the effectiveness of the proposed approach.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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