Research Article

Flexural Vibration Analysis of Nonuniform Double-Beam System with General Boundary and Coupling Conditions

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In this paper, an analytical modeling approach for the flexural vibration analysis of the nonuniform double-beam system is proposed via an improved Fourier series method, in which both types of translational and rotational springs are introduced to account for the mechanical coupling on the interface as well as boundary restraints. Energy formulation is employed for the dynamic description of the coupling system. With the aim to treat the varying thickness across the beam in a unified pattern, the relevant variables are all expanded into Fourier series. Supplementary terms with the smoothed characteristics are introduced to the standard Fourier series for the construction of displacement admissible function for each beam. In conjunction with the Rayleigh–Ritz procedure, the transverse modal characteristics of nonuniform double-beam system can be obtained by solving a standard eigenvalue problem. Instead of solving the certain value of nonideal boundary conditions, the continuous spring stiffnesses of the boundary conditions are considered, and the rotational restraints are introduced in the coupling beam interface. Numerical results are then presented to demonstrate the reliability of the current model and study the influence of various parameters, such as taper ratio, boundary, and coupling strength on the free vibration characteristics, with the emphasis put on the rotational restraining coefficients on the beam interface. This work can provide an efficient modeling framework for the vibration characteristics study of the complex double-beam system, especially with arbitrary varying thickness and coupling stiffness.

1. Introduction

The multiple-beam system has been extensively studied due to its wide application in various branches, such as mechanical and aeronautical engineering. A good understanding on its dynamic characteristics will be of great importance for the efficient design as well as vibration control of such complex system. For this reason, a lot of research attention has been devoted to the vibration behavior of multiple-beam structure by many researchers in the past decades.

For these multiple-beam structures, mechanical coupling between each beam component is usually taken into account through the introduction of translational spring across the beam interface [1]. Seelig and Hoppmann [2] formulated the differential equation of motion and obtained the solution for vibration analysis of the n elastically connected parallel beams. Rao [3] solved the differential governing equation for the flexural vibration of elastically connected parallel bars based on the Timoshenko beam theory, in which the effects of rotary inertia and shear deformation are considered. Li and Hua [4] employed the dynamic stiffness method to analyze the free vibration characteristics of a three-beam system with the elastic spring and dashpot on the coupling interface. Moreover, Winkler elastic layers are also accoutered for in the vibration analysis of coupling interface of the multiple-beam system [5–7]. Deng et al. [8] also studied the double functionally graded Timoshenko beam system resting on Winkler–Pasternak elastic foundation using Hamilton’s principle.

In many occasions, the nonuniform beam component will be encountered with the background of optimal design, in which a better or more suitable distribution of mass and...
strength than the uniform beam is of great desire. Thus, vibration analysis of the nonuniform beam has been studied continuously [9–11]. Abrate [12] presented a method to transform the motion equation of a nonuniform beam into its uniform counterpart. Lee and Lee [13] developed a transfer-matrix method to investigate the free vibration characteristics of a tapered Bernoulli–Euler beam. Bessel functions are widely used in dealing with the nonuniform beam system with different boundary conditions. Rosa and Auciello [14] solved the nonuniform beam-governing equation and the elastic boundary condition by substituting the four flexibility coefficients of the constructed functions. Auciello and Ercolano [15] investigated the transverse vibration of a beam, for which one part was tapered, and the remaining part is the uniform thickness. Torabi et al. [16] derived an exact closed-form solution for the free vibration of Euler–Bernoulli conical and tapered beams with any number of attached masses, which are described by Dirac’s delta function. Furthermore, Chen and Pan Liu [17] employed the Bessel function and the numerical assembly method (NAM) to investigate the vibration characteristics of a tapered beam with multiple arbitrarily placed rotational dampers.

In the current studies on the vibration analysis of the multiple-beam system, most of them are devoted to the uniform ones, and just little exception can be found in the literature for the vibration studies of nonuniform multiple-beam structure. For example, Mabie and Rogers [18, 19] obtained an accurate solution for determining the first five frequencies of the cantilevered double-tampered beam system. Takahashi and Yoshioka [20] utilized the transfer-matrix approach to analyze the vibration and stability of a double-cantilever beam, in which only translational spring is introduced to restrain the relevant displacement. Such mechanical interaction is coupled with other through the elastic restraining springs on the coupling interfaces. Such mechanical interaction is represented using two types of springs, namely translational and rotational one. Any boundary condition can be easily obtained by setting the relevant restraining spring stiffnesses. For the beam vibration, there are two freedom degrees at each field point, namely translation and rotation. Then, a full-coupling restraint should include the translational and rotational spring distributions, accordingly. In this way, the familiar Winkler-type of elastic interface can be readily derived by setting the coefficients of rotational springs into zero.

2. Theoretical Formulations

2.1. Model Description. Consider an elastically connected nonuniform double-beam system, as illustrated in Figure 1. The beam member with arbitrary boundary conditions is coupled with other through the elastic restraining springs on the coupling interfaces. Such mechanical interaction is described from the viewpoint of energy. Although the double-beam system is considered here, the current modeling framework is suitable for the general N-beam system. Without losing the generality, the subsequent formulation will be given for such N-beam coupling system. Lagrangian function is used for the description of system dynamic behavior, which can be written as

\[
L = V - T = \sum_{i=1}^{N} V_i + V_{\text{coupling}} - \sum_{i=1}^{N} T_i
\]

where \(L\) is the system Lagrangian, \(V\) and \(T\) are the total potential energy and kinetic energy, \(V_i\) is the potential energy associated with the \(i\)th beam member, \(V_{\text{coupling}}\) is

\[
\sum_{i=1}^{N} T_i
\]
the potential energy stored in the interface coupling springs, and $T_i$ is the kinetic energy due to the $i^{th}$ vibrating beam.

For the $i^{th}$ beam member with elastically restrained edges, its potential energy $V_i$ is

$$ V_i = \frac{1}{2} \int_0^L E_i I_i(x) \left( \frac{\partial^2 w_i(x)}{\partial x^2} \right)^2 dx + \frac{1}{2} k_{T_0} w_i^2(x)_{x=0} + \frac{1}{2} K_{R_0} \left( \frac{\partial w_i(x)}{\partial x} \right)^2_{x=0} + \frac{1}{2} K_{R_1} \left( \frac{\partial w_i(x)}{\partial x} \right)^2_{x=L} ,$$

where $w(x)$ is the transverse vibration displacement field function, $K_{T_0}$ and $K_{R_0}$ are respectively the stiffness coefficients for the translational and rotational springs at the end $x=0$, and similar meaning can be deduced for the right end of $x=L$. The subscript $i$ means that this variable is associated with the $i^{th}$ beam member. $I_i(x)$ is the moment of inertia of the nonuniform beam.

The total kinetic energy of the $i^{th}$ beam structure is

$$ T_i = \frac{1}{2} \int_0^L \rho_i S_i(x) \left( \frac{\partial w_i(x,t)}{\partial t} \right)^2 dx = \frac{1}{2} \omega^2 \int_0^L \rho_i S_i(x) w_i^2(x) dx, $$

where $\omega$ is the radian frequency and $\rho_i$ and $S_i(x)$ are respectively the mass density and cross section area of the $i^{th}$ beam member.

The coupling potential energy between the interfaces can be written as

$$ V_{\text{coupling}} = \frac{1}{2} \int_0^L K_c (w_i - w_{i+1})^2 + K_r \left( \frac{\partial w_i}{\partial x} - \frac{\partial w_{i+1}}{\partial x} \right)^2 dx, $$

in which $K_c$ and $K_r$ are respectively the translational and rotational coupling spring stiffnesses distributed across the interface between the $i^{th}$ and $i + 1^{st}$ beam member.

In this work, in order to treat the cross-section area variation in a most general unified pattern, variables, such as arbitrary inertia $I_i(x)$ and cross-section area $S_i(x)$, associated with the nonuniform thickness variation are expanded into Fourier cosine series, with the corresponding expansion coefficients defined as follows:

$$ S_i(x) = S_0 + \sum_{m=1}^{\infty} c_m \cos \lambda_m x = S_0 + \sum_{m=0}^{\infty} c_m \cos \lambda_m x, $$

$$ c_m = \begin{cases} \frac{1}{L} \int_0^L f_i(x) dx, & m = 0, \\ \frac{2}{L} \int_0^L f_i(x) \cos \frac{m\pi}{L} x dx, & m \neq 0, \end{cases} $$

and

$$ I_i(x) = I_0 + \sum_{m=1}^{\infty} d_m \cos \lambda_m x, $$

$$ d_m = \begin{cases} \frac{1}{L} \int_0^L f_2(x) dx, & m = 0, \\ \frac{2}{L} \int_0^L f_2(x) \cos \frac{m\pi}{L} x dx, & m \neq 0, \end{cases} $$

in which $\lambda_m = m\pi/L$, and $m$ is the Fourier series term.

For various kinds of cross sections, the difference between them is merely the Fourier coefficients and the items which can be obtained easily through the Fourier transformation. For the traditional uniform beam, $S_i(x)$ and $I_i(x)$ will be constant across the beam length.

Once the system Lagrangian is obtained, the other thing is to construct the appropriate admissible function. Differential continuity of the constructed function has significant effect on the final convergence and accuracy. Here, an improved Fourier series method is employed for this purpose, in which the additional functions are introduced to the standard Fourier series to remove all the discontinuities associated with the spatial differentiation of the displacement field functions. For each beam member, its flexural vibrating displacement function is expanded as [22]

$$ w(x) = \sum_{n=0}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) + b_1 \zeta_1(x) + b_2 \zeta_2(x) + b_3 \zeta_3(x) + b_4 \zeta_4(x), $$

in which

$$ \zeta_1(x) = \frac{9L}{4n^2} \sin \left( \frac{n\pi x}{2L} \right) - \frac{L}{12n^2} \sin \left( \frac{3n\pi x}{2L} \right), $$

$$ \zeta_2(x) = -\frac{9L}{4n^2} \cos \left( \frac{n\pi x}{2L} \right) - \frac{L}{12n^2} \cos \left( \frac{3n\pi x}{2L} \right), $$

$$ \zeta_3(x) = \frac{L^3}{n^3} \sin \left( \frac{n\pi x}{2L} \right) - \frac{L^3}{3n^3} \sin \left( \frac{3n\pi x}{2L} \right), $$

$$ \zeta_4(x) = -\frac{L^3}{n^3} \cos \left( \frac{n\pi x}{2L} \right) - \frac{L^3}{3n^3} \cos \left( \frac{3n\pi x}{2L} \right). $$

$$ S_i(x) = S_0 f_1(x) = S_0 \left( \frac{c_2}{2} + \sum_{m=1}^{\infty} c_m \cos \lambda_m x \right) $$

$$ = S_0 \sum_{m=0}^{\infty} c_m \cos \lambda_m x, $$

$$ \zeta_1(x) = \frac{9L}{4n^2} \sin \left( \frac{n\pi x}{2L} \right) - \frac{L}{12n^2} \sin \left( \frac{3n\pi x}{2L} \right), $$

$$ \zeta_2(x) = -\frac{9L}{4n^2} \cos \left( \frac{n\pi x}{2L} \right) - \frac{L}{12n^2} \cos \left( \frac{3n\pi x}{2L} \right), $$

$$ \zeta_3(x) = \frac{L^3}{n^3} \sin \left( \frac{n\pi x}{2L} \right) - \frac{L^3}{3n^3} \sin \left( \frac{3n\pi x}{2L} \right), $$

$$ \zeta_4(x) = -\frac{L^3}{n^3} \cos \left( \frac{n\pi x}{2L} \right) - \frac{L^3}{3n^3} \cos \left( \frac{3n\pi x}{2L} \right). $$
It can be easily proven that the current constructed trigonometric function can satisfy the displacement and its higher-order differentiation continuity requirement in the interval \((0, L)\). It should be pointed out that the choice of the supplementary functions is not unique, while the appropriate form will be helpful for simplifying the subsequent mathematical formulations.

Substituting the admissible function Equation (9) into the elastically connected double-beam system Lagrangian Equations (1)–(4), minimizing it with respect to all the unknown Fourier series coefficients and truncating the Fourier series into finite number \(n = N\), one will obtain the system characteristic equation in the matrix form:

\[
(K - \omega^2 M) A = 0,
\]

(11)

where \(K\) and \(M\) are the stiffness and mass matrices for the elastically connected double-beam system, respectively, and \(A\) is the unknown Fourier series coefficient vector. By solving such standard eigenvalue problem, all the modal parameters can be easily obtained.

### 3. Numerical Examples and Discussions

In this section, the aforementioned modeling framework is programmed in the MATLAB environment. Several numerical results of different kinds of cross sections with various boundary conditions will be presented to demonstrate the effectiveness and advantage of the proposed model. As no research has been published about the free vibration of double-beam system with uniform cross sections, the results of the current method will be the first compared with those from other relevant literatures to validate the correctness. In current solution framework, the elastic boundary conditions are easily obtained by setting the restraining stiffness coefficient into various values accordingly. Similarly, variation of the arbitrary inertia \(I_i(x)\) and cross-section area \(S_i(x)\) can be easily handled using the same MATLAB program by just changing the mapping parameters of the nonuniform profile in Fourier space. In the following analysis, the nondimensional frequency parameter will be used, with their definitions as \(\Omega = \omega L^2 / \sqrt{\rho S/ET}\).

#### 3.1. Double-Beam System with Uniform Cross Section

Here, by setting the relevant parameters to constant in Equations (5) and (7), several vibration results can be obtained for a uniform double-beam system, which has been investigated in some former studies. With the comparison purpose, modal parameters are kept the same as those used in Ref. [23], namely, \(E_1I_1 = 4 \times 10^6\) Nm², \(E_2I_2 = 2 \times E_1I_1\), \(\rho_1A_1 = 100 \) kg/m, \(\rho_2A_2 = 2 \times \rho_1A_1\), \(L = 10\) m. From the comparison tabulated in Table 1, it can be observed that the current results can agree well with those from other approaches in which the parameters of beam 1 are used for the calculation of nondimensional frequency \(\Omega\). Comparing the results of Tables 1 and 2, it can be found that the variation of coupling stiffness will not affect the odd-mode frequency in the uniform double-beam system.

#### 3.2. Single Nonuniform Beam with Different Cross-Section Parameters

In order to validate the proposed method for the case of variable cross sections, the coupling translational spring \(k_c\) and rotational spring \(K_c\) are set to zero which transfer the double-beam system to two single-beams with the same vibration characteristics. The polynomial function of inertia \(I_i(x)\) and cross-section area \(S_i(x)\) is as follows:

\[
S_i(x) = S_0 \left( a - b \frac{x}{L} \right)^r,
\]

(12)

and

\[
I_i(x) = I_0 \left( a - b \frac{x}{L} \right)^{r+2},
\]

(13)

where \(b\) is the taper ratio of the cross-section area.

Table 3 shows the results of one single nonuniform beam with clamped-free (C-F) boundary condition, in which the data taken form Ref. [24] are also presented to validate the correctness of the present method in dealing with different cross sections. With the increase of the taper ratio \(b\), the frequency will raise which represents the decrease of the beam’s stiffness.

Moreover, a single beam with three-step changes in cross section is analyzed. The geometrical dimensions and material properties are \(L_R = 1\) m, \(m_R = 1\) kg/m, \(E_R = 1\) Nm², \(\mu_1 = 1.0\), \(\mu_2 = 0.8\), \(\mu_3 = 0.65\), \(\mu_4 = 0.25\), \(L_1 = 0.25L_R\), \(L_2 = 0.3L_R\), \(L_3 = 0.25L_R\), and \(L_4 = 0.2L_R\). Table 4 lists the first four frequency parameters for various combinations of boundary conditions. The results can agree well with those from the reference approaches.

#### 3.3. Nonuniform Double-Beam System with Uniform Cross Section of Beam 2

The nonuniform double beam system is considered with C-F boundary condition for both beams in which the upper beam’s cross-section function is the same as defined in Equations (12) and (13) in Section 3.2. The lower beam’s cross section is uniform with the same parameter of the upper beam at \(x = 0\). Other material parameters are the same for the two beams. Table 5 lists the results of different coupling stiffness boundaries. When the coupling translational spring \(k_c\) and rotational spring \(K_c\) are both zero, the frequencies are from the two single beams in which the odd-mode results of the lower uniform beam is larger than the even-mode results of the upper taper beam because the nonuniform cross section makes the beam’s stiffness smaller. After the coupling stiffness \(k_c\) and \(K_c\) becomes bigger, the frequencies for both beams becomes larger. As the value of \(k_c\) and \(K_c\) are both \(10^7\), the frequencies are similar as the single beam in which the value is bigger than odd- and even-mode results.

The influence of different coupling springs \(k_c\) and \(K_c\) on the fundamental frequency of the double-beam system is presented in Figure 2 which can illustrate that the influence of rotational spring \(K_c\) is obvious than that of translational spring \(k_c\) which means the rotational spring cannot be ignored in the double-beam system.
Finally, the nonuniform double-beam system is considered, in which both beams are tapered beams as defined in Section 3.2. Here, the boundary condition is clamped at $x = 0$ and elastic at $x = L$. The influence of elastic boundary stiffness $K_2$ ($K_2 = K_{TL} = K_{RL}$) at $x = L$ of both beams and coupling spring $K_C$ ($K_C = k_c = k_c$) are plotted in Figure 3. As discussed in Section 3.3, the rotational spring cannot be ignored, and the translational and rotational springs of the coupling spring are set at the same value. From Figure 3, when the elastic boundary stiffness increases, the frequencies will become bigger, and there will be a sensitive area in which the influence to the fundamental frequency is uniform. Otherwise, the fundamental frequencies will not

### Table 1: The first six nondimensional frequency $\Omega$ for the uniform double-beams with different boundary conditions ($k_c = 1 \times 10^5$ N/m², $K_c = 0$).

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Beam 1 $\Omega_1$</th>
<th>Beam 2 $\Omega_2$</th>
<th>$\Omega_3$</th>
<th>$\Omega_4$</th>
<th>$\Omega_5$</th>
<th>$\Omega_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S</td>
<td>9.8683</td>
<td>21.7345</td>
<td>39.4577</td>
<td>43.9572</td>
<td>88.7208</td>
<td>90.8309</td>
</tr>
<tr>
<td>C-C</td>
<td>22.3011</td>
<td>29.5464</td>
<td>61.4070</td>
<td>64.4416</td>
<td>120.1731</td>
<td>121.9079</td>
</tr>
<tr>
<td>C-F</td>
<td>3.5113</td>
<td>19.6809</td>
<td>21.9986</td>
<td>29.3130</td>
<td>61.5645</td>
<td>64.5643</td>
</tr>
<tr>
<td>S-S</td>
<td>8.7062</td>
<td>18.5912</td>
<td>25.7651</td>
<td>42.4380</td>
<td>62.6190</td>
<td>90.1489</td>
</tr>
<tr>
<td>S-S</td>
<td>16.2054</td>
<td>26.5768</td>
<td>42.3472</td>
<td>62.4802</td>
<td>90.1361</td>
<td>120.5933</td>
</tr>
<tr>
<td>C-C</td>
<td>15.1683</td>
<td>29.4314</td>
<td>42.1633</td>
<td>63.6958</td>
<td>89.9281</td>
<td>121.3566</td>
</tr>
<tr>
<td>C-F</td>
<td>10.8006</td>
<td>22.9898</td>
<td>29.1021</td>
<td>61.7083</td>
<td>64.3423</td>
<td>120.5406</td>
</tr>
</tbody>
</table>

*Results from Ref. [23].

### Table 2: The first six nondimensional frequency $\Omega$ for the uniform double-beams with different boundary conditions ($k_c = 0$, $K_c = 1 \times 10^5$ N/m²).

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Beam 1 $\Omega_1$</th>
<th>Beam 2 $\Omega_2$</th>
<th>$\Omega_3$</th>
<th>$\Omega_4$</th>
<th>$\Omega_5$</th>
<th>$\Omega_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S</td>
<td>9.8683</td>
<td>11.5923</td>
<td>39.4576</td>
<td>43.9572</td>
<td>88.7208</td>
<td>90.8309</td>
</tr>
<tr>
<td>C-C</td>
<td>22.3008</td>
<td>23.3205</td>
<td>61.4048</td>
<td>64.4416</td>
<td>120.1731</td>
<td>121.9079</td>
</tr>
<tr>
<td>C-F</td>
<td>3.5113</td>
<td>5.3243</td>
<td>21.9985</td>
<td>24.5854</td>
<td>61.5645</td>
<td>64.5643</td>
</tr>
<tr>
<td>S-S</td>
<td>22.3421</td>
<td>23.6691</td>
<td>61.4904</td>
<td>63.3157</td>
<td>120.2885</td>
<td>122.2928</td>
</tr>
<tr>
<td>S-S</td>
<td>10.3911</td>
<td>20.2322</td>
<td>28.4861</td>
<td>42.4389</td>
<td>63.6832</td>
<td>89.9532</td>
</tr>
<tr>
<td>S-S</td>
<td>10.9830</td>
<td>23.0436</td>
<td>42.1633</td>
<td>63.6958</td>
<td>89.9281</td>
<td>121.3566</td>
</tr>
</tbody>
</table>

*Results from Ref. [23].

### Table 3: Nondimensional natural frequencies for a tapered nonuniform beam with clamped-free (C-F) boundary condition and different cross-section parameters.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
<th>$\Omega_3$</th>
<th>$\Omega_4$</th>
<th>$\Omega_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.60566</td>
<td>20.6030</td>
<td>56.1232</td>
<td>3.85241</td>
<td>21.0389</td>
</tr>
<tr>
<td>0.4</td>
<td>3.60827</td>
<td>20.6210</td>
<td>56.1923</td>
<td>3.83511</td>
<td>21.0568</td>
</tr>
<tr>
<td>0.6</td>
<td>3.73471</td>
<td>19.0993</td>
<td>50.2941</td>
<td>4.31621</td>
<td>20.0358</td>
</tr>
<tr>
<td>0.8</td>
<td>3.73708</td>
<td>19.1138</td>
<td>50.3537</td>
<td>4.31878</td>
<td>20.0500</td>
</tr>
</tbody>
</table>

3.4. Nonuniform Double-Beam System with Elastic Boundary Condition. Finally, the nonuniform double-beam system is considered, in which both beams are tapered beams as defined in Section 3.2. Here, the boundary condition is clamped at $x = 0$ and elastic at $x = L$. The influence of elastic boundary stiffness $K_2$ ($K_2 = K_{TL} = K_{RL}$) at $x = L$ of both beams and coupling spring $K_C$ ($K_C = k_c = k_c$) are plotted in Figure 3. As discussed in Section 3.3, the rotational spring cannot be ignored, and the translational and rotational springs of the coupling spring are set at the same value. From Figure 3, when the elastic boundary stiffness increases, the frequencies will become bigger, and there will be a sensitive area in which the influence to the fundamental frequency is uniform. Otherwise, the fundamental frequencies will not
change when there is an increase in the coupling stiffness in Figure 3(a) which illustrates the same characteristic as in Section 3.1 for the uniform double-beam system. In Figure 3(b), the second mode frequency will increase along with the increase of elastic boundary and coupling spring. With the increase of coupling spring KC, there is a rapid change when the coupling spring KC is around $10^5$ N/m², which means the influence of coupling spring on the frequency is more obvious than that of elastic boundary.

In order to analyze the coupling influence of elastic boundary and taper ratio of the nonuniform beam, the fundamental frequency of the same structure is shown in Table 6, and the corresponding results are plotted in Figure 4. The sensitive area of the influence of elastic boundary will move when the taper ratio changes. At the left side of the sensitive which means the elastic boundary stiffness is smaller, the fundamental frequency will increase along with the raise of the taper ratio, and the contrary tendency will occur when on the right side. In the sensitive area, each line in Figure 4 will intersect with each other which means the fundamental frequency are equal at the certain value of elastic boundary condition. This phenomenon implies that when the change of elastic boundary is difficult to achieve at, one can get the same result by changing the taper ratios of the beams.

### 4. Conclusion

In this paper, an efficient modeling approach for the vibration analysis of nonuniform double-beam system with general boundary condition is established, in which the full-coupling on the common interface are taken into account by introducing both the translational and rotational restraining springs across the beams. The elastic boundary condition of the beams and various rotational restraints in the beam’s interface can be studied easily in the current modeling. In order to treat the nonuniform beam profile in the most general pattern, the arbitrary thickness variation functions are all expanded into Fourier series. Energy formulation is employed for the description of double-beam dynamics,

![Figure 2: The influence of the coupling translational spring $k_c$ and rotational spring $K_c$ to the nondimensional fundamental frequencies for the nonuniform double-beam system with $c = 0.2$ and $r = 2$.](image-url)
with the admissible function constructed as the superposition of Fourier series and boundary smoothed auxiliary terms. In conjunction with the Rayleigh–Ritz procedure, all the modal parameters can be derived by solving a standard eigenvalue problem.

Numerical examples are then presented to demonstrate the correctness and reliability of the current model for predicting the modal frequencies of the uniform double-beam and nonuniform beam structures. Based on the model established, the influence of boundary condition and coupling strength on the modal characteristics of the non-uniform beam is investigated and addressed. The results show that the rotational coupling stiffness can also play an important role in affecting the double-beam system, which has received little research attention in the existing literature. It can be also found that the variation of the thickness profile can be utilized to adjust the system modal parameters when the change of boundary and/or coupling conditions may be difficult to perform. Although just simulation is implemented for the double-beam system, this approach can be very easy for handling the multiple-beam structure of any number. This work can provide an efficient modeling approach for the dynamic study of the multiple-beam system with complex boundary, coupling, and thickness variation conditions.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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