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Research Article

Free Vibration Analysis of Rings via Wave Approach

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Free vibration of rings is presented via wave approach theoretically. Firstly, based on the solutions of out-of-plane vibration, propagation, reflection, and coordination matrices are derived for the case of a fixed boundary at inner surface and a free boundary at outer surface. Then, assembling these matrices, characteristic equation of natural frequency is obtained. Wave approach is employed to study the free vibration of these ring structures. Natural frequencies calculated by wave approach are compared with those obtained by classical method and Finite Element Method (FEM). Afterwards natural frequencies of four type boundaries are calculated. Transverse vibration transmissibility of rings propagating from outer to inner and from inner to outer is investigated. Finally, the effects of structural and material parameters on free vibration are discussed in detail.

1. Introduction

Noise and vibration problem of rings has been a hotspot that makes it attract lots of attention because of the wide applications, such as gear transmission system and aircraft structures. Since most of components within these structures can be regarded as a simple model of ring structure, dynamic properties of these structures in recent years have been the subject of present studies. Based on the Stodola–Vianello method, Gutierrez et al. [1] considered the free vibration of annular membranes with continuously variable density by using Rayleigh method, differential quadrature method, and finite elements simulation. By adopting the dynamic stiffness method, Jabareen and Eisenberger [2] investigated the natural characteristics of the nonhomogeneous annular structure. Wang [3] discussed the natural frequency with the boundaries of fixed, free, and simply supported. And the researches indicated clearly that complex modes would switch correspondingly with the radius increasing gradually. Based on the first-order shear deformation theory, Roshan and Rashmi [4] analyzed the free vibration of axisymmetric sandwich circular plate with relatively stiff thickness. Oveisi and Shakeri [5] constructed a sandwich composite circular plate containing piezoelectric material, finding that the feedback control gain had an effective control for suppressing the transverse vibration. Hosseini-Hashemi et al. [6] presented the free vibration of functionally graded circular plate with stepped thickness via Mindlin plate theory. By employing harmonic differential quadrature method for obtaining numerical solution of circular plate, Civalek and Uelker [7] made a further analysis for the behavior of the boundary conditions of fixed and simply supported. Moreover, the results obtained by finite difference method were compared with those calculated by harmonic differential quadrature method, whose feasibility is verified by Bakhshi Khaniki and Hosseini-Hashemi [8]. Liu et al. [9] construct the composite thin annular plate by wave approach. However, composite structures with multilayer are not studied. Different boundaries and parameter effects are also not analyzed.

Additionally, in terms of the vibration of structures, there is an alternative method called “wave approach” which is very efficient and widely used for calculating the natural frequency of structures, such as beams, plates, rings, and periodic structure by describing waves in the matrix form. For example, as early as 1984, Mace [10] applied wave approach to analyze the wave behaviors in Euler beam. By dividing the waves into propagation and attenuation matrices, he derived the reflection matrices under three boundary conditions, which established a theoretical foundation for wave approach. Adopting wave approach, Mei [11] analyzed the flexural vibration with added mass for Timoshenko beam, and the influence of lumped mass on natural frequency is also discussed in detail. Kang et al. [12] divided the real and imaginary parts of wave solutions of curved beam into four cases, and they calculated the natural frequencies by combining propagation, transmission and reflection matrices. Lee et al. [13, 14] considered the power flow when wave propagated in curved beam. Furthermore, they applied the Flugge theory to analyze the free vibration of a single curved beam, and their result was compared with Kang et al., which verified the correctness of the numerical results. Huang et al. [15] investigated the free vibration of planar rotating rings. The effect of cross section on natural frequency was also discussed. Bahrami and Teimourian [16] studied the free vibration of composite plates consisting of two layers, and they also made a comparison between classical results and wave propagation results. Tan and Kang [17] concentrated on the free vibration of rotating Timoshenko shaft with axial force and discussed the effect of continuous condition and cross section on natural frequencies. From the wave point of view, Bahrami and Teimourian [18] analyzed the free vibration of nanobeams for the first time. Ilkhani et al. [19] studied the free vibration of thin rectangular plate. It should be noted that the above scholars have done lots of studies for free vibration of structures by using wave approach, while few reports for the analysis of natural frequency for transverse vibration of rings can be found. In fact, it is well known that the nature of vibration is the propagation of waves. Analyzing free vibration in terms of wave propagation and attenuating can have a better understanding for us. Moreover, one advantage of using wave approach to analyze the free vibration is its conciseness of matrices that makes the natural frequencies be calculated easily. Wave approach is a strong tool for studying the behavior of wave transmission and reflection in waveguides, providing a practical engineering application such as filters.

The emphasis of this paper is focused on the free vibration of rings. This paper is organized into five parts. Section 1 is introduction. In Section 2, propagation, coordination, and reflection matrices are deduced in forms of matrix. In addition, the characteristic equation of natural frequency is obtained using classical method and wave approach. In Section 3, natural frequencies of rings are calculated by combining these matrices. Meanwhile, vibration transmissibility of rings propagating from outer to inner and from inner to outer is obtained. In Section 4, the influence of structural and material parameters on natural frequencies is discussed. Section 5 is the conclusion.

2. Theoretical Analysis

2.1. Classical Method for Free Vibration

2.1.1. Solution of Transverse Vibration. Consider sandwich rings consisting of two different materials depicted in Figure 1. Adhesive can be employed for connecting the rings. Same material is selected for the first and third layers. The other material is selected for the second layer. Radius of the first and third layers is r_0 and r_c. Radius of the intermediate

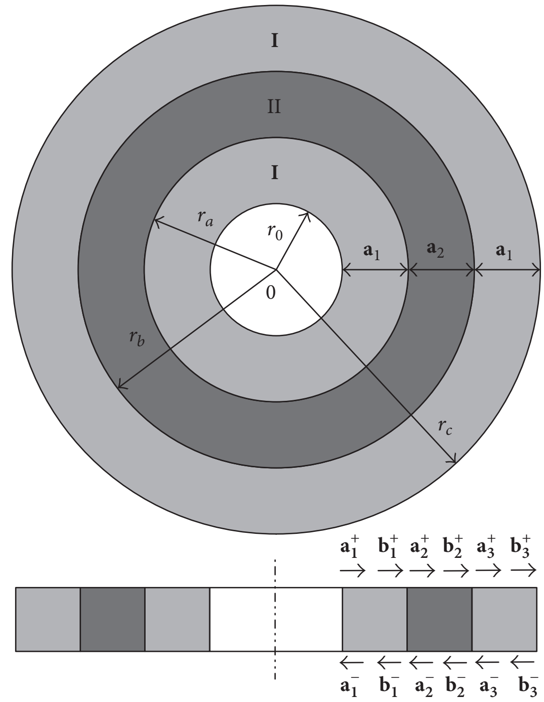


Figure 1: Composite rings.

layers is r_a and r_b. Radial span of the first and third layers is $a1$. Radial span of the second layer is $a2$. w is bending deflection. h is thickness. At the boundaries of $r = r_a$ and $r = r_b$, positive–going and negative–going wave vectors are $\mathbf{b}_1^+$, $\mathbf{b}_1^-$, $\mathbf{a}_2^+$, $\mathbf{a}_2^-$, $\mathbf{b}_2^+$, $\mathbf{b}_2^-$, $\mathbf{a}_3^+$, $\mathbf{a}_3^-$. Also, considering another two boundaries at $r = r_0$ and $r = r_c$, positive–going and negative–going wave vectors are $\mathbf{a}_1^+$, $\mathbf{a}_1^-$, $\mathbf{b}_3^+$, $\mathbf{b}_3^-$. In cylindrical coordinates, the radius is assumed to be large enough compared to thickness which means that it satisfies the small deformation theory. Transverse solution is given by Wang [3]:

$$W = A_1 J_0(kr) + B_2 Y_0(kr) + C_3 I_0(kr) + D_4 K_0(kr), \quad (1)$$

where A_1, B_2, C_3, D_4 are constants which are determined by boundaries. $J_0(kr)$ and $Y_0(kr)$ are Bessel functions of first and second kinds, respectively. $I_0(kr)$ and $K_0(kr)$ are modified Bessel functions of first and second kinds. $k = (4\pi^2 f^2 \rho h/D)^{0.25}$ is wave number, and $D = Eh^3/12(1-\sigma^2)$ is stiffness.

2.1.2. Solution of Classical Bessel Method. With regard to rings subjected to bending excitation, expression of transverse displacement, rotational angle, shear force, and bending moment within the first and third layers can be written as

$$\begin{aligned} W_1(r) &= A_{11} J_0(k_1 r) + B_{11} Y_0(k_1 r) + C_{11} I_0(k_1 r) \\ &\quad + D_{11} K_0(k_1 r), \\ \frac{\partial W}{\partial r}_1(r) &= -k_1 A_{11} J_1(k_1 r) - k_1 B_{11} Y_1(k_1 r) \\ &\quad + k_1 C_{11} I_1(k_1 r) - k_1 D_{11} K_1(k_1 r), \end{aligned}$$

$$\begin{aligned} M_1(r) = D\Big\{ & A_{11}\left[\frac{k_1}{r}J_1(k_1 r) - \frac{\sigma_1 k_1}{r}J_1(k_1 r) - k_1^{\ 2}J_0(k_1 r)\right] + B_{11}\left[\frac{k_1}{r}Y_1(k_1 r) - \frac{\sigma_1 k_1}{r}Y_1(k_1 r) \right. \\ & \left. - k_1^{\ 2}Y_0(k_1 r)\right] + C_{11}\left[k_1^{\ 2}I_0(k_1 r) - \frac{k_1}{r}I_1(k_1 r) + \frac{\sigma_1 k_1}{r}I_1(k_1 r)\right] + D_{11}\left[k_1^{\ 2}K_0(k_1 r) + \frac{k_1}{r}K_1(k_1 r) \right. \\ & \left. - \frac{\sigma_1 k_1}{r}K_1(k_1 r)\right]\Big\}, \end{aligned}$$

$$\begin{aligned} Q_1(r) = D\big\{ & A_{11}k_1^{\ 3}J_1(k_1 r) + B_{11}k_1^{\ 3}Y_1(k_1 r) \\ & + C_{11}k_1^{\ 3}I_1(k_1 r) - D_{11}k_1^{\ 3}K_1(k_1 r)\big\}. \end{aligned} \tag{2}$$

Applying fixed boundary condition at $r = r_0$ obtains

$$[J_0(k_1 r_0) \;\; Y_0(k_1 r_0) \;\; I_0(k_1 r_0) \;\; K_0(k_1 r_0)]\,\mathbf{\Psi_{11}} = 0 \tag{3}$$

$$[-k_1 J_1(k_1 r_0) \;\; -k_1 Y_1(k_1 r_0) \;\; k_1 I_1(k_1 r_0) \;\; -k_1 K_1(k_1 r_0)]\,\mathbf{\Psi_{11}} = 0, \tag{4}$$

where $\mathbf{\Psi}_{11} = [A_{11} \; B_{11} \; C_{11} \; D_{11}]^T$.

Free boundary condition is selected at $r = r_c$; then

$$D\left\{\begin{matrix} \left[\frac{k_1}{r_c}J_1(k_1 r_c) - \frac{\sigma_1 k_1}{r_c}J_1(k_1 r_c) - k_1^{\ 2}J_0(k_1 r_c)\right] & \left[\frac{k_1}{r_c}Y_1(k_1 r_c) - \frac{\sigma_1 k_1}{r_c}Y_1(k_1 r_c) - k_1^{\ 2}Y_0(k_1 r_c)\right] \\ \left[k_1^{\ 2}I_0(k_1 r_c) - \frac{k_1}{r_c}I_1(k_1 r_c) + \frac{\sigma_1 k_1}{r_c}I_1(k_1 r_c)\right] & \left[k_1^{\ 2}K_0(k_1 r_c) + \frac{k_1}{r_c}K_1(k_1 r_c) - \frac{\sigma_1 k_1}{r_c}K_1(k_1 r_c)\right] \end{matrix}\right\}\mathbf{\Psi}_{13} = 0$$

$$D\left[k_1^{\ 3}J_1(k_1 r_c) \;\; k_1^{\ 3}Y_1(k_1 r_c) \;\; k_1^{\ 3}I_1(k_1 r_c) \;\; -k_1^{\ 3}K_1(k_1 r_c)\right]\mathbf{\Psi_{13}} = 0, \tag{5}$$

where $\mathbf{\Psi_{13}} = [A_{13} \; B_{13} \; C_{13} \; D_{13}]^T$.

In order to obtain the natural frequencies, substituting (A.9) into (5) and combining (3)-(4), it reduces to

$$\begin{bmatrix} J_0(k_1 r_0) & Y_0(k_1 r_0) & I_0(k_1 r_0) & K_0(k_1 r_0) \\ -k_1 J_1(k_1 r_0) & -k_1 Y_1(k_1 r_0) & k_1 I_1(k_1 r_0) & -k_1 K_1(k_1 r_0) \\ \mathbf{J_1} & \mathbf{Y_1} & \mathbf{I_1} & \mathbf{K_1} \\ k_1^{\ 3}J_1(k_1 r_c)\times\mathbf{T_{13}} & k_1^{\ 3}Y_1(k_1 r_c)\times\mathbf{T_{13}} & k_1^{\ 3}I_1(k_1 r_c)\times\mathbf{T_{13}} & -k_1^{\ 3}K_1(k_1 r_c)\times\mathbf{T_{13}} \end{bmatrix}\mathbf{\Psi}_{11} = 0, \tag{6}$$

where the specific theoretical derivation of $\mathbf{T_{13}}$ in (6) is presented in the Appendix. And each element is defined as

$$\mathbf{J_1} = \left[\frac{k_1}{r_c}J_1(k_1 r_c) - \frac{\sigma_1 k_1}{r_c}J_1(k_1 r_c) - k_1^{\ 2}J_0(k_1 r_c)\right] \times \mathbf{T_{13}}, \tag{7a}$$

$$\mathbf{Y_1} = \left[\frac{k_1}{r_c}Y_1(k_1 r_c) - \frac{\sigma_1 k_1}{r_c}Y_1(k_1 r_c) - k_1^{\ 2}Y_0(k_1 r_c)\right] \times \mathbf{T_{13}}, \tag{7b}$$

$$\mathbf{I_1} = \left[k_1^{\ 2}I_0(k_1 r_c) - \frac{k_1}{r_c}I_1(k_1 r_c) + \frac{\sigma_1 k_1}{r_c}I_1(k_1 r_c)\right] \times \mathbf{T_{13}}, \tag{7c}$$

$$\mathbf{K_1} = \left[k_1^{\ 2}K_0(k_1 r_c) + \frac{k_1}{r_c}K_1(k_1 r_c) - \frac{\sigma_1 k_1}{r_c}K_1(k_1 r_c)\right] \times \mathbf{T_{13}}. \tag{7d}$$

Therefore, (6) can be written as a 4×4 determinant:

$$\begin{vmatrix} J_0(k_1 r_0) & Y_0(k_1 r_0) & I_0(k_1 r_0) & K_0(k_1 r_0) \\ -k_1 J_1(k_1 r_0) & -k_1 Y_1(k_1 r_0) & k_1 I_1(k_1 r_0) & -k_1 K_1(k_1 r_0) \\ \mathbf{J_1} & \mathbf{Y_1} & \mathbf{I_1} & \mathbf{K_1} \\ k_1^{\ 3}J_1(k_1 r_c)\times\mathbf{T_{13}} & k_1^{\ 3}Y_1(k_1 r_c)\times\mathbf{T_{13}} & k_1^{\ 3}I_1(k_1 r_c)\times\mathbf{T_{13}} & -k_1^{\ 3}K_1(k_1 r_c)\times\mathbf{T_{13}} \end{vmatrix} = 0, \tag{8}$$

where (8) is the characteristic equation of natural frequency. By searching the root, natural frequency of rings can be calculated with a fixed boundary at inner surface and a free boundary at outer surface.

2.1.3. Solution of Classical Hankel Method. The solution is obtained in (1). However, it also can be expressed in a Hankel form:

$$W = A_1^+ H_0^{(2)}(k_1 r) + A_1^- H_0^{(1)}(k_1 r) + B_1^+ K_0(k_1 r) + B_1^- I_0(k_1 r), \tag{9}$$

where $H_0^{(1)}(k_1 r)$ and $H_0^{(2)}(k_1 r)$ are the Hankel functions of second and first kinds, respectively. They can be defined as

$$\begin{aligned} H_0^{(1)}(k_1 r) &= J_0(k_1 r) + iY_0(k_1 r) \\ H_0^{(2)}(k_1 r) &= J_0(k_1 r) - iY_0(k_1 r). \end{aligned} \tag{10}$$

Similarly, expression of parameters within the first and third layers can be written as

$$\begin{aligned} W(r) = {} & A_1^+ [J_0(k_1 r) - iY_0(k_1 r)] + A_1^- [J_0(k_1 r) \\ & + iY_0(k_1 r)] + B_1^+ K_0(k_1 r) + B_1^- I_0(k_1 r) \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{\partial W}{\partial r}_1 (r) = {} & A_1^+ [-k_1 J_1(k_1 r) + ik_1 Y_1(k_1 r)] \\ & + A_1^- [-k_1 J_1(k_1 r) - ik_1 Y_1(k_1 r)] - B_1^+ k_1 K_1(k_1 r) \\ & + B_1^- k_1 I_1(k_1 r), \end{aligned} \tag{12}$$

$$\begin{aligned} M_1(r) = D \Bigg\{ & A_1^+ \bigg[\frac{k_1}{r} J_1(k_1 r) - k_1{}^2 J_0(k_1 r) \\ & + ik_1^2 Y_0(k_1 r) - \frac{ik_1}{r} Y_1(k_1 r) \\ & + \frac{\sigma}{r} [-k_1 J_1(k_1 r) + ik_1 Y_1(k_1 r)] \bigg] \\ & + A_1^- \bigg[\frac{k_1}{r} J_1(k_1 r) - k_1{}^2 J_0(k_1 r) - ik_1^2 Y_0(k_1 r) \\ & + \frac{ik_1}{r} Y_1(k_1 r) + \frac{\sigma}{r} [-k_1 J_1(k_1 r) - ik_1 Y_1(k_1 r)] \bigg] \\ & + B_1^+ \bigg[k_1{}^2 K_0(k_1 r) + \frac{k_1}{r} K_1(k_1 r) - \frac{\sigma}{r} k_1 K_1(k_1 r) \bigg] \\ & + B_1^- \bigg[k_1{}^2 I_0(k_1 r) - \frac{k_1}{r} I_1(k_1 r) + \frac{\sigma_1 k_1}{r} I_1(k_1 r) \bigg] \Bigg\} \end{aligned} \tag{13}$$

$$\begin{aligned} Q_1(r) = D \big\{ & A_1^+ [-ik_1^3 Y_1(k_1 r) + k_1^3 J_1(k_1 r)] \\ & - B_1^+ k_1^3 K_1(k_1 r) + A_1^- [ik_1^3 Y_1(k_1 r) + k_1^3 J_1(k_1 r)] \\ & + B_1^- k_1{}^3 I_1(k_1 r) \big\}. \end{aligned} \tag{14}$$

Natural frequencies of transverse vibration can be calculated using classical Hankel method. Characteristic equation of natural frequency can be deuced like the process of (3)–(8). In order to avoid repeating, herein, it is ignored.

2.2. Wave Approach for Free Vibration. In this section, the solution is presented in terms of cylindrical waves for this ring. Meanwhile, positive–going propagation, negative–going propagation, coordination, and reflection matrices are also deduced. By combining these matrices, natural frequencies are calculated using wave approach.

2.2.1. Propagation Matrices. Wave propagates along the positive–going and negative–going directions when propagating within structures, as is shown in Figure 1. Waves will not propagate at the boundaries but only can be reflected. Moreover, parameters are continuous for the connection. In recent years, many researchers describe the waves in the matrix forms [8–17].

By considering (11), positive–going waves can be described as

$$\mathbf{a}_1^+ = \begin{bmatrix} A_1^+ \{J_0(k_1 r_0) - iY_0(k_1 r_0)\} \\ B_1^+ K_0(k_1 r_0) \end{bmatrix}, \tag{15a}$$

$$\mathbf{b}_1^+ = \begin{bmatrix} A_1^+ \{J_0(k_1 r_a) - iY_0(k_1 r_a)\} \\ B_1^+ K_0(k_1 r_a) \end{bmatrix}, \tag{15b}$$

$$\mathbf{a}_2^+ = \begin{bmatrix} A_1^+ \{J_0(k_2 r_a) - iY_0(k_2 r_a)\} \\ B_1^+ K_0(k_2 r_a) \end{bmatrix}, \tag{16a}$$

$$\mathbf{b}_2^+ = \begin{bmatrix} A_1^+ \{J_0(k_2 r_b) - iY_0(k_2 r_b)\} \\ B_1^+ K_0(k_2 r_b) \end{bmatrix}, \tag{16b}$$

$$\mathbf{a}_3^+ = \begin{bmatrix} A_1^+ \{J_0(k_1 r_c) - iY_0(k_1 r_c)\} \\ B_1^+ K_0(k_1 r_c) \end{bmatrix}, \tag{17a}$$

$$\mathbf{b}_3^+ = \begin{bmatrix} A_1^+ \{J_0(k_1 r_c) - iY_0(k_1 r_c)\} \\ B_1^+ K_0(k_1 r_c) \end{bmatrix}. \tag{17b}$$

These wave vectors are related by

$$\mathbf{b}_1^+ = \mathbf{f}_1^+ (\mathbf{r_a} - \mathbf{r_0}) \mathbf{a}_1^+ \tag{18a}$$

$$\mathbf{b}_2^+ = \mathbf{f}_2^+ (\mathbf{r_b} - \mathbf{r_a}) \mathbf{a}_2^+ \tag{18b}$$

$$\mathbf{b}_3^+ = \mathbf{f}_3^+ (\mathbf{r_c} - \mathbf{r_b}) \mathbf{a}_3^+. \tag{18c}$$

Substituting matrices (15a), (15b), (16a), (16b), (17a), (17b) into (18a)–(18c), positive–going propagation matrices are obtained as

$$\mathbf{f}_1^+ = \begin{bmatrix} \dfrac{J_0(k_1 r_a) - iY_0(k_1 r_a)}{J_0(k_1 r_0) - iY_0(k_1 r_0)} & 0 \\ 0 & \dfrac{K_0(k_1 r_a)}{K_0(k_1 r_0)} \end{bmatrix} \tag{19a}$$

$$\mathbf{f}_2^+ = \begin{bmatrix} \dfrac{J_0(k_1 r_b) - iY_0(k_1 r_b)}{J_0(k_1 r_a) - iY_0(k_1 r_a)} & 0 \\ 0 & \dfrac{K_0(k_1 r_b)}{K_0(k_1 r_a)} \end{bmatrix} \tag{19b}$$

$$\mathbf{f}_3^+ = \begin{bmatrix} \dfrac{J_0(k_1 r_c) - iY_0(k_1 r_c)}{J_0(k_1 r_b) - iY_0(k_1 r_b)} & 0 \\ 0 & \dfrac{K_0(k_1 r_c)}{K_0(k_1 r_b)} \end{bmatrix}. \tag{19c}$$

Similarly, negative–going waves can be rewritten as

$$\mathbf{a_1^-} = \begin{bmatrix} A_1^- \{J_0(k_1r_0) + iY_0(k_1r_0)\} \\ B_1^- I_0(k_1r_0) \end{bmatrix} \tag{20a}$$

$$\mathbf{b_1^-} = \begin{bmatrix} A_1^- \{J_0(k_1r_a) + iY_0(k_1r_a)\} \\ B_1^- I_0(k_1r_a) \end{bmatrix} \tag{20b}$$

$$\mathbf{a_2^-} = \begin{bmatrix} A_1^- \{J_0(k_2r_a) + iY_0(k_2r_a)\} \\ B_1^- I_0(k_2r_a) \end{bmatrix} \tag{21a}$$

$$\mathbf{b_2^-} = \begin{bmatrix} A_1^- \{J_0(k_2r_b) + iY_0(k_2r_b)\} \\ B_1^- I_0(k_2r_b) \end{bmatrix} \tag{21b}$$

$$\mathbf{a_3^-} = \begin{bmatrix} A_1^- \{J_0(k_1r_c) + iY_0(k_1r_c)\} \\ B_1^- I_0(k_1r_c) \end{bmatrix} \tag{22a}$$

$$\mathbf{b_3^-} = \begin{bmatrix} A_1^- \{J_0(k_1r_c) + iY_0(k_1r_c)\} \\ B_1^- I_0(k_1r_c) \end{bmatrix}. \tag{22b}$$

These wave vectors are related by

$$\mathbf{a_1^-} = \mathbf{f_1^-}(\mathbf{r_0} - \mathbf{r_a})\,\mathbf{b_1^-} \tag{23a}$$

$$\mathbf{a_2^-} = \mathbf{f_2^-}(\mathbf{r_a} - \mathbf{r_b})\,\mathbf{b_2^-} \tag{23b}$$

$$\mathbf{a_3^-} = \mathbf{f_3^-}(\mathbf{r_b} - \mathbf{r_c})\,\mathbf{b_3^-}. \tag{23c}$$

Substituting matrices (20a), (20b), (21a), (21b), (22a), (22b) into (23a)–(23c), negative–going propagation matrices are obtained:

$$\mathbf{f_1^-} = \begin{bmatrix} \dfrac{J_0(k_1r_0) + iY_0(k_1r_0)}{J_0(k_1r_a) + iY_0(k_1r_a)} & 0 \\ 0 & \dfrac{I_0(k_1r_0)}{I_0(k_1r_a)} \end{bmatrix} \tag{24a}$$

$$\mathbf{f_2^-} = \begin{bmatrix} \dfrac{J_0(k_1r_a) + iY_0(k_1r_a)}{J_0(k_1r_b) + iY_0(k_1r_b)} & 0 \\ 0 & \dfrac{I_0(k_1r_a)}{I_0(k_1r_b)} \end{bmatrix} \tag{24b}$$

$$\mathbf{f_3^-} = \begin{bmatrix} \dfrac{J_0(k_1r_b) + iY_0(k_1r_b)}{J_0(k_1r_c) + iY_0(k_1r_c)} & 0 \\ 0 & \dfrac{I_0(k_1r_b)}{I_0(k_1r_c)} \end{bmatrix}. \tag{24c}$$

2.2.2. Reflection Matrices. Keeping the boundary condition of $r = r_0$ fixed, thus, displacements and rotational angle are taken as

$$\begin{aligned} &A_1^+ [J_0(k_1r_0) - iY_0(k_1r_0)] \\ &\quad + A_1^- [J_0(k_1r_0) + iY_0(k_1r_0)] + B_1^+ K_0(k_1r_0) \\ &\quad + B_1^- I_0(k_1r_0) = 0 \\ &A_1^+ [-k_1J_1(k_1r_0) + ik_1Y_1(k_1r_0)] \\ &\quad + A_1^- [-k_1J_1(k_1r_0) - ik_1Y_1(k_1r_0)] \\ &\quad - B_1^+ k_1K_1(k_1r_0) + B_1^- k_1I_1(k_1r_0) = 0. \end{aligned} \tag{25}$$

The relationship of incident wave $\mathbf{a_1^+}$ and reflected wave $\mathbf{a_1^-}$ is related by

$$\mathbf{a_1^+} = \mathbf{R_0}\mathbf{a_1^-}. \tag{26}$$

Substituting (25) into (26), the reflection matrices can be obtained as follows:

$$\begin{aligned} &\mathbf{R_0} \\ &= -\begin{bmatrix} \dfrac{J_0(k_1r_0) - iY_0(k_1r_0)}{J_0(k_1r_0) - iY_0(k_1r_0)} & \dfrac{K_0(k_1r_0)}{K_0(k_1r_0)} \\ \dfrac{-k_1J_1(k_1r_0) + ik_1Y_1(k_1r_0)}{J_0(k_1r_0) - iY_0(k_1r_0)} & \dfrac{-k_1K_1(k_1r_0)}{K_0(k_1r_0)} \end{bmatrix}^{-1} \\ &\cdot \begin{bmatrix} \dfrac{J_0(k_1r_0) + iY_0(k_1r_0)}{J_0(k_1r_0) + iY_0(k_1r_0)} & \dfrac{I_0(k_1r_0)}{I_0(k_1r_0)} \\ \dfrac{-k_1J_1(k_1r_0) - ik_1Y_1(k_1r_0)}{J_0(k_1r_0) + iY_0(k_1r_0)} & \dfrac{k_1I_1(k_1r_0)}{I_0(k_1r_0)} \end{bmatrix}. \end{aligned} \tag{27}$$

Keeping the boundary condition of $r = r_c$ free gives

$$\begin{aligned} &D\left\{A_1^+\left[\frac{k_1}{r_c}J_1(k_1r_c) - {k_1}^2J_0(k_1r_c) + ik_1^2Y_0(k_1r_c)\right.\right. \\ &\quad \left. - \frac{ik_1}{r_c}Y_1(k_1r_c) + \frac{\sigma}{r_c}[-k_1J_1(k_1r_c) + ik_1Y_1(k_1r_c)]\right] \\ &\quad + A_1^-\left[\frac{k_1}{r_c}J_1(k_1r_c) - {k_1}^2J_0(k_1r_c) - ik_1^2Y_0(k_1r_c)\right. \\ &\quad \left. + \frac{ik_1}{r_c}Y_1(k_1r_c) + \frac{\sigma}{r_c}[-k_1J_1(k_1r_c) - ik_1Y_1(k_1r_c)]\right] \\ &\quad + B_1^+\left[{k_1}^2K_0(k_1r_c) + \frac{k_1}{r_c}K_1(k_1r_c)\right. \\ &\quad \left. - \frac{\sigma}{r}k_1K_1(k_1r_c)\right] + B_1^-\left[{k_1}^2I_0(k_1r_c) - \frac{k_1}{r_c}I_1(k_1r_c)\right. \\ &\quad \left.\left. + \frac{\sigma_1k_1}{r_c}I_1(k_1r_c)\right]\right\} = 0. \\ &D\left\{A_1^+\left[-ik_1^3Y_1(k_1r_c) + k_1^3J_1(k_1r_c)\right] - B_1^+k_1^3K_1(k_1r_c)\right. \\ &\quad + A_1^-\left[ik_1^3Y_1(k_1r_c) + k_1^3J_1(k_1r_c)\right] \\ &\quad \left. + B_1^-{k_1}^3I_1(k_1r_c)\right\} = 0 \end{aligned} \tag{28}$$

The relationship of incident wave $\mathbf{b_3^+}$ and reflected wave $\mathbf{b_3^-}$ is

$$\mathbf{b_3^-} = \mathbf{R_3}\mathbf{b_3^+}. \tag{29}$$

Substituting (28) into (29), reflection matrices are calculated as

$$\mathbf{R}_3 = -\begin{bmatrix} \dfrac{(k_1/r_c) J_1(k_1 r_c) - k_1{}^2 J_0(k_1 r_c) - ik_1^2 Y_0(k_1 r_c) + (ik_1/r_c) Y_1(k_1 r_c) + (\sigma/r_c)[-k_1 J_1(k_1 r_c) - ik_1 Y_1(k_1 r_c)]}{J_0(k_1 r_c) + iY_0(k_1 r_c)} & \dfrac{k_1{}^2 I_0(k_1 r_c) - (k_1/r_c) I_1(k_1 r_c) + (\sigma_1 k_1/r_c) I_1(k_1 r_c)}{I_0(k_1 r_c)} \\ \dfrac{ik_1^3 Y_1(k_1 r_c) + k_1^3 J_1(k_1 r_c)}{J_0(k_1 r_c) + iY_0(k_1 r_c)} & \dfrac{k_1{}^3 I_1(k_1 r_c)}{I_0(k_1 r_c)} \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} \dfrac{(k_1/r_c) J_1(k_1 r_c) - k_1{}^2 J_0(k_1 r_c) + ik_1^2 Y_0(k_1 r_c) - (ik_1/r_c) Y_1(k_1 r_c) + (\sigma/r_c)[-k_1 J_1(k_1 r_c) + ik_1 Y_1(k_1 r_c)]}{J_0(k_1 r_c) - iY_0(k_1 r_c)} & \dfrac{k_1{}^2 K_0(k_1 r_c) + (k_1/r_c) K_1(k_1 r_c) - (\sigma/r) k_1 K_1(k_1 r_c)}{K_0(k_1 r_c)} \\ \dfrac{-ik_1^3 Y_1(k_1 r_c) + k_1^3 J_1(k_1 r_c)}{J_0(k_1 r_c) - iY_0(k_1 r_c)} & \dfrac{-k_1^3 K_1(k_1 r_c)}{K_0(k_1 r_c)} \end{bmatrix}. \tag{30}$$

2.2.3. Coordination Matrices. By imposing the geometric continuity at $r = r_a$ yields

$$\begin{aligned}
&\begin{bmatrix} J_0(k_1 r_a) - iY_0(k_1 r_a) & K_0(k_1 r_a) \\ -k_1 J_1(k_1 r_a) + ik_1 Y_1(k_1 r_a) & -k_1 K_1(k_1 r_a) \end{bmatrix} \mathbf{b}_1^+ + \begin{bmatrix} J_0(k_1 r_a) + iY_0(k_1 r_a) & I_0(k_1 r_a) \\ -k_1 J_1(k_1 r_a) - ik_1 Y_1(k_1 r_a) & k_1 I_1(k_1 r_a) \end{bmatrix} \mathbf{b}_1^- = \begin{bmatrix} J_0(k_2 r_a) - iY_0(k_2 r_a) & K_0(k_2 r_a) \\ -k_2 J_1(k_2 r_a) + ik_2 Y_1(k_2 r_a) & -k_2 K_1(k_2 r_a) \end{bmatrix} \mathbf{a}_2^+ \\
&\quad + \begin{bmatrix} J_0(k_2 r_a) + iY_0(k_2 r_a) & I_0(k_2 r_a) \\ -k_2 J_1(k_2 r_a) - ik_2 Y_1(k_2 r_a) & k_2 I_1(k_2 r_a) \end{bmatrix} \mathbf{a}_2^- \\
&\begin{bmatrix} \frac{k_1}{r_a} J_1(k_1 r_a) - k_1{}^2 J_0(k_1 r_a) + ik_1^2 Y_0(k_1 r_a) - \frac{ik_1}{r_a} Y_1(k_1 r_a) + \frac{\sigma_1}{r_a}[-k_1 J_1(k_1 r_a) + ik_1 Y_1(k_1 r_a)] & k_1{}^2 K_0(k_1 r_a) + \frac{k_1}{r_a} K_1(k_1 r_a) - \frac{\sigma_1}{r_a} k_1 K_1(k_1 r_a) \\ -ik_1^3 Y_1(k_1 r_a) + k_1^3 J_1(k_1 r_a) & -k_1^3 K_1(k_1 r_a) \end{bmatrix} \mathbf{b}_1^+ \\
&\quad + \begin{bmatrix} \frac{k_1}{r_a} J_1(k_1 r_a) - k_1{}^2 J_0(k_1 r_a) - ik_1^2 Y_0(k_1 r_a) + \frac{ik_1}{r_a} Y_1(k_1 r_a) + \frac{\sigma_1}{r_a}[-k_1 J_1(k_1 r_a) - ik_1 Y_1(k_1 r_a)] & k_1{}^2 I_0(k_1 r_a) - \frac{k_1}{r_a} I_1(k_1 r_a) + \frac{\sigma_1 k_1}{r_a} I_1(k_1 r_a) \\ ik_1^3 Y_1(k_1 r_a) + k_1^3 J_1(k_1 r_a) & k_1{}^3 I_1(k_1 r_a) \end{bmatrix} \mathbf{b}_1^- \\
&\quad = \begin{bmatrix} \frac{k_2}{r_a} J_1(k_2 r_a) - k_2{}^2 J_0(k_2 r_a) + ik_2^2 Y_0(k_2 r_a) - \frac{ik_2}{r_a} Y_1(k_2 r_a) + \frac{\sigma_2}{r_a}[-k_2 J_1(k_2 r_a) + ik_2 Y_1(k_2 r_a)] & k_2{}^2 K_0(k_2 r_a) + \frac{k_2}{r_a} K_1(k_2 r_a) - \frac{\sigma_2}{r_a} k_2 K_1(k_2 r_a) \\ -ik_2^3 Y_1(k_2 r_a) + k_2^3 J_1(k_2 r_a) & -k_2^3 K_1(k_2 r_a) \end{bmatrix} \mathbf{a}_2^+ \\
&\quad + \begin{bmatrix} \frac{k_2}{r_a} J_1(k_2 r_a) - k_2{}^2 J_0(k_2 r_a) - ik_2^2 Y_0(k_2 r_a) + \frac{ik_2}{r_a} Y_1(k_2 r_a) + \frac{\sigma_2}{r_a}[-k_2 J_1(k_2 r_a) - ik_2 Y_1(k_2 r_a)] & k_2{}^2 I_0(k_2 r_a) - \frac{k_2}{r_a} I_1(k_2 r_a) + \frac{\sigma_2 k_2}{r_a} I_1(k_2 r_a) \\ ik_2^3 Y_1(k_2 r_a) + k_2^3 J_1(k_2 r_a) & k_2{}^3 I_1(k_2 r_a) \end{bmatrix} \mathbf{a}_2^-.
\end{aligned} \tag{31}$$

Equations (31) can be rewritten as

$$\begin{aligned} \mathbf{R}_{\mathbf{a1}}^+ \mathbf{b}_1^+ + \mathbf{R}_{\mathbf{a1}}^- \mathbf{b}_1^- &= \mathbf{T}_{\mathbf{a2}}^+ \mathbf{a}_2^+ + \mathbf{T}_{\mathbf{a2}}^- \mathbf{a}_2^- \\ \mathbf{R}_{\mathbf{a3}}^+ \mathbf{b}_1^+ + \mathbf{R}_{\mathbf{a3}}^- \mathbf{b}_1^- &= \mathbf{T}_{\mathbf{a4}}^+ \mathbf{a}_2^+ + \mathbf{T}_{\mathbf{a4}}^- \mathbf{a}_2^-. \end{aligned} \tag{32}$$

According to the continuity at $r = r_b$, shear force and bending moment are required that

$$\begin{aligned}
&\begin{bmatrix} J_0(k_2 r_b) - iY_0(k_2 r_b) & K_0(k_2 r_b) \\ -k_2 J_1(k_2 r_b) + ik_2 Y_1(k_2 r_b) & -k_2 K_1(k_2 r_b) \end{bmatrix} \mathbf{b}_2^+ + \begin{bmatrix} J_0(k_2 r_b) + iY_0(k_2 r_b) & I_0(k_2 r_b) \\ -k_2 J_1(k_2 r_b) - ik_2 Y_1(k_2 r_b) & k_2 I_1(k_2 r_b) \end{bmatrix} \mathbf{b}_2^- = \begin{bmatrix} J_0(k_1 r_1) - iY_0(k_1 r_b) & K_0(k_1 r_b) \\ -k_1 J_1(k_1 r_b) + ik_1 Y_1(k_1 r_b) & -k_1 K_1(k_1 r_b) \end{bmatrix} \mathbf{a}_3^+ \\
&\quad + \begin{bmatrix} J_0(k_1 r_b) + iY_0(k_1 r_b) & I_0(k_1 r_b) \\ -k_1 J_1(k_1 r_b) - ik_1 Y_1(k_1 r_b) & k_1 I_1(k_1 r_b) \end{bmatrix} \mathbf{a}_3^- \\
&\begin{bmatrix} \frac{k_2}{r_b} J_1(k_2 r_b) - k_2{}^2 J_0(k_2 r_b) + ik_2^2 Y_0(k_2 r_b) - \frac{ik_2}{r_2} Y_1(k_2 r_b) + \frac{\sigma_2}{r_b}[-k_2 J_1(k_2 r_b) + ik_2 Y_1(k_2 r_b)] & k_2{}^2 K_0(k_2 r_b) + \frac{k_2}{r_b} K_1(k_2 r_b) - \frac{\sigma_2}{r_b} k_2 K_1(k_2 r_b) \\ -ik_2^3 Y_1(k_2 r_b) + k_2^3 J_1(k_2 r_b) & -k_2^3 K_1(k_2 r_b) \end{bmatrix} \mathbf{b}_2^+ \\
&\quad + \begin{bmatrix} \frac{k_2}{r_b} J_1(k_2 r_b) - k_2{}^2 J_0(k_2 r_b) - ik_2^2 Y_0(k_2 r_b) + \frac{ik_2}{r_b} Y_1(k_2 r_b) + \frac{\sigma_2}{r_b}[-k_2 J_1(k_2 r_b) - ik_2 Y_1(k_2 r_b)] & k_2{}^2 I_0(k_2 r_b) - \frac{k_2}{r_b} I_1(k_2 r_b) + \frac{\sigma_2 k_2}{r_b} I_1(k_2 r_b) \\ ik_2^3 Y_1(k_2 r_b) + k_2^3 J_1(k_2 r_b) & k_2{}^3 I_1(k_2 r_b) \end{bmatrix} \mathbf{b}_2^- \\
&\quad = \begin{bmatrix} \frac{k_1}{r_b} J_1(k_2 r_b) - k_1{}^2 J_0(k_1 r_b) + ik_1^2 Y_0(k_1 r_b) - \frac{ik_1}{r_b} Y_1(k_1 r_b) + \frac{\sigma_1}{r_b}[-k_1 J_1(k_1 r_b) + ik_1 Y_1(k_1 r_b)] & k_1{}^2 K_0(k_1 r_b) + \frac{k_1}{r_b} K_1(k_1 r_b) - \frac{\sigma_1}{r_b} k_1 K_1(k_1 r_b) \\ -ik_1^3 Y_1(k_1 r_a) + k_1^3 J_1(k_1 r_b) & -k_1^3 K_1(k_1 r_b) \end{bmatrix} \mathbf{a}_3^+ \\
&\quad + \begin{bmatrix} \frac{k_1}{r_b} J_1(k_1 r_b) - k_1{}^2 J_0(k_1 r_b) - ik_1^2 Y_0(k_1 r_b) + \frac{ik_1}{r_b} Y_1(k_1 r_b) + \frac{\sigma_1}{r_b}[-k_1 J_1(k_1 r_b) - ik_1 Y_1(k_1 r_b)] & k_1{}^2 I_0(k_1 r_b) - \frac{k_1}{r_b} I_1(k_1 r_b) + \frac{\sigma_1 k_1}{r_b} I_1(k_1 r_b) \\ ik_1^3 Y_1(k_1 r_b) + k_1^3 J_1(k_1 r_b) & k_1{}^3 I_1(k_1 r_b) \end{bmatrix} \mathbf{a}_3^-
\end{aligned} \tag{33}$$

Equations (33) can be written as

$$\begin{aligned}\mathbf{R}_{b1}^{+}\mathbf{b}_1^{+} + \mathbf{R}_{b1}^{-}\mathbf{b}_1^{-} &= \mathbf{T}_{b2}^{+}\mathbf{a}_2^{+} + \mathbf{T}_{b2}^{-}\mathbf{a}_2^{-}\\ \mathbf{R}_{b3}^{+}\mathbf{b}_1^{+} + \mathbf{R}_{b3}^{-}\mathbf{b}_1^{-} &= \mathbf{T}_{b4}^{+}\mathbf{a}_2^{+} + \mathbf{T}_{b4}^{-}\mathbf{a}_2^{-}.\end{aligned} \tag{34}$$

2.2.4. Characteristic Equation of Natural Frequency. Combining propagation matrices, reflection matrices, and coordination matrices derived in Section 2.2, natural frequencies of composite rings can be calculated smoothly. Figure 1 presents a clear description of incident and reflected waves. Thus, the wave matrices described by (18a)–(18c), (23a)–(23c), (26), (29), (32), and (34) are assembled as

$$\begin{aligned}\mathbf{b}_1^{+} &= \mathbf{f}_1^{+}(\mathbf{r}_a - \mathbf{r}_0)\,\mathbf{a}_1^{+}\\ \mathbf{a}_1^{-} &= \mathbf{f}_1^{-}(\mathbf{r}_0 - \mathbf{r}_a)\,\mathbf{b}_1^{-}\\ \mathbf{b}_3^{+} &= \mathbf{f}_3^{+}(\mathbf{r}_c - \mathbf{r}_b)\,\mathbf{a}_3^{+}\\ \mathbf{a}_1^{+} &= \mathbf{R}_0\mathbf{a}_1^{-}\\ \mathbf{b}_2^{+} &= \mathbf{f}_2^{+}(\mathbf{r}_b - \mathbf{r}_a)\,\mathbf{a}_2^{+}\\ \mathbf{a}_2^{-} &= \mathbf{f}_2^{-}(\mathbf{r}_a - \mathbf{r}_b)\,\mathbf{b}_2^{-}\\ \mathbf{a}_3^{-} &= \mathbf{f}_3^{-}(\mathbf{r}_b - \mathbf{r}_c)\,\mathbf{b}_3^{-}\\ \mathbf{b}_3^{-} &= \mathbf{R}_3\mathbf{b}_3^{+}\\ \mathbf{R}_{a1}^{+}\mathbf{b}_1^{+} + \mathbf{R}_{a1}^{-}\mathbf{b}_1^{-} &= \mathbf{T}_{a2}^{+}\mathbf{a}_2^{+} + \mathbf{T}_{a2}^{-}\mathbf{a}_2^{-}\\ \mathbf{R}_{b1}^{+}\mathbf{b}_2^{+} + \mathbf{R}_{b1}^{-}\mathbf{b}_2^{-} &= \mathbf{T}_{b2}^{+}\mathbf{a}_3^{+} + \mathbf{T}_{b2}^{-}\mathbf{a}_3^{-}\\ \mathbf{R}_{a3}^{+}\mathbf{b}_1^{+} + \mathbf{R}_{a3}^{-}\mathbf{b}_1^{-} &= \mathbf{T}_{a4}^{+}\mathbf{a}_2^{+} + \mathbf{T}_{a4}^{-}\mathbf{a}_2^{-}\\ \mathbf{R}_{b3}^{+}\mathbf{b}_2^{+} + \mathbf{R}_{b3}^{-}\mathbf{b}_2^{-} &= \mathbf{T}_{b4}^{+}\mathbf{a}_3^{+} + \mathbf{T}_{b4}^{-}\mathbf{a}_3^{-}.\end{aligned} \tag{35}$$

In order to obtain the natural frequency, (35) can be rewritten in a matrix form

$$\begin{bmatrix} -\mathbf{I}_{2\times2} & \mathbf{R}_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\mathbf{I}_{2\times2} & 0 & \mathbf{f}_1^{-} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\mathbf{I}_{2\times2} & 0 & \mathbf{f}_2^{-} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{I}_{2\times2} & 0 & \mathbf{f}_3^{-}\\ \mathbf{f}_1^{+} & 0 & -\mathbf{I}_{2\times2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbf{f}_2^{+} & 0 & -\mathbf{I}_{2\times2} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{f}_3^{+} & 0 & -\mathbf{I}_{2\times2} & 0\\ 0 & 0 & \mathbf{R}_{a1}^{+} & \mathbf{R}_{a1}^{-} & \mathbf{T}_{a2}^{+} & \mathbf{T}_{a2}^{-} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{R}_{a3}^{+} & \mathbf{R}_{a3}^{-} & \mathbf{T}_{a4}^{+} & \mathbf{T}_{a4}^{-} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{R}_{b1}^{+} & \mathbf{R}_{b1}^{-} & \mathbf{T}_{b2}^{+} & \mathbf{T}_{b2}^{-} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{R}_{b3}^{+} & \mathbf{R}_{b3}^{-} & \mathbf{T}_{b4}^{+} & \mathbf{T}_{b4}^{-} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{R}_3 & -\mathbf{I}_{2\times2} \end{bmatrix} \begin{bmatrix}\mathbf{a}_1^{+}\\ \mathbf{a}_1^{-}\\ \mathbf{b}_1^{+}\\ \mathbf{b}_1^{-}\\ \mathbf{a}_2^{+}\\ \mathbf{a}_2^{-}\\ \mathbf{b}_2^{+}\\ \mathbf{b}_2^{-}\\ \mathbf{a}_3^{+}\\ \mathbf{a}_3^{-}\\ \mathbf{b}_3^{+}\\ \mathbf{b}_3^{-}\end{bmatrix} = F(f) \begin{bmatrix}\mathbf{a}_1^{+}\\ \mathbf{a}_1^{-}\\ \mathbf{b}_1^{+}\\ \mathbf{b}_1^{-}\\ \mathbf{a}_2^{+}\\ \mathbf{a}_2^{-}\\ \mathbf{b}_2^{+}\\ \mathbf{b}_2^{-}\\ \mathbf{a}_3^{+}\\ \mathbf{a}_3^{-}\\ \mathbf{b}_3^{+}\\ \mathbf{b}_3^{-}\end{bmatrix} = 0. \tag{36}$$

$F(f)$ is a matrix of 12×12. If (36) has solution, it requires that

$$|F(f)| = 0. \tag{37}$$

By solving the roots of characteristic equation (37), one can calculate the real and imaginary parts. It is important here to note that the natural frequencies can be found by searching the intersections in x-axis.

3. Numerical Results and Discussion

In this section, free vibration of rings is calculated by using wave approach, and the results are also compared with those obtained by classical method. Material RESIN is selected for the first and third layers. Material STEEL is selected for the middle layers. Material and structural parameters are given in Table 1.

Based on Bessel and Hankel solutions calculated by classical method theoretically, natural frequency curves are presented by solving characteristic equation (8) depicted in Figure 2. Furthermore, (37) is calculated using wave approach. It can be seen that the real and imaginary parts intersect at multiple points simultaneously in x-axis. It is important, here, to note that the roots of the characteristic curves are natural frequencies when the values of longitudinal coordinates are zero.

In Figure 2, two different natural frequencies can be clearly presented in the range of 450–1500 Hz, that is, 1244.22 Hz and 1443.31 Hz. However, the values are very small in the range of 0–450 Hz. In order to find whether the

Table 1: Material and structural parameters.

Material parameters	Density ρ (kg/m^3)	Young modulus E (Pa)	Poisson's ratio
I (RESIN)	1180	0.435×10^{10}	0.3679
II (STEEL)	7780	21.06×10^{10}	0.3
Structural parameters	$r_a = 9r_0$	$r_c = r_b + 40$	h
(mm)	45	125	1

Table 2: Results calculated by classical method, wave approach, and FEM.

Method	1st mode	2nd mode	3rd mode	4th mode	5th mode
Classical Bessel	37.65 Hz	167.54 Hz	414.27 Hz	1244.22 Hz	1443.31 Hz
Classical Hankel	37.65 Hz	167.54 Hz	414.27 Hz	1244.22 Hz	1443.31 Hz
Wave approach	37.65 Hz	167.54 Hz	414.27 Hz	1244.22 Hz	1443.31 Hz
FEM	37.76 Hz	168.30 Hz	415.19 Hz	1247.90 Hz	1448.11 Hz

Table 3: Comparison of free vibration by FEM for four type boundaries.

Different boundaries	1st mode	2nd mode	3rd mode	4th mode	5th mode
Inner free, outer free	143.40 Hz	334.10 Hz	569.55 Hz	1336.50 Hz	1845.16 Hz
Inner fixed, outer free	37.76 Hz	168.30 Hz	415.19 Hz	1247.90 Hz	1448.10 Hz
Inner free, outer fixed	70.44 Hz	321.96 Hz	572.94 Hz	1388.86 Hz	1848.67 Hz
Inner fixed, outer fixed	101.07 Hz	413.28 Hz	1272.37 Hz	1479.74 Hz	2927.38 Hz

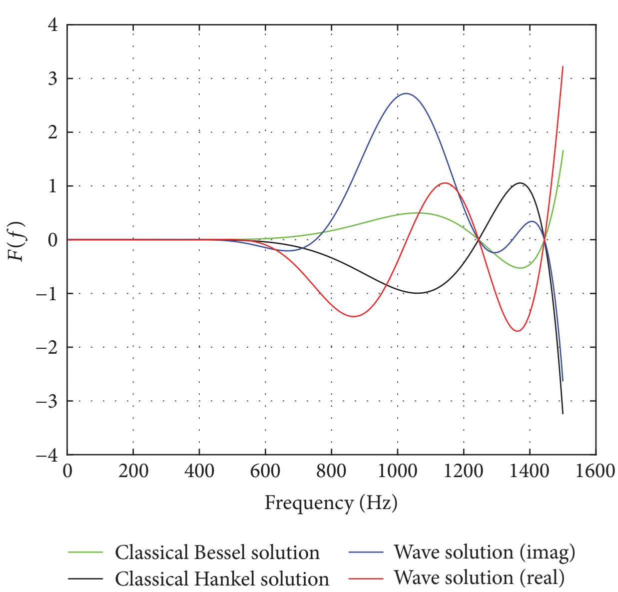


Figure 2: Natural frequency obtained by classical method and wave approach.

values in this range also intersect at one point, three zoomed figures are drawn for the purpose of better illustration about the natural frequencies of characteristic curves which are described in Figure 3.

Natural frequencies calculated by these two methods are compared. Modal analysis is carried out by FEM. The natural frequencies are presented in Table 2 from which it can be observed that the first five-order modes calculated by these three methods are in good agreement. Obviously, it also can be found that natural frequencies obtained by ANSYS software are larger than the results calculated by classic method and wave approach, which is mainly caused by the mesh and simplified solid model in FEM. However, these errors are within an acceptable range, which verifies the correctness of theoretical calculations. To assess the deformation of rings, Figure 4 is employed to describe the mode shape. It can be found that the maximum deformations of the first three mode shapes occur in the outermost surface. The fourth and fifth mode shapes appear in the innermost surface.

Adopting FEM method, the first five natural frequencies are calculated for four type boundaries, as is shown in Table 3. It shows that the first natural frequency is 37.76 Hz (Min) at the case of inner boundary fixed and outer boundary free. The first natural frequency is 143.40 Hz (Max) at the case of inner and outer boundaries both free.

Harmonic Response Analysis of rings is carried out by using ANSYS 14.5 software. RESIN is chosen for the first and third layer. The second layer is selected as STEEL. Element can be selected as Solid 45, which is shown in red and blue in Figure 5(a). Through loading transverse displacement onto the innermost layer and picking the transverse displacement onto the outermost layer, vibration transmissibility of rings propagating from inner to outer is obtained by using formula $dB = 20\log(d_{\text{outer}}/d_{\text{inner}})$. Similarly, through loading transverse displacement onto the outermost layer and picking the transverse displacement onto the innermost layer, vibration transmissibility propagating from outer to inner is obtained by using formula $dB = 20\log(d_{\text{inner}}/d_{\text{outer}})$.

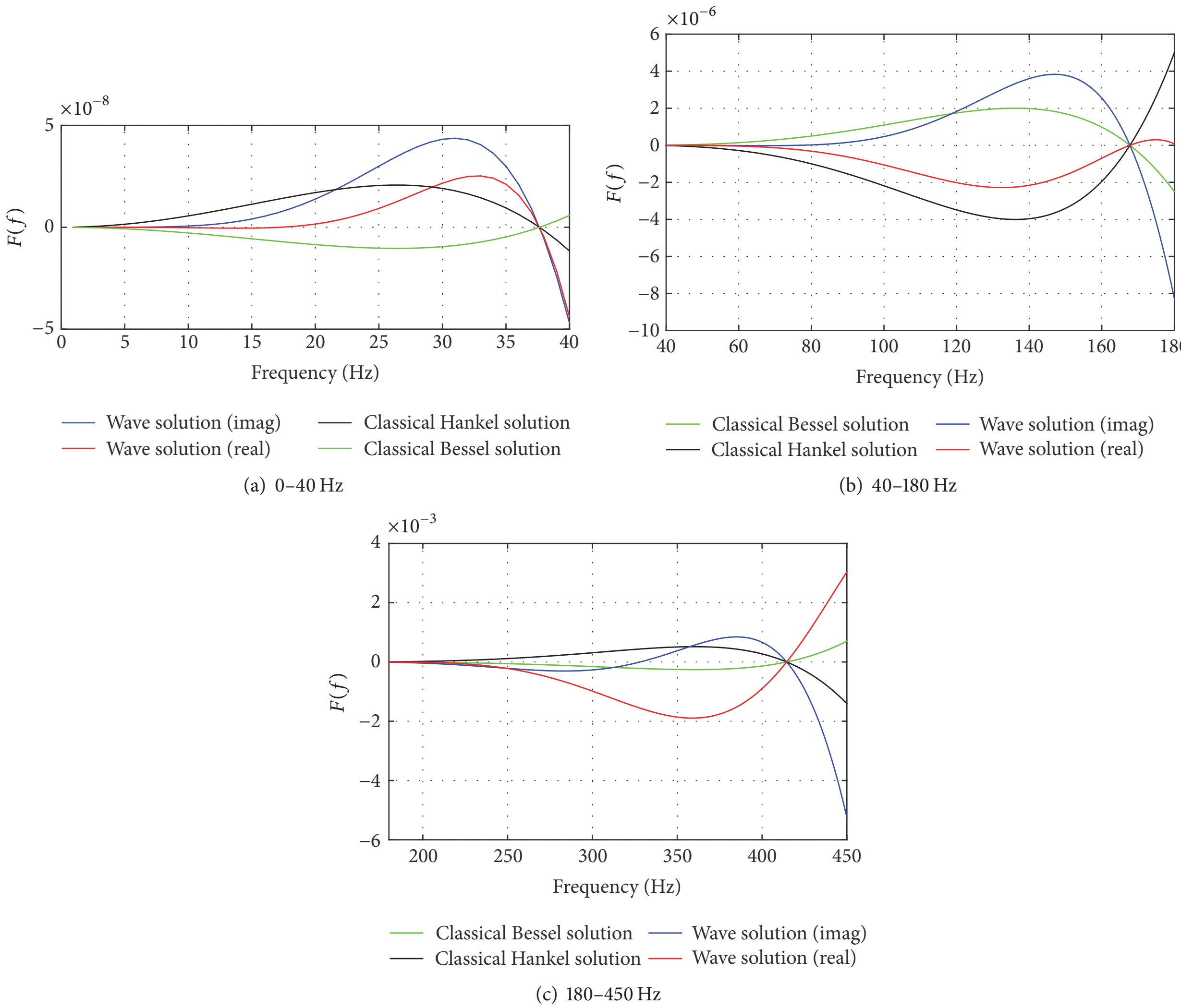


Figure 3: Characteristic curves in the range of 0–450 Hz.

Figure 5(b) indicates that there is no vibration attenuation in the range of 0–1500 Hz when transverse vibration propagates from outer to inner. Also, four resonance frequencies appear, namely, 70.44 Hz, 321.96 Hz, 572.94 Hz, 1388.86 Hz, which coincide with the first four-order natural frequencies in Table 3 at the case of innermost layer free and outermost layer fixed. Compared with the case of vibration propagation from outer to inner, there is vibration attenuation when vibration propagates from inner to outer. In addition, five resonance frequencies also appear, namely, 37.76 Hz, 168.30 Hz, 415.19 Hz, 1247.9 Hz, and 1448.1 Hz, which coincide with the results obtained by wave approach, classical Hankel, and classical Bessel methods shown in Table 2.

4. Effects of Structural and Material Parameters

4.1. Structural Parameters. The effects of structural parameters such as thickness, inner radius, and radial span are investigated in Figure 6. Adopting single variable principle, herein, only change one parameter. Figure 6(a) shows clearly that, with thickness increasing, the first modes change from 37.76 Hz to 188.15 Hz, and the remaining three modes increase obviously, which indicates that thickness has great effect on the first four natural frequencies. In fact, characteristic equation of natural frequency is determined by thickness, density, and elastic modulus, which is shown by the expression of wave number $k = (4\pi^2 f^2 \rho h/D)^{0.25}$ and stiffness $D = Eh^3/12(1-\sigma^2)$. Therefore, thickness is used to adjust the natural frequency directly through varying wave number $k = (4\pi^2 f^2 \rho h/D)^{0.25}$ in (36).

From the wave number $k = (4\pi^2 f^2 \rho h/D)^{0.25}$, it can be found that inner radius is not related to the natural frequency. Thus, inner radius almost has no effect on the natural frequency shown in Figure 6(b).

In Figure 6(c), there are five different types analyzed for the radial span ratios of RESIN and STEEL, that is, $a_1 : a_2 = 1 \times 0.04/2.2 : 1.2 \times 0.04/2.2$, $a_1 : a_2 = 1 \times 0.04/2.1 : 1.1 \times 0.04/2.1$, $a_1 : a_2 = 0.04 : 0.04$, $a_1 : a_2 = 1.1 \times 0.04/2.1 : 1 \times 0.04/2.1$, $a_1 : a_2 = 1.2 \times 0.04/2.2 : 1 \times 0.04/2.2$, respectively.

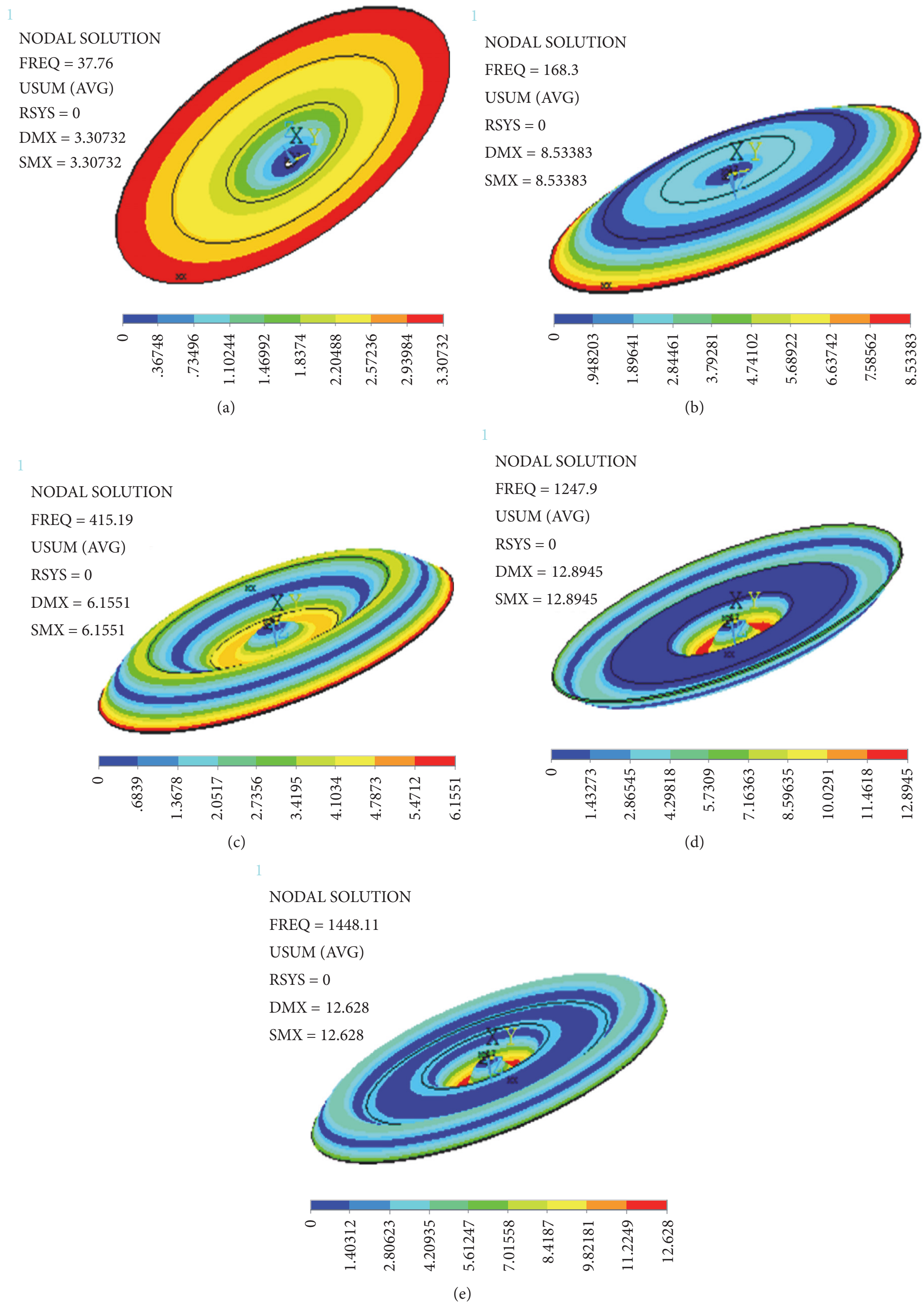
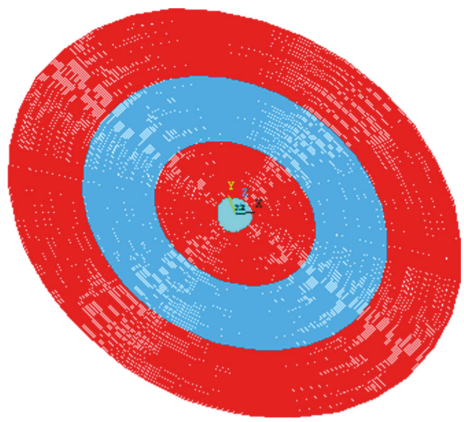
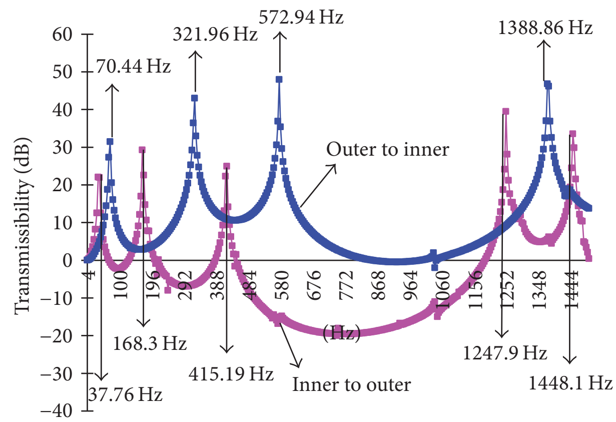


Figure 4: Mode shapes of natural frequencies. (a) First mode. (b) Second mode. (c) Third mode. (d) Fourth mode. (e) Fifth mode.

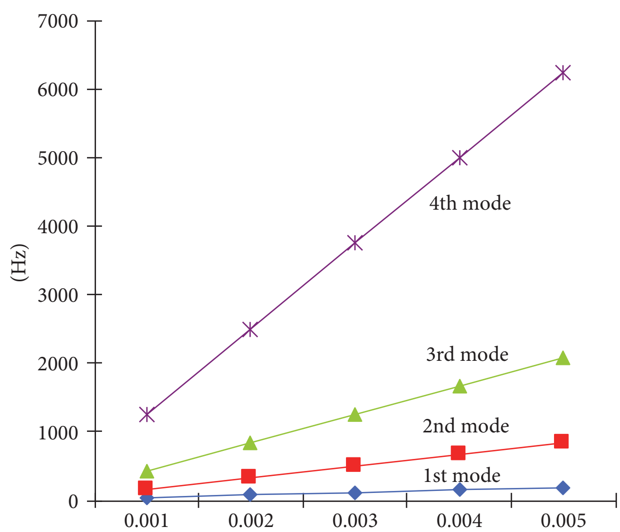


(a) The meshing mode

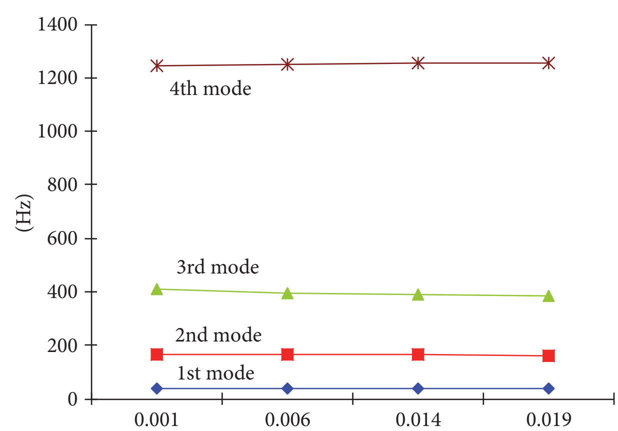


(b) Vibration response

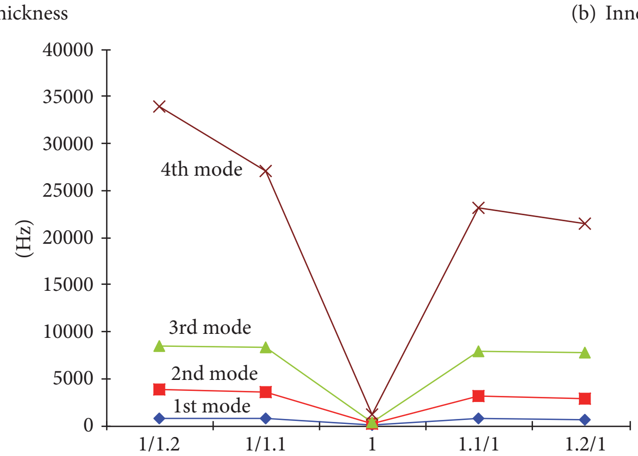
Figure 5



(a) Thickness



(b) Inner radius

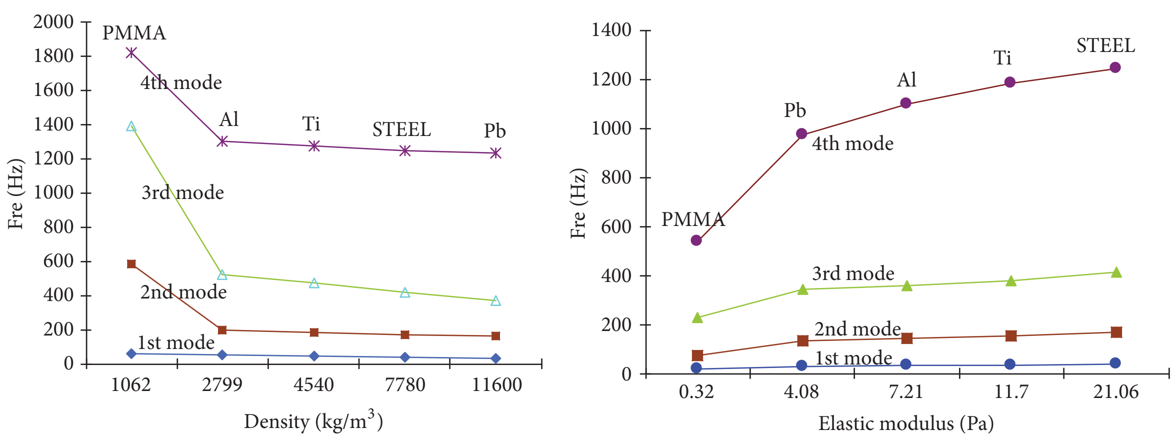


(c) Radial span

Figure 6: Effect of structural parameters.

Table 4: Material parameters.

Method	Density ρ (kg/m^3)	Young modulus E (Pa)	Poisson's ratio
PMMA	1062	0.32×10^{10}	0.3333
Al	2799	7.21×10^{10}	0.3451
Pb	11600	4.08×10^{10}	0.3691
Ti	4540	11.7×10^{10}	0.32



(a) Density

(b) Elastic modulus

Figure 7: Effect of material parameters.

When radial span is equal to 1, this means that the size of RESIN and STEEL is 1:1, namely, $a_1 : a_2 = 0.04 : 0.04$. For this case, the total size of composite ring is max, so the mode is min. Additionally, symmetrical five types cause the approximate symmetry of Figure 6(c). It also can be found that as radial span increases, natural frequencies appear as a similar trend, namely, decrease afterwards increase.

4.2. Material Parameters. Adopting single variable principle, density of middle material STEEL is replaced by the density of PMMA, Al, Pb, Ti. Similar with the study on effects of structural parameters, the effects of density and elastic modulus are studied for the case of keeping the material and structural parameters unchanged. Also, material parameters of PMMA, Al, Pb, and Ti are presented in Table 4.

Figure 7(a) indicates that as density increases, the first mode decreases but not very obviously. However, the second, third, and fourth modes reduce significantly. Figure 7(b) shows that when elastic modulus increases gradually, the first mode increases but not significantly. The second, third, and fourth modes increase rapidly.

5. Conclusion

This paper focuses on calculating natural frequency for rings via classical method and wave approach. Based on the solutions of transverse vibration, expression of rotational angle, shear force, and bending moment are obtained. Wave propagation matrices within structure, coordination matrices between the two materials, and reflection matrices at the boundary conditions are also deduced. Additionally, characteristic equation of natural frequencies is obtained by assembling these wave matrices. The real and imaginary parts calculated by wave approach intersect at the same point with the results obtained by classical method, which verifies the correctness of theoretical calculations.

A further analysis for the influence of different boundaries on natural frequencies is discussed. It can be found that the first natural frequency is Min 37.76 Hz at the case of inner boundary fixed and outer boundary free. In addition, it also shows that there exists vibration attenuation when vibration propagates from inner to outer. However, there is no vibration attenuation when vibration propagates from outer to inner. Structural and material parameters have strong sensitivity for the free vibration.

Finally, the behavior of wave propagation is studied in detail which is of great significance to the design of natural frequency for the vibration analysis of rotating rings and shaft systems.

Appendix

Derivation of the Transfer Matrix

Due to the continuity at $r = r_a$, the following is obtained:

$$\begin{aligned} W_1(r_a) &= W_2(r_a) \\ \frac{\partial W}{\partial r}_1(r_a) &= \frac{\partial W}{\partial r}_2(r_a) \\ M_1(r_a) &= M_2(r_a) \\ Q_1(r_a) &= Q_2(r_a). \end{aligned} \tag{A.1}$$

Equation (A.1) can be organized as

$$\begin{bmatrix} J_0(\bar{k}_1 r_a) & Y_0(\bar{k}_1 r_a) & I_0(k_1 r_a) & K_0(k_1 r_a) \\ -k_1 J_1(k_1 r_a) & -k_1 Y_1(k_1 r_a) & k_1 I_1(k_1 r_a) & -k_1 K_1(k_1 r_a) \\ \mathbf{J_2} & \mathbf{Y_2} & \mathbf{I_2} & \mathbf{K_2} \\ k_1^3 J_1(k_1 r_a) & k_1^3 Y_1(k_1 r_a) & k_1^3 I_1(k_1 r_a) & -k_1^3 K_1(k_1 r_a) \end{bmatrix} \mathbf{\Psi_{11}}$$
$$= \begin{bmatrix} J_0(k_2 r_a) & Y_0(k_2 r_a) & I_0(k_2 r_a) & K_0(k_2 r_a) \\ -k_2 J_1(k_2 r_a) & -k_2 Y_1(k_2 r_a) & k_2 I_1(k_2 r_a) & -k_2 K_1(k_2 r_a) \\ \mathbf{J_3} & \mathbf{Y_3} & \mathbf{I_3} & \mathbf{K_3} \\ k_2^3 J_1(k_2 r_a) & k_2^3 Y_1(k_2 r_a) & k_2^3 I_1(k_2 r_a) & -k_2^3 K_1(k_2 r_a) \end{bmatrix} \mathbf{\Psi_{12}}, \quad \text{(A.2)}$$

where $\mathbf{\Psi_{12}} = [A_{12}\ B_{12}\ C_{12}\ D_{12}]^T$, and each element is defined as

$$\mathbf{J_2} = \frac{k_1}{r_a}\{J_1(k_1 r_a) - \sigma_1 J_1(k_1 r_a)\} - k_1^2 J_0(k_1 r_a),$$

$$\mathbf{Y_2} = \frac{k_1}{r_a}\{Y_1(k_1 r_a) - \sigma_1 Y_1(k_1 r_a)\} - k_1^2 Y_0(k_1 r_a),$$

$$\mathbf{I_2} = \frac{k_1}{r_a}\{\sigma_1 I_1(k_1 r_a) - I_1(k_1 r_a)\} + k_1^2 I_0(k_1 r_a),$$

$$\mathbf{K_2} = \frac{k_1}{r_a}\{K_1(k_1 r_a) - \sigma_1 K_1(k_1 r_a)\} + k_1^2 K_0(k_1 r_a),$$

$$\mathbf{J_3} = \frac{k_2}{r_a}\{J_1(k_2 r_a) - \sigma_2 J_1(k_2 r_a)\} - k_2^2 J_0(k_2 r_a),$$

$$\mathbf{Y_3} = \frac{k_2}{r_a}\{Y_1(k_2 r_a) - \sigma_2 Y_1(k_2 r_a)\} - k_2^2 Y_0(k_2 r_a),$$

$$\mathbf{I_3} = \frac{k_2}{r_a}\{\sigma_2 I_1(k_2 r_a) - I_1(k_2 r_a)\} + k_2^2 I_0(k_2 r_a),$$

$$\mathbf{K_3} = \frac{k_2}{r_a}\{K_1(k_2 r_a) - \sigma_2 K_1(k_2 r_a)\} + k_2^2 K_0(k_2 r_a). \quad \text{(A.3)}$$

Hence, (A.2) can be written as

$$\mathbf{H_1 \Psi_{11}} = \mathbf{K_1 \Psi_{12}}. \quad \text{(A.4)}$$

Similarly, by imposing the geometric continuity at $r = r_b$, the following is obtained:

$$\begin{gathered} W_2(r_b) = W_1(r_b) \\ \frac{\partial W}{\partial r}_2 (r_b) = \frac{\partial W}{\partial r}_1 (r_b) \\ M_2(r_b) = M_1(r_b) \\ Q_2(r_b) = Q_1(r_b). \end{gathered} \quad \text{(A.5)}$$

Arranging (A.5) yields

$$\begin{bmatrix} J_0(k_2 r_b) & Y_0(k_2 r_b) & I_0(k_2 r_b) & K_0(k_2 r_b) \\ -k_2 J_1(k_2 r_b) & -k_2 Y_1(k_2 r_b) & k_2 I_1(k_2 r_b) & -k_2 K_1(k_2 r_b) \\ \mathbf{J_4} & \mathbf{Y_4} & \mathbf{I_4} & \mathbf{K_4} \\ k_2^3 J_1(k_2 r_b) & k_2^3 Y_1(k_2 r_b) & k_2^3 I_1(k_2 r_b) & -k_2^3 K_1(k_2 r_b) \end{bmatrix} \mathbf{\Psi_{12}}$$
$$= \begin{bmatrix} J_0(\bar{k}_1 r_b) & Y_0(\bar{k}_1 r_b) & I_0(k_1 r_b) & K_0(k_1 r_b) \\ -k_1 J_1(k_1 r_b) & -k_1 Y_1(k_1 r_b) & k_1 I_1(k_1 r_b) & -k_1 K_1(k_1 r_b) \\ \mathbf{J_5} & \mathbf{Y_5} & \mathbf{I_5} & \mathbf{K_5} \\ k_1^3 J_1(k_1 r_b) & k_1^3 Y_1(k_1 r_b) & k_1^3 I_1(k_1 r_b) & -k_1^3 K_1(k_1 r_b) \end{bmatrix} \mathbf{\Psi_{13}}, \quad \text{(A.6)}$$

and each element is defined as

$$\begin{aligned}
\mathbf{J_4} &= \frac{k_2}{r_b}\left\{J_1\left(k_2 r_b\right) - \sigma_2 J_1\left(k_2 r_b\right)\right\} - k_2^2 J_0\left(k_2 r_b\right),\\
\mathbf{Y_4} &= \frac{k_2}{r_b}\left\{Y_1\left(k_2 r_b\right) - \sigma_2 Y_1\left(k_2 r_b\right)\right\} - k_2^2 Y_0\left(k_2 r_b\right),\\
\mathbf{I_4} &= \frac{k_2}{r_b}\left\{\sigma_2 I_1\left(k_2 r_b\right) - I_1\left(k_2 r_b\right)\right\} + k_2^2 I_0\left(k_2 r_b\right),\\
\mathbf{K_4} &= \frac{k_2}{r_b}\left\{K_1\left(k_2 r_b\right) - \sigma_2 K_1\left(k_2 r_b\right)\right\} + k_2^2 K_0\left(k_2 r_b\right),\\
\mathbf{J_5} &= \frac{k_1}{r_b}\left\{J_1\left(k_1 r_b\right) - \sigma_1 J_1\left(k_1 r_b\right)\right\} - k_1^2 J_0\left(k_1 r_b\right),\\
\mathbf{Y_5} &= \frac{k_1}{r_b}\left\{Y_1\left(k_1 r_b\right) - \sigma_1 Y_1\left(k_1 r_b\right)\right\} - k_1^2 Y_0\left(k_1 r_b\right),\\
\mathbf{I_5} &= \frac{k_1}{r_b}\left\{\sigma_1 I_1\left(k_1 r_b\right) - I_1\left(k_1 r_b\right)\right\} + k_1^2 I_0\left(k_1 r_b\right),\\
\mathbf{K_5} &= \frac{k_1}{r_b}\left\{K_1\left(k_1 r_b\right) - \sigma_1 K_1\left(k_1 r_b\right)\right\} + k_1^2 K_0\left(k_1 r_b\right).
\end{aligned} \tag{A.7}$$

Equation (A.6) can be simplified as

$$\mathbf{K_2 \Psi_{12}} = \mathbf{H_2 \Psi_{13}}. \tag{A.8}$$

Combining (A.4) and (A.8) gives

$$\mathbf{\Psi_{13}} = \mathbf{T_{13}\Psi_{11}} = \mathbf{H_2^{-1} K_2 K_1^{-1} H_1 \Psi_{11}}, \tag{A.9}$$

where $\mathbf{T_{13}}$ is the transfer matrix of flexural wave from inner to outer.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this paper.

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