

Research Article

Flexural Vibration Analysis of Nonuniform Double-Beam System with General Boundary and Coupling Conditions

Lujun Chen,^{1,2} Deshui Xu ,³ Jingtao Du ,³ and Chengwen Zhong ¹

¹School of Aeronautics, Northwestern Polytechnical University, Xi'an 710072, China

²China Aerodynamics Research and Development Center, Mianyang 621000, China

³College of Power and Energy Engineering, Harbin Engineering University, Harbin 150001, China

Correspondence should be addressed to Deshui Xu; xudeshui2010@163.com

Received 18 May 2018; Revised 1 August 2018; Accepted 26 August 2018; Published 1 October 2018

Academic Editor: Pedro Museros

Copyright © 2018 Lujun Chen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, an analytical modeling approach for the flexural vibration analysis of the nonuniform double-beam system is proposed via an improved Fourier series method, in which both types of translational and rotational springs are introduced to account for the mechanical coupling on the interface as well as boundary restraints. Energy formulation is employed for the dynamic description of the coupling system. With the aim to treat the varying thickness across the beam in a unified pattern, the relevant variables are all expanded into Fourier series. Supplementary terms with the smoothed characteristics are introduced to the standard Fourier series for the construction of displacement admissible function for each beam. In conjunction with the Rayleigh–Ritz procedure, the transverse modal characteristics of nonuniform double-beam system can be obtained by solving a standard eigenvalue problem. Instead of solving the certain value of nonideal boundary conditions, the continuous spring stiffnesses of the boundary conditions are considered, and the rotational restraints are introduced in the coupling beam interface. Numerical results are then presented to demonstrate the reliability of the current model and study the influence of various parameters, such as taper ratio, boundary, and coupling strength on the free vibration characteristics, with the emphasis put on the rotational restraining coefficients on the beam interface. This work can provide an efficient modeling framework for the vibration characteristics study of the complex double-beam system, especially with arbitrary varying thickness and coupling stiffness.

1. Introduction

The multiple-beam system has been extensively studied due to its wide application in various branches, such as mechanical and aeronautical engineering. A good understanding on its dynamic characteristics will be of great importance for the efficient design as well as vibration control of such complex system. For this reason, a lot of research attention has been devoted to the vibration behavior of multiple-beam structure by many researchers in the past decades.

For these multiple-beam structures, mechanical coupling between each beam component is usually taken into account through the introduction of translational spring across the beam interface [1]. Seelig and Hoppmann [2] formulated the differential equation of motion and obtained the solution for vibration analysis of the n elastically connected parallel beams.

Rao [3] solved the differential governing equation for the flexural vibration of elastically connected parallel bars based on the Timoshenko beam theory, in which the effects of rotary inertia and shear deformation are considered. Li and Hua [4] employed the dynamic stiffness method to analyze the free vibration characteristics of a three-beam system with the elastic spring and dashpot on the coupling interface. Moreover, Winkler elastic layers are also accounted for in the vibration analysis of coupling interface of the multiple-beam system [5–7]. Deng et al. [8] also studied the double functionally graded Timoshenko beam system resting on Winkler–Pasternak elastic foundation using Hamilton's principle.

In many occasions, the nonuniform beam component will be encountered with the background of optimal design, in which a better or more suitable distribution of mass and

strength than the uniform beam is of great desire. Thus, vibration analysis of the nonuniform beam has been studied continuously [9–11]. Abrate [12] presented a method to transform the motion equation of a nonuniform beam into its uniform counterpart. Lee and Lee [13] developed a transfer-matrix method to investigate the free vibration characteristics of a tapered Bernoulli–Euler beam. Bessel functions are widely used in dealing with the nonuniform beam system with different boundary conditions. Rosa and Auciello [14] solved the nonuniform beam-governing equation and the elastic boundary condition by substituting the four flexibility coefficients of the constructed functions. Auciello and Ercolano [15] investigated the transverse vibration of a beam, for which one part was tapered, and the remaining part is the uniform thickness. Torabi et al. [16] derived an exact closed-form solution for the free vibration of Euler–Bernoulli conical and tapered beams with any number of attached masses, which are described by Dirac’s delta function. Furthermore, Chen and Pan Liu [17] employed the Bessel function and the numerical assembly method (NAM) to investigate the vibration characteristics of a tapered beam with multiple arbitrarily placed rotational dampers.

In the current studies on the vibration analysis of the multiple-beam system, most of them are devoted to the uniform ones, and just little exception can be found in the literature for the vibration studies of nonuniform multiple-beam structure. For example, Mabie and Rogers [18, 19] obtained an accurate solution for determining the first five frequencies of the cantilevered double-tapered beam system. Takahashi and Yoshioka [20] utilized the transfer-matrix approach to analyze the vibration and stability of a double-cantilever beam, in which only translational spring is used for the description of coupling strength between two beams. From the above literature review, it can be found that the current study mainly considered the classical boundary conditions and just one type of restraining spring on the beam interface.

From the practical point of view, elastic boundary restraint and other types of mechanical coupling strength should be considered. There is a clear gap in the literature on this aspect. In analysis of the real system, the nearest ideal boundary conditions such as clamped, free, and simply support is selected for the modeling. However, small deviations from ideal conditions in real systems indeed occur. For example, a beam connected at the ends to rigid supports by pins is modeled using simply supported boundary conditions which require deflections and moments to be zero. But the hole and pin assembly may have small gaps and/or friction which is called the nonideal boundary. In this paper, the spring stiffnesses of the boundary conditions are different and continuous, so the elastic boundary is used here which includes the aforementioned nonideal boundary. Moreover, when the beam is under axial force or in axial moving, the minimal rotations are emerged, and the rotation restrains are important. For the double beam system, when the similar rotation movements happen, rotational coupling spring is introduced to restrain the relevant displacement. Claeys et al. [21] presents a direct comparison of measured

and predicted nonlinear vibrations of a clamped-clamped steel beam with nonideal boundary conditions, and the results show that the nonideal boundary of transverse spring and rotational spring have significant effect for the beam and a correct estimation is necessary to simulate the real structure’s characteristic.

In this work, motivated by the current limitation in literature, an efficient modeling approach for the vibration analysis of the nonuniform double-beam system with general elastic boundary condition is proposed, in which both the translational and rotational restraining springs are taken into account to describe the dynamic interaction between each beam interface. Energy principle is formulated for the analysis of the system motion equation, with the nonuniform thickness variation expanded into the Fourier series in a unified pattern. The transverse displacement admissible function is constructed as the superposition of the standard Fourier series and the boundary smoothed auxillary terms. In conjunction with the Rayleigh–Ritz procedure, all the modal parameters can be derived by solving an eigenvalue matrix. Numerical examples are then given to demonstrate the reliability and effectiveness of the current model. Finally, some concluding remarks are made.

2. Theoretical Formulations

2.1. Model Description. Consider an elastically connected nonuniform double-beam system, as illustrated in Figure 1. The beam member with arbitrary boundary conditions is coupled with other through the elastic restraining springs on the coupling interfaces. Such mechanical interaction is represented using two types of springs, namely translational and rotational one. Any boundary condition can be easily obtained by setting the relevant restraining spring stiffnesses. For the beam vibration, there are two freedom degrees at each field point, namely translation and rotation. Then, a full-coupling restraint should include the translational and rotational spring distributions, accordingly. In this way, the familiar Winkler-type of elastic interface can be readily derived by setting the coefficients of rotational springs into zero.

2.2. Double-Beam System Dynamics and Its Solution. For the coupled beam structure as shown in Figure 1, it will be described from the viewpoint of energy. Although the double-beam system is considered here, the current modeling framework is suitable for the general N beam system. Without losing the generality, the subsequent formulation will be given for such N -beam coupling system. Lagrangian function is used for the description of system dynamic behavior, which can be written as

$$L = V - T = \sum_{i=1}^N V_i + V_{\text{coupling}} - \sum_{i=1}^N T_i, \quad (1)$$

where L is the system Lagrangian, V and T are the total potential energy and kinetic energy, V_i is the potential energy associated with the i^{th} beam member, V_{coupling} is

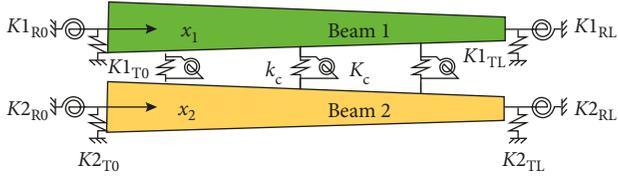


FIGURE 1: Nonuniform double-beam system with arbitrary boundary conditions, in which both the translational and rotational coupling effects are taken into account.

the potential energy stored in the interface coupling springs, and T_i is the kinetic energy due to the i^{th} vibrating beam.

For the i^{th} beam member with elastically restrained edges, its potential energy V_i is

$$\begin{aligned}
 V_i = & \frac{1}{2} \int_0^L E_i I_i(x) \left[\frac{\partial^2 w_i(x)}{\partial x^2} \right]^2 dx + \frac{1}{2} K_{T0}^i w_i^2(x) \Big|_{x=0} \\
 & + \frac{1}{2} K_{R0}^i \left[\frac{\partial w_i(x)}{\partial x} \right]^2 \Big|_{x=0} + \frac{1}{2} K_{TL}^i w_i^2(x) \Big|_{x=L} \\
 & + \frac{1}{2} K_{RL}^i \left[\frac{\partial w_i(x)}{\partial x} \right]^2 \Big|_{x=L}, \quad (2)
 \end{aligned}$$

where $w(x)$ is the transverse vibration displacement field function, K_{T0} and K_{R0} are respectively the stiffness coefficients for the translational and rotational springs at the end $x = 0$, and similar meaning can be deduced for the right end of $x = L$. The subscript i means that this variable is associated with the i^{th} beam member. $I_i(x)$ is the moment of inertia of the nonuniform beam.

The total kinetic energy of the i^{th} beam structure is

$$T_i = \frac{1}{2} \int_0^L \rho_i S_i(x) \left[\frac{\partial w_i(x, t)}{\partial t} \right]^2 dx = \frac{1}{2} \omega^2 \int_0^L \rho_i S_i(x) w_i^2(x) dx, \quad (3)$$

where ω is the radian frequency and ρ_i and $S_i(x)$ are respectively the mass density and cross section area of the i^{th} beam member.

The coupling potential energy between the interfaces can be written as

$$V_{\text{coupling}} = \frac{1}{2} \int_0^L \left[k_c (w_i - w_{i+1})^2 + K_c \left(\frac{\partial w_i}{\partial x} - \frac{\partial w_{i+1}}{\partial x} \right)^2 \right] dx, \quad (4)$$

in which k_c and K_c are respectively the translational and rotational coupling spring stiffnesses distributed across the interface between the i^{th} and $i + 1^{\text{st}}$ beam member.

In this work, in order to treat the cross-section area variation in a most general unified pattern, variables, such as arbitrary inertia $I_i(x)$ and cross-section area $S_i(x)$, associated with the nonuniform thickness variation are expanded into Fourier cosine series, with the corresponding expansion coefficients defined as follows:

$$\begin{aligned}
 S_i(x) = S_0 f_1(x) &= S_0 \left(\frac{c_0}{2} + \sum_{m=1}^{\infty} c_m \cos \lambda_m x \right) \\
 &= S_0 \sum_{m=0}^{\infty} c_m \cos \lambda_m x, \quad (5)
 \end{aligned}$$

$$c_m = \begin{cases} \frac{1}{L} \int_0^L f_1(x) dx, & m = 0, \\ \frac{2}{L} \int_0^L f_1(x) \cos \frac{m\pi}{L} x dx, & m \neq 0, \end{cases} \quad (6)$$

and

$$\begin{aligned}
 I_i(x) = I_0 f_2(x) &= I_0 \left(\frac{d_0}{2} + \sum_{m=1}^{\infty} d_m \cos \lambda_m x \right) \\
 &= I_0 \sum_{m=0}^{\infty} d_m \cos \lambda_m x, \quad (7)
 \end{aligned}$$

$$d_m = \begin{cases} \frac{1}{L} \int_0^L f_2(x) dx, & m = 0, \\ \frac{2}{L} \int_0^L f_2(x) \cos \frac{m\pi}{L} x dx, & m \neq 0, \end{cases} \quad (8)$$

in which $\lambda_m = m\pi/L$, and m is the Fourier series term.

For various kinds of cross sections, the difference between them is merely the Fourier coefficients and the items which can be obtained easily through the Fourier transformation. For the traditional uniform beam, $S_i(x)$ and $I_i(x)$ will be constant across the beam length.

Once the system Lagrangian is obtained, the other thing is to construct the appropriate admissible function. Differential continuity of the constructed function has significant effect on the final convergence and accuracy. Here, an improved Fourier series method is employed for this purpose, in which the additional functions are introduced to the standard Fourier series to remove all the discontinuities associated with the spatial differentiation of the displacement field functions. For each beam member, its flexural vibrating displacement function is expanded as [22]

$$\begin{aligned}
 w(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + b_1 \zeta_1(x) \\
 + b_2 \zeta_2(x) + b_3 \zeta_3(x) + b_4 \zeta_4(x), \quad (9)
 \end{aligned}$$

in which

$$\begin{aligned}
 \zeta_1(x) &= \frac{9L}{4\pi} \sin\left(\frac{\pi x}{2L}\right) - \frac{L}{12\pi} \sin\left(\frac{3\pi x}{2L}\right), \\
 \zeta_2(x) &= -\frac{9L}{4\pi} \cos\left(\frac{\pi x}{2L}\right) - \frac{L}{12\pi} \cos\left(\frac{3\pi x}{2L}\right), \\
 \zeta_3(x) &= \frac{L^3}{\pi^3} \sin\left(\frac{\pi x}{2L}\right) - \frac{L^3}{3\pi^3} \sin\left(\frac{3\pi x}{2L}\right), \\
 \zeta_4(x) &= -\frac{L^3}{\pi^3} \cos\left(\frac{\pi x}{2L}\right) - \frac{L^3}{3\pi^3} \cos\left(\frac{3\pi x}{2L}\right). \quad (10)
 \end{aligned}$$

It can be easily proven that the current constructed trigonometric function can satisfy the displacement and its higher-order differentiation continuity requirement in the interval $(0, L)$. It should be pointed out that the choice of the supplementary functions is not unique, while the appropriate form will be helpful for simplifying the subsequent mathematical formulations.

Substituting the admissible function Equation (9) into the elastically connected double-beam system Lagrangian Equations (1)–(4), minimizing it with respect to all the unknown Fourier series coefficients and truncating the Fourier series into finite number $n = N$, one will obtain the system characteristic equation in the matrix form:

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{A} = \mathbf{0}, \quad (11)$$

where \mathbf{K} and \mathbf{M} are the stiffness and mass matrices for the elastically connected double-beam system, respectively, and \mathbf{A} is the unknown Fourier series coefficient vector. By solving such standard eigenvalue problem, all the modal parameters can be easily obtained.

3. Numerical Examples and Discussions

In this section, the aforementioned modeling framework is programmed in the MATLAB environment. Several numerical results of different kinds of cross sections with various boundary conditions will be presented to demonstrate the effectiveness and advantage of the proposed model. As no research has been published about the free vibration of double-beam system with uniform cross sections, the results of the current method will be the first compared with those from other relevant literatures to validate the correctness. In current solution framework, the elastic boundary conditions are easily obtained by setting the restraining stiffness coefficient into various values accordingly. Similarly, variation of the arbitrary inertia $I_i(x)$ and cross-section area $S_i(x)$ can be easily handled using the same MATLAB program by just changing the mapping parameters of the nonuniform profile in Fourier space. In the following analysis, the nondimensional frequency parameter will be used, with their definitions as $\Omega = \omega L^2 \sqrt{\rho S / EI}$.

3.1. Double-Beam System with Uniform Cross Section. Here, by setting the relevant parameters to constant in Equations (5) and (7), several vibration results can be obtained for a uniform double-beam system, which has been investigated in some former studies. With the comparison purpose, modal parameters are kept the same as those used in Ref. [23], namely, $E_1 I_1 = 4 \times 10^6 \text{ Nm}^2$, $E_2 I_2 = 2 \times E_1 I_1$, $\rho_1 A_1 = 100 \text{ kg/m}$, $\rho_2 A_2 = 2 \times \rho_1 A_1$, $L = 10 \text{ m}$. From the comparison tabulated in Table 1, it can be observed that the current results can agree well with those from other approaches in which the parameters of beam 1 are used for the calculation of nondimensional frequency Ω . Comparing the results of Tables 1 and 2, it can be found that the variation of coupling stiffness will not affect the odd-mode frequency in the uniform double-beam system.

3.2. Single Nonuniform Beam with Different Cross-Section Parameters. In order to validate the proposed method for the case of variable cross sections, the coupling translational spring k_c and rotational spring K_c are set to zero which transfer the double-beam system to two single-beams with the same vibration characteristics. The polynomial function of inertia $I_i(x)$ and cross-section area $S_i(x)$ is as follows:

$$S_i(x) = S_0 \left(a - b \frac{x}{L} \right)^r, \quad (12)$$

and

$$I_i(x) = I_0 \left(a - b \frac{x}{L} \right)^{r+2}, \quad (13)$$

where b is the taper ratio of the cross-section area.

Table 3 shows the results of one single nonuniform beam with clamped-free (C-F) boundary condition, in which the data taken from Ref. [24] are also presented to validate the correctness of the present method in dealing with the different cross sections. With the increase of the taper ratio b , the frequency will raise which represents the decrease of the beam's stiffness.

Moreover, a single beam with three-step changes in cross section is analyzed. The geometrical dimensions and material properties are $L_R = 1 \text{ m}$, $m_R = 1 \text{ kg/m}$, $EI_R = 1 \text{ Nm}^2$, $\mu_1 = 1.0$, $\mu_2 = 0.8$, $\mu_3 = 0.65$, $\mu_4 = 0.25$, $L_1 = 0.25L_R$, $L_2 = 0.3L_R$, $L_3 = 0.25L_R$, and $L_4 = 0.2L_R$. Table 4 lists the first four frequency parameters for various combinations of boundary conditions. The results can agree well with those from the reference approaches.

3.3. Nonuniform Double-Beam System with Uniform Cross Section of Beam 2. The nonuniform double beam system is considered with C-F boundary condition for both beams in which the upper beam's cross-section function is the same as defined in Equations (12) and (13) in Section 3.2. The lower beam's cross section is uniform with the same parameter of the upper beam at $x = 0$. Other material parameters are the same for the two beams. Table 5 lists the results of different coupling stiffness boundaries. When the coupling translational spring k_c and rotational spring K_c are both zero, the frequencies are from the two single beams in which the odd-mode results of the lower uniform beam is larger than the even-mode results of the upper taper beam because the nonuniform cross section makes the beam's stiffness smaller. After the coupling stiffness k_c and K_c becomes bigger, the frequencies for both beams becomes larger. As the value of k_c and K_c are both 10^7 , the frequencies are similar as the single beam in which the value is bigger than odd- and even-mode results.

The influence of different coupling springs k_c and K_c on the fundamental frequency of the double-beam system is presented in Figure 2 which can illustrate that the influence of rotational spring K_c is obvious than that of translational spring k_c which means the rotational spring cannot be ignored in the double-beam system.

TABLE 1: The first six nondimensional frequency Ω for the uniform double-beams with different boundary conditions ($k_c = 1 \times 10^5 \text{ N/m}^2$, $K_c = 0$).

Boundary conditions		Nondimensional frequency Ω_i					
Beam 1	Beam 2	1	2	3	4	5	6
S-S	S-S	9.8683	21.7345	39.4577	43.9572	88.7208	90.8309
		9.8696 ^a	21.7350	39.4784	43.9721	88.8265	90.9128
C-C	C-C	22.3011	29.5464	61.4070	64.4416	120.1731	121.9079
		22.3733 ^a	29.5900	61.6728	64.6416	120.9034	122.4444
C-F	C-F	3.5113	19.6809	21.9986	29.3130	61.5645	64.5643
		3.5160 ^a	19.6815	22.0345	29.3346	61.6972	64.6649
S-S	C-F	8.7062	18.5912	25.7651	42.4830	62.6190	90.1489
		10.3911 ^b	20.2322	28.4861	42.4398	63.6832	89.9532
S-S	C-C	16.2054	26.5768	42.3472	62.4802	90.1361	120.5933
		15.1683 ^b	29.4314	42.1633	63.6958	89.9281	121.3566
C-C	C-F	10.8006	22.9898	29.1021	61.7083	64.3423	120.5406
		10.8090 ^a	23.0286	29.1356	61.8652	64.4954	120.9652

^aResults from Ref. [23]. ^bResults from FEM with 500 elements.

TABLE 2: The first six nondimensional frequency Ω for the uniform double-beams with different boundary conditions ($k_c = 0$, $K_c = 1 \times 10^5 \text{ N/m}^2$).

Boundary conditions		Nondimensional frequency Ω_i					
Beam 1	Beam 2	1	2	3	4	5	6
S-S	S-S	9.8683	11.5923	39.4576	41.2919	88.7207	90.5930
		9.8649 ^a	12.1168	39.4240	41.8523	88.5772	91.0371
C-C	C-C	22.3008	23.3205	61.4048	62.8443	120.1720	121.8760
		22.3421 ^a	23.6691	61.4904	63.3157	120.2885	122.2928
C-F	C-F	3.5113	5.3243	21.9985	24.5854	61.5643	63.8939
		3.5148 ^a	5.7683	22.0091	25.3841	61.5438	64.5943
S-S	C-F	4.2204	10.9595	22.9299	40.6538	62.3415	89.9755
		4.8060 ^a	10.8866	23.8391	40.5768	63.1366	89.7648
S-S	C-C	11.0137	22.6417	40.6693	61.8348	89.9830	120.5744
		10.9830 ^a	23.0436	40.6083	62.4392	89.7773	121.3189
C-C	C-F	4.2334	22.2624	23.6115	61.6101	63.1334	120.5412
		4.7805 ^a	22.3691	24.3769	61.6224	63.8560	120.4110

^aResults from Ref. [23].

TABLE 3: Nondimensional natural frequencies for a tapered nonuniform beam with clamped-free (C-F) boundary condition and different cross-section parameters.

b		$c = 1$			$c = 2$		
		Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.2	Current	3.60566	20.6030	56.1232	3.85241	21.0389	56.5570
	Ref. [24]	3.60827	20.6210	56.1923	3.85511	21.0568	56.6303
0.4	Current	3.73471	19.0993	50.2941	4.31621	20.0358	51.2654
	Ref. [24]	3.73708	19.1138	50.3537	4.31878	20.0500	51.3346
0.6	Current	3.93214	17.4767	43.9735	5.00654	19.0544	45.6680
	Ref. [24]	3.93428	17.4878	44.0248	5.00904	19.0649	45.7384
0.8	Current	4.29053	15.7350	36.8378	6.19373	18.3793	39.7350
	Ref. [24]	4.29249	15.7427	36.8846	6.19639	18.3855	39.8336

3.4. *Nonuniform Double-Beam System with Elastic Boundary Condition.* Finally, the nonuniform double-beam system is considered, in which both beams are tapered beams as defined in Section 3.2. Here, the boundary condition is clamped at $x = 0$ and elastic at $x = L$. The influence of elastic boundary stiffness K_2 ($K_2 = K_{TL} = K_{RL}$) at $x = L$ of both beams and coupling spring KC ($KC = k_c = K_c$) are plotted in

Figure 3. As discussed in Section 3.3, the rotational spring cannot be ignored, and the translational and rotational springs of the coupling spring are set at the same value. From Figure 3, when the elastic boundary stiffness increases, the frequencies will become bigger, and there will be a sensitive area in which the influence to the fundamental frequency is uniform. Otherwise, the fundamental frequencies will not

TABLE 4: The first four dimensionless frequencies Ω for the three-stepped beams.

BC	Ω_1	Ω_2	Ω_3	Ω_4
SS-SS	3.097	6.184	9.343	12.605
	3.096 ^a	6.184	9.343	12.605
SS-F	4.313	7.331	10.240	13.297
	4.313 ^a	7.331	10.240	13.297
C-C	4.536	7.663	10.808	14.068
	4.541 ^a	7.660	10.809	14.064
C-F	2.287	5.133	8.083	10.978
	2.285 ^a	5.133	8.083	10.978

^aResults from [25].

TABLE 5: Nondimensional natural frequencies for the nonuniform double-beam system under different coupling spring stiffness with $c = 0.2$ and $n = 2$ of beam 2.

Ω	$k_c = K_c = 0$ (N/m ²)	$k_c = 0, K_c = 10^4$ (N/m ²)	$k_c = 10^4, K_c = 0$ (N/m ²)	$k_c = K_c = 10^4$ (N/m ²)	$k_c = K_c = 10^7$ (N/m ²)
1	3.5132	3.5972	3.6543	3.6545	3.6641
2	3.8524	4.1375	8.6261	8.7880	21.6424
3	21.0389	21.2602	21.4288	21.4751	59.5703
4	22.0129	22.2475	22.9277	23.3122	116.1607
5	56.5570	56.7903	56.8261	57.0392	191.3209
6	61.6177	61.7810	61.8310	62.0116	244.6118
7	109.6314	109.8539	109.7742	109.9922	267.2176
8	120.6885	120.8388	120.7933	120.9473	285.3005
9	180.0414	180.2551	180.1284	180.3406	313.1571
10	199.3790	199.5227	199.4420	199.5869	379.2597

change when there is an increase in the coupling stiffness in Figure 3(a) which illustrates the same characteristic as in Section 3.1 for the uniform double-beam system. In Figure 3(b), the second mode frequency will increase along with the increase of elastic boundary and coupling spring. With the increase of coupling spring KC , there is a rapid change when the coupling spring KC is around 10^5 N/m², which means the influence of coupling spring on the frequency is more obvious than that of elastic boundary.

In order to analyze the coupling influence of elastic boundary and taper ratio of the nonuniform beam, the fundamental frequency of the same structure is shown in Table 6, and the corresponding results are plotted in Figure 4. The sensitive area of the influence of elastic boundary will move when the taper ratio changes. At the left side of the sensitive which means the elastic boundary stiffness is smaller, the fundamental frequency will increase along with the raise of the taper ratio, and the contrary tendency will occur when on the right side. In the sensitive area, each line in Figure 4 will intersect with each other which means the fundamental frequency are equal at the certain value of elastic boundary condition. This phenomenon implies that when the change of elastic boundary is difficult to achieve at, one can get the same result by changing the taper ratios of the beams.

4. Conclusion

In this paper, an efficient modeling approach for the vibration analysis of nonuniform double-beam system with general boundary condition is established, in which the full-

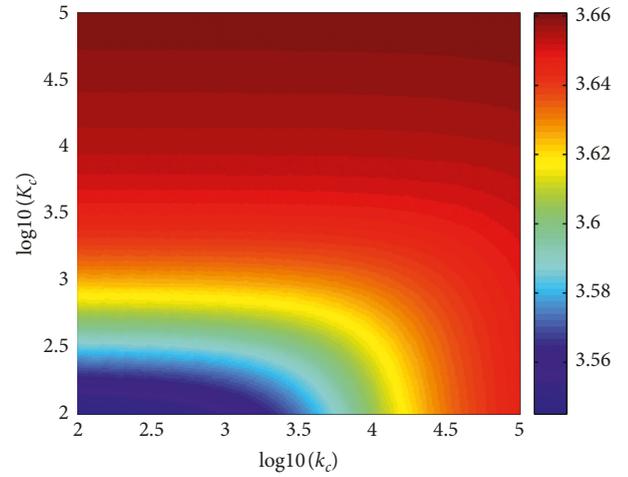


FIGURE 2: The influence of the coupling translational spring k_c and rotational spring K_c to the nondimensional fundamental frequencies for the nonuniform double-beam system with $c = 0.2$ and $r = 2$.

coupling on the common interface are taken into account by introducing both the translational and rotational restraining springs across the beams. The elastic boundary condition of the beams and various rotational restrains in the beam's interface can be studied easily in the current modeling. In order to treat the nonuniform beam profile in the most general pattern, the arbitrary thickness variation functions are all expanded into Fourier series. Energy formulation is employed for the description of double-beam dynamics,

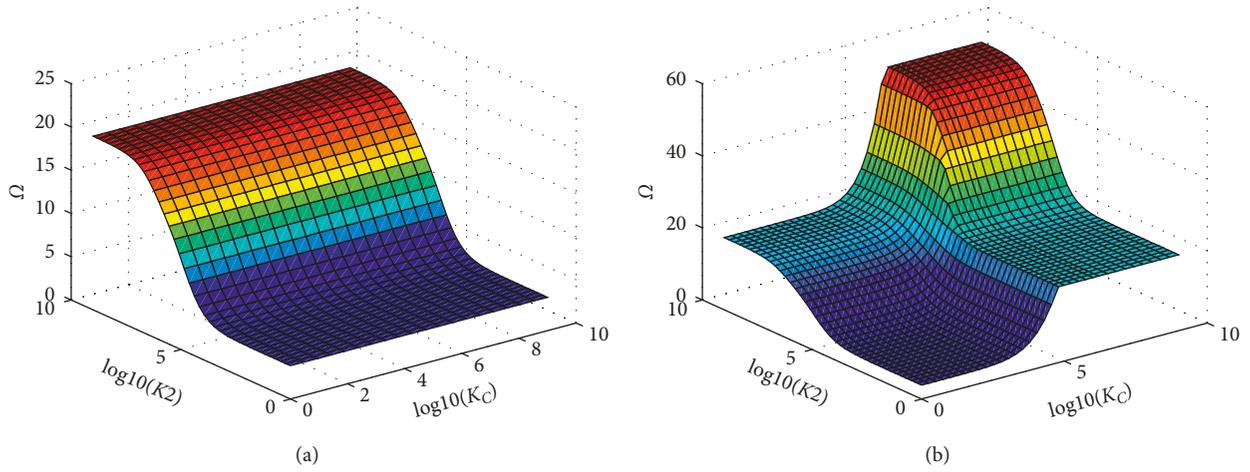


FIGURE 3: Nondimensional fundamental and second mode frequencies of the nonuniform double-beam system with $c = 0.2$ and $r = 2$. (a) Mode 1; (b) mode 2.

TABLE 6: Nondimensional system fundamental frequencies for the non-uniform double-beam system under elastic boundary with $r = 2$ and $k_c = s_c = 100 \text{ N/m}^2$.

$K_2 \text{ (N/m}^2\text{)}$	c				
	0	0.2	0.4	0.6	0.8
0	3.5133	3.8526	4.3165	5.0070	6.1947
10	3.5146	3.8544	4.3191	5.0113	6.2036
10^2	3.5276	3.8720	4.3448	5.0537	6.2906
10^3	3.6548	4.0431	4.5915	5.4483	7.0016
10^4	4.7161	5.4056	6.3962	7.8291	9.5231
10^5	9.4720	10.5052	11.4273	12.0812	11.7939
10^6	15.7713	15.8984	15.6945	14.5994	12.3139
10^7	20.5987	19.2828	17.4287	15.1154	12.3731
10^8	22.1393	19.9912	17.6770	15.1730	12.3791
10^9	22.3299	20.0700	17.7029	15.1788	12.3797

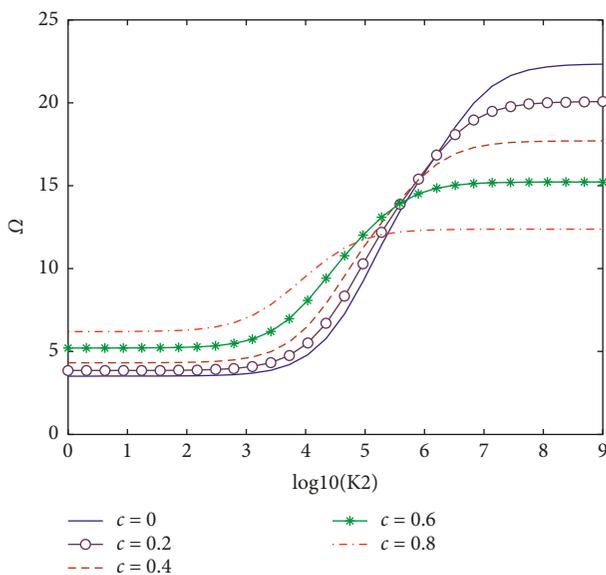


FIGURE 4: Nondimensional fundamental frequencies of the non-uniform double-beam system with $r = 2$ and $k_c = K_c = 100 \text{ N/m}^2$.

with the admissible function constructed as the superposition of Fourier series and boundary smoothed auxiliary terms. In conjunction with the Rayleigh–Ritz procedure, all the modal parameters can be derived by solving a standard eigenvalue problem.

Numerical examples are then presented to demonstrate the correctness and reliability of the current model for predicting the modal frequencies of the uniform double-beam and nonuniform beam structures. Based on the model established, the influence of boundary condition and coupling strength on the modal characteristics of the non-uniform beam is investigated and addressed. The results show that the rotational coupling stiffness can also play an important role in affecting the double-beam system, which has received little research attention in the existing literature. It can be also found that the variation of the thickness profile can be utilized to adjust the system modal parameters when the change of boundary and/or coupling conditions may be difficult to perform. Although just simulation is implemented for the double-beam system, this approach can be very easy for handling the multiple-beam structure of any number. This work can provide an efficient modeling approach for the dynamic study of the multiple-beam system with complex boundary, coupling, and thickness variation conditions.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Fok Ying Tung Education Foundation (Grant no. 161049).

References

- [1] Z. Oniszczuk, "Free transverse vibrations of an elastically connected complex beam-string system," *Journal of Sound and Vibration*, vol. 254, no. 4, pp. 703–715, 2002.
- [2] J. M. Seelig and W. H. Hoppmann, "Normal mode vibrations of systems of elastically connected parallel bars," *Journal of the Acoustical Society of America*, vol. 36, no. 1, pp. 93–99, 1964.
- [3] S. S. Rao, "Natural vibrations of systems of elastically connected Timoshenko beams," *Journal of the Acoustical Society of America*, vol. 55, no. 6, pp. 1232–1237, 1974.
- [4] J. Li and H. X. Hua, "Dynamic stiffness vibration analysis of an elastically connected three-beam system," *Applied Acoustics*, vol. 69, no. 7, pp. 591–600, 2008.
- [5] Z. Oniszczuk, "Free transverse vibrations of elastically connected simply supported double-beam complex system," *Journal of Sound and Vibration*, vol. 232, no. 2, pp. 387–403, 2000.
- [6] S. G. Kelly and S. Srinivas, "Free vibrations of elastically connected stretched beams," *Journal of Sound and Vibration*, vol. 326, no. 3–5, pp. 883–893, 2009.
- [7] M. A. De Rosa and M. Lippiello, "Non-classical boundary conditions and DQM for double-beams," *Mechanics Research Communications*, vol. 34, no. 7–8, pp. 538–544, 2007.
- [8] H. Deng, W. Cheng, and S. G. Zhao, "Vibration and buckling analysis of double-functionally graded Timoshenko beam system on Winkler-Pasternak elastic foundation," *Composite Structures*, vol. 160, pp. 152–168, 2017.
- [9] J. R. Banerjee and F. W. Williams, "Exact Bernoulli–Euler dynamic stiffness matrix for a range of tapered beams," *International Journal for Numerical Methods in Engineering*, vol. 21, no. 12, pp. 2289–2302, 1985.
- [10] E. M. Cem, M. Aydogdu, and V. Taskin, "Vibration of a variable cross-section beam," *Mechanics Research Communications*, vol. 34, no. 1, pp. 78–84, 2007.
- [11] S. Lenci, F. Clementi, and C. E. N. Mazzilli, "Simple formulas for the natural frequencies of non-uniform cables and beams," *International Journal of Mechanical Sciences*, vol. 77, pp. 155–163, 2013.
- [12] S. Abrate, "Vibration of non-uniform rods and beams," *Journal of Sound and Vibration*, vol. 185, no. 4, pp. 703–716, 1995.
- [13] J. W. Lee and J. Y. Lee, "Free vibration analysis using the transfer-matrix method on a tapered beam," *Computers & Structures*, vol. 164, pp. 75–82, 2016.
- [14] M. A. Rosa and N. M. Auciello, "Free vibrations of tapered beams with flexible ends," *Computers & Structures*, vol. 60, no. 2, pp. 197–202, 1996.
- [15] N. M. Auciello and A. Ercolano, "Exact solution for the transverse vibration of a beam a part of which is a taper beam and other part is a uniform beam," *International Journal of Solids and Structures*, vol. 34, no. 17, pp. 2115–2129, 1997.
- [16] K. Torabi, H. Afshari, M. Sadeghi, and H. Toghian, "Exact closed-form solution for vibration analysis of truncated conical and tapered beams carrying multiple concentrated masses," *Journal of Solid Mechanics*, vol. 9, no. 4, pp. 760–782, 2017.
- [17] Y. Chen and L. Pan Liu, "A complex-valued solution of free vibration for tapered beams with any number of rotational dampers," *Applied Mathematics & Information Sciences*, vol. 6, no. 3, pp. 749–758, 2012.
- [18] H. H. Mabie and C. B. Rogers, "Transverse vibrations of double-tapered cantilever beams," *Journal of the Acoustical Society of America*, vol. 51, no. 5B, pp. 1771–1774, 1972.
- [19] H. H. Mabie and C. B. Rogers, "Transverse vibrations of double-tapered cantilever beams with end support and with end mass," *Journal of the Acoustical Society of America*, vol. 55, no. 5, pp. 986–991, 1974.
- [20] I. Takahashi and T. Yoshioka, "Vibration and stability of a non-uniform double-beam subjected to follower forces," *Computers & Structures*, vol. 59, no. 6, pp. 1033–1038, 1996.
- [21] M. Claeys, J. J. Sinou, J. P. Lambelin, and B. Alcoverro, "Multi-harmonic measurements and numerical simulations of nonlinear vibrations of a beam with non-ideal boundary conditions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 12, pp. 4196–4212, 2014.
- [22] W. L. Li, X. F. Zhang, J. T. Du, and Z. G. Liu, "An exact series solution for the transverse vibration of rectangular plates with general elastic boundary supports," *Journal of Sound and Vibration*, vol. 321, no. 1–2, pp. 254–269, 2009.
- [23] Q. B. Mao, "Free vibration analysis of elastically connected multiple-beams by using the Adomian modified decomposition method," *Journal of Sound and Vibration*, vol. 331, no. 11, pp. 2532–2542, 2012.
- [24] J. R. Banerjee, H. Su, and D. R. Jackson, "Free vibration of rotating tapered beams using the dynamic stiffness method," *Journal of Sound and Vibration*, vol. 298, no. 4–5, pp. 1034–1054, 2006.
- [25] X. Wang and Y. Wang, "Free vibration analysis of multiple-stepped beams by the differential quadrature element method," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 5802–5810, 2013.

