Research Article

An Analytical Model for a Pneumatic Vibration Isolator with the Stiffness Effect of the Elastomeric Diaphragm

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The elastomeric diaphragm is widely used in pneumatic vibration isolators, and the relevant model is often ignored due to its complexity. Considering that the ignored model of the elastomeric diaphragm in pneumatic vibration isolators plays an important role in the discrepancy between the predicted and practical behavior, this paper develops an analytical model for the elastomeric diaphragm using the Mooney-Rivlin modeling method and elastomeric theory. Specifically, the elastomeric diaphragm consists of several segments in the axial section. After considering the structural restriction, each segment can be simplified as uniaxial stretching, and the force-strain equation can be established for each segment. By combining the equations of all segments, an analytical model of the elastomeric diaphragm can be built and solved via numerical methods. The developed model is added to the standard model of pneumatic cylinders to supply a complete analytical model for pneumatic vibration isolators. The experimental results demonstrate that the analytically predicted behavior is similar to the practical behavior. The proposed analytical model can be used as a guide for the parameter design of pneumatic isolators in practice.

1. Introduction

Pneumatic vibration isolators are widely applied in optics, semiconductors, and other fields [1, 2]. Many efforts have been devoted to the design of active control systems, which require an accurate mathematical model of the pneumatic vibration isolator [3, 4]. Harris [5] assumed that the airflow in the damping hole was laminar and first offered a linear model for the pneumatic vibration isolators. In 1984, Debra and Bryan [6] established a simpler linear model of the volume ratio of the upper chamber to the lower chamber and the inherent frequency. Erin and Wilson [7] proposed an analytical modeling method for the pneumatic vibration isolator based on the ideal gas state equation. In all previous studies, the discrepancies between the model-predicted behavior and the practically observed behavior of pneumatic vibration isolators exceeded 50% because the effect of the elastomeric diaphragm was ignored.

To obtain a more accurate model of the pneumatic isolator, Erin and Wilson proposed an improved model for pneumatic isolators by incorporating the stiffness and damping effect of the elastomeric diaphragm based on experimental investigation. Lee and Kim [8, 9] reported a nonlinear complex stiffness model that reflected the frequency and excitation amplitude, which was validated by comparison with experimental measurements. Pu et al. [10] obtained the stiffness and damping characteristics of the elastomeric diaphragm indirectly by comparing the experimental and theoretical results. Additionally, Pu established the stiffness and damping model using a similar approach in other research. Since the 1990s, analog simulation has become an attractive method for examining the pneumatic spring, and the stiffness characteristic is often studied using the finite element method (FEM) [11]. However, experimental modification or the FEM is inconvenient for the parameter optimization design of a pneumatic isolator. Significant error

FIGURE 1: Pneumatic vibration isolator.

typically still exists between the actual behavior and the presented model after the experimental modification or FEM study. Therefore, a direct theoretical derivation method must be established for the development of an analytical model for a pneumatic vibration isolator.

Zhu et al. [12] and Li [13] proposed an analytic model of an elastomeric belt in a magnetic-levitation vibration isolator using the Mooney-Rivlin model, and the model was validated experimentally. In that research, the elastomeric belt was chosen as the research object, and an accurate model was obtained according to direct theoretical derivation. Inspired by this work, we use the same method and theory to study and analyze the elastomeric diaphragm in pneumatic vibration absorbers and develop an analytical model of the elastomeric diaphragm in this paper. Specifically, because the elastomeric diaphragm is composed of several segments in the axial section, each segment can be simplified as uniaxial stretching after considering the structural restriction, and a force-strain equation can be established for each segment. The analytical model of the elastomeric diaphragm can be built by combining all segments, and the model can be resolved by numerical solution. Experimental tests were conducted, and the results demonstrate that the analytically predicted behavior is similar to the practically observed behavior. The error in this method can be reduced to less than 10% after the elastomeric diaphragm model is added to the standard pneumatic cylinder model. Although certain scholars have proposed several improved models that also imitate the practical behavior, this paper supplies a direct theoretical derivation method with a notably small error. The proposed analytical model facilitates the practical parameter optimization design of pneumatic isolators.

2. Mathematic Model of the Elastomeric Diaphragm

2.1. Configuration of the Pneumatic Vibration Isolator. The configuration of a typical pneumatic vibration isolator is shown in Figure 1(a), which includes an air-filled cylinder, elastomeric diaphragm, and piston. The cylinder consists of a top chamber and bottom chamber that are both filled

with compressed air. The elastomeric diaphragm is located between the piston and cylinder, which is the seal component of the cylinder. The payload is connected to the piston, and the flow restrictor supplies damping for the vibration isolator.

Figure 1(b) illustrates that the payload is supported by two components: the pressurized air in the pneumatic cylinder and the elastomeric diaphragm, the model of which is often ignored. This paper only analyzes the stiffness effect of the elastomeric diaphragm without considering the damping effect because the vibration damping effect of the elastomeric diaphragm is a highly complex phenomenon. Therefore, the total stiffness of the pneumatic spring isolators can be obtained by adding the vertical stiffness of the elastomeric diaphragm (the model typically ignored in previous analysis and research) to the vertical stiffness of the pressurized air. The total stiffness of the pneumatic spring can be expressed as follows [14]:

$$K_{total} = K_m + K_a \quad (1)$$

where K_{total} represents the total stiffness of the pneumatic spring isolators, K_a represents the vertical stiffness of the compressed air, and K_m represents the vertical stiffness of the elastomeric diaphragm. K_m has typically been neglected in previous analyses of the pneumatic spring, which causes a certain error between the theory and experiments. This error in the total vertical stiffness can result in a series of problems. For example, the actual inherent frequency of a pneumatic vibration absorber cannot be easily estimated with high accuracy. After considering the effect of elastomeric diaphragm, a complete analytical model for a pneumatic vibration isolator is developed, and the developed analytical model can be used in the parameter optimization design of a pneumatic vibration isolator when the total stiffness and inherent frequency of the pneumatic vibration absorber are both known.

Figure 2 shows a cross-sectional view of an elastomeric diaphragm, and the parameters of the elastomeric diaphragm are listed in Table 1. After the elastomeric diaphragm is divided into four segments, the vertical stiffness of the segments, expressed as K_1 , K_2 , K_3 , and K_4 , is deduced such

TABLE 1: Parameters of the elastomeric diaphragm.

Symbol	Description
R	Radius of the piston (radius of the inner circle of the elastomeric diaphragm)
r	Radius of the arc of the elastomeric diaphragm
h_1	Height of the inner circle of the elastomeric diaphragm
h_3	Height of the outer circle
d_0	Thickness of the elastomeric diaphragm
"1"	Inner circle of the elastomeric diaphragm
"2"	Inner arc of the elastomeric diaphragm
"3"	Outer circle of the elastomeric diaphragm
"4"	Outer arc of the elastomeric diaphragm

FIGURE 2: Elastomeric diaphragm.

that the total vertical stiffness of the elastomeric diaphragm can be determined.

2.2. Force Analysis of the Elastomeric Diaphragm. The forces of the piston and elastomeric diaphragm in the vertical direction are shown in Figure 3; then,

$$F_a + F_1 = G + F_0 \quad (2)$$

$$F_1 = F'_1 = F_3 = F'_3 = \frac{F_P}{2} \quad (3)$$

where F_1 is the pulling force of part "2" acting on part "1" and F_3 is the pulling force of part "4" acting on part "3". In addition, F'_1 and F'_3 are the reaction forces of F_1 and F_3 , respectively, and F_a , F_0 and G can be rewritten as

$$F_a = P_a \cdot \pi (R + d_0 \lambda_{31})^2 \quad (4)$$

$$F_0 = P_0 \cdot \pi (R + d_0 \lambda_{31})^2 \quad (5)$$

$$G = mg \quad (6)$$

where F_a is the vertical force of the air in the chamber acting on the effective piston area, F_0 is the vertical force of the air outside the chamber acting on the effective piston area, G is the gravity of the load, P_0 is the atmospheric pressure, P_a is the air pressure in the chamber, m is the payload mass, g is

the gravitational acceleration, and λ_{31} is the corresponding stretch ratio of a unit of part "1", which is explained later. The stretch ratios in this paper are defined as the ratios of the stretched length to the unstretched length or original length of a small-volume element, which is a cube when it is unstretched or in the original state.

As shown in Figure 3(b), F_P is the force acting on the arc of the elastomeric diaphragm in the vertical direction and is expressed as

$$F_P = (P_a - P_0) \cdot s_t \quad (7)$$

where s_t is the projection area of the arc of the elastomeric diaphragm, as shown in Figure 4. Then, s_t can be expressed as

$$s_t = \pi \cdot (R + 2d_0 + 2r - d_0 \lambda_{33})^2 - \pi \cdot (d_0 \lambda_{31} + R)^2 \quad (8)$$

where λ_{33} is the corresponding stretch ratio of a unit of part "3", which is explained later.

Substituting (8) into (7) leads to

$$F_P = \pi (P_a - P_0) \cdot [(R + 2d_0 + 2r - d_0 \lambda_{33})^2 - (d_0 \lambda_{31} + R)^2] \quad (9)$$

FIGURE 3: Force analysis of the piston and elastomeric diaphragm.

FIGURE 4: Cross-section of the dual-chamber pneumatic spring.

By substituting (9) into (3) and combining the result with (2), (4), (5), and (6), the air pressure in the chamber can be obtained as

$$P_a = \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0\lambda_{33})^2 + (d_0\lambda_{31} + R)^2]} + P_0. \quad (10)$$

By substituting (10) into (9) and noting (3), one can obtain

$$F_1 = F_3 = \frac{F_p}{2} = \frac{mg [(R + 2d_0 + 2r - d_0\lambda_{33})^2 - (d_0\lambda_{31} + R)^2]}{(R + 2d_0 + 2r - d_0\lambda_{33})^2 + (d_0\lambda_{31} + R)^2}. \quad (11)$$

2.3. Mooney-Rivlin Model. Mooney-Rivlin [15, 16] assumed that the elastic behavior was isotropic and that simple stretching satisfied Hooke's law; then, they obtained the rubber energy density function W under the premise of not changing the rubber state and compressing its volume; i.e.,

$$W = W(J_1, J_2, J_3) \quad (12)$$

and

$$\begin{aligned} J_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \\ J_2 &= (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2 - 3 \\ J_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 - 1 \end{aligned} \quad (13)$$

where λ_1 , λ_2 , and λ_3 represent the stretch ratios in three different directions and are defined as the ratios of the stretched length to the unstretched length of a small-volume element, which is a cube when it is unstretched. These three directions, which are related to λ_1 , λ_2 , and λ_3 , are expressed under generalized XYZ coordinates. If ε_i is defined as a corresponding stretched length, the stretch ratio can be expressed as $\lambda_i = 1 + \varepsilon_i$. Herein, the rubber satisfies the incompressibility condition, i.e., $\lambda_1\lambda_2\lambda_3 = 1$, which means that $J_3 = 0$. The corresponding deformation energy density W only depends on J_1 and J_2 . Under the condition of sufficiently small deformation, the form of deformation energy function can be expressed as

$$W = C_1 J_1 + C_2 J_2 \quad (14)$$

$$\begin{aligned} C_1 &= \frac{\partial W}{\partial J_1}, \\ C_2 &= \frac{\partial W}{\partial J_2} \end{aligned} \quad (15)$$

where C_1 and C_2 represent different material parameters.

2.4. Stiffness Analysis of the Elastomeric Diaphragm

2.4.1. Part "1" Stiffness of the Elastomeric Diaphragm. As shown in Figure 5(a), when an elastomeric diaphragm is stretched, the tension F_1 acting on part "1" can be expressed as

$$F_1 = \delta_{11} s_{t1} \quad (16)$$

where s_{t1} is the sectional area of part "1", which is shown in Figure 4, and s_{t1} can be expressed as

$$s_{t1} = \pi [(R + d_0\lambda_{31})^2 - R^2] \quad (17)$$

where λ_{31} is the corresponding stretch ratio of a unit of part "1". In this paper, the letters of the subscript contain two numbers i and j , where $i(1,2,3)$ represents the corresponding principal stress surface and $j(1,2,3,4)$ represents the corresponding component of the elastomeric diaphragm, i.e., the inner circle of the elastomeric diaphragm, the inner arc of

FIGURE 5: Force analysis of part “1” of the elastomeric diaphragm.

FIGURE 6: Cross-section of one unit of part “1”.

the elastomeric diaphragm, the outer circle of the elastomeric diaphragm, and the outer arc of the elastomeric diaphragm, respectively.

Figure 5(b) shows the unit of part “1”, and the unit stress in three directions can be expressed as

$$\delta_{11} = 2 [C_1 \lambda_{11}^2 - C_2 \lambda_{11}^{-2}] + P_1 \quad (18)$$

$$\delta_{21} = 2 [C_1 \lambda_{21}^2 - C_2 \lambda_{21}^{-2}] + P_1 \quad (19)$$

$$\delta_{31} = 2 [C_1 \lambda_{31}^2 - C_2 \lambda_{31}^{-2}] + P_1 = P_a \quad (20)$$

where λ_{ij} is the corresponding stretch ratio and P_1 is the unknown pressure. The stretch ratio can be expressed as $\lambda_{11} = (h_1 + \varepsilon_1)/h_1$, where ε_1 is the length variable in the vertical direction of part “1”.

The following equation can be obtained by combining (18) with (20):

$$\delta_{11} = P_a + 2 [C_1 \lambda_{11}^2 - C_2 \lambda_{11}^{-2}] - 2 [C_1 \lambda_{31}^2 - C_2 \lambda_{31}^{-2}]. \quad (21)$$

Substituting (17) and (21) into (16) yields

$$F_1 = \pi \{ P_a + 2 [C_1 \lambda_{11}^2 - C_2 \lambda_{11}^{-2}] - 2 [C_1 \lambda_{31}^2 - C_2 \lambda_{31}^{-2}] \} [(R + d_0 \lambda_{31})^2 - R^2]. \quad (22)$$

The force on the unit in the cross-section is shown in Figure 6. The force of the piston acting on the unit of part “1” can be expressed as

$$F_{N1} = F_{P1} + 2F_{21} \cdot \sin\left(\frac{\gamma_1}{2}\right) \quad (23)$$

where F_{P1} is the third principal stress surface of part “1” subjected to the force of the gas chamber, F_{21} is the second principal stress surface of part “1” subjected to the force, and γ_1 is the angle of the unit in the cross-section of the dual-chamber pneumatic spring.

F_{P1} and F_{21} are defined as

$$F_{P1} = P_a \cdot h_1 \lambda_{11} \cdot b_1 \lambda_{21} \quad (24)$$

$$F_{21} = \delta_{21} \cdot h_1 \lambda_{11} \cdot d_0 \lambda_{31} \quad (25)$$

where $h_1 \lambda_{11} \cdot b_1 \lambda_{21}$ is the area of the third principal stress surface of part “1”, δ_{21} is the stress of the second principal stress surface of part “1”, and $h_1 \lambda_{11} \cdot d_0 \lambda_{31}$ is the area of the second principal stress surface of part “1”, as shown in Figure 5(b).

Substituting (24) and (25) into (23) yields

$$F_{N1} = P_a \cdot h_1 \lambda_{11} \cdot b_1 \lambda_{21} + 2\delta_{21} \cdot h_1 \lambda_{11} \cdot d_0 \lambda_{31} \cdot \sin\left(\frac{\gamma_1}{2}\right) \quad (26)$$

which indicates that $F_{N1} > 0$. Therefore, part “1” of the elastomeric diaphragm remains in contact with the piston wall, and the size of part “1” of the elastomeric diaphragm does not change when it is stretched in the circumferential direction; i.e., $\lambda_{21} = 1$. The incompressibility of rubber leads to $\lambda_{11} \lambda_{21} \lambda_{31} = 1$, and the following expression can be obtained:

$$\lambda_{31} = \lambda_{11}^{-1} = \left(\frac{h_1 + \varepsilon_1}{h_1} \right)^{-1}. \quad (27)$$

Substituting (10) and (27) into (22) yields

$$F_1 = \pi \left\{ \frac{2mg}{\pi \left[(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2 \right]} + P_0 \right. \\ \left. + 2(C_1 + C_2) \left[\left(\frac{h_1 + \varepsilon_1}{h_1} \right)^2 - \left(\frac{h_1}{h_1 + \varepsilon_1} \right)^2 \right] \right\} \left[\left(R + \frac{h_1 d_0}{h_1 + \varepsilon_1} \right)^2 - R^2 \right]. \quad (28)$$

To obtain a function for the stiffness K_1 of part “1” with respect to variable ε_1 , the following equation can be obtained by differentiating (28) with respect to ε_1 :

$$K_1 = \frac{dF_1}{d\varepsilon_1}. \quad (29)$$

The complete equation (29) is extremely long and is provided in Appendix B.

2.4.2. Part “2” Stiffness of the Elastomeric Diaphragm. As shown in Figure 7, the tension F_2 acting on part “2” can be expressed as

$$F_2 = \int_0^{\pi/2} \int_0^{2\pi} \delta_2 ds_{t2} \cdot d\alpha \quad (30)$$

where F_2 originates from the pressure difference of the top chamber and ambient environment in the vertical direction.

For the projected area of the unit, ds_{t2} can be written as

$$ds_{t2} = (R + r(1 - \cos \alpha)) d\theta \cdot d_0 \lambda_{32} \cdot \cos \alpha \quad (31)$$

where $R + r(1 - \cos \alpha)$ is the length l_2 from the analysis point to the center axis, $(R + r(1 - \cos \alpha))d\theta$ is the side length d_2 of the unit in part “2”, and $d_0 \lambda_{32}$ is the thickness of the unit in part “2”. θ is shown in Figure 7(a).

The vertical stress δ_2 of the unit of part “2” can be expressed as

$$\delta_2 = \delta_{12} \cos \alpha + \delta_{32} \sin \alpha \quad (32)$$

where δ_{12} and δ_{32} can be written as

$$\delta_{12} = 2 [C_1 \lambda_{12}^2 - C_2 \lambda_{12}^{-2}] + P_2 \quad (33)$$

$$\delta_{32} = 2 [C_1 \lambda_{32}^2 - C_2 \lambda_{32}^{-2}] + P_2 = P_a - P_0 \quad (34)$$

where δ_{12} and δ_{32} are the stresses of the corresponding principal stress surface and P_2 is the unknown pressure. The stretch ratio can be expressed as $\lambda_{12} = (r + \varepsilon_2)/r$, where ε_2 is the length variable in the vertical direction. The size of part “2” does not change when it is stretched in the circumferential direction; i.e., $\lambda_{22} = 1$. The incompressibility of rubber leads to $\lambda_{12} \lambda_{22} \lambda_{32} = 1$. Thus, the following equation can be obtained:

$$\lambda_{32} = \lambda_{12}^{-1} = \left(\frac{r + \varepsilon_2}{r} \right)^{-1} \quad (35)$$

The following equation can be obtained by transferring (32) to (35):

$$\delta_2 = \left\{ P_a - P_0 \right. \\ \left. + 2(C_1 + C_2) \left[\left(\frac{r + \varepsilon_2}{r} \right)^2 - \left(\frac{r}{r + \varepsilon_2} \right)^2 \right] \right\} \cos \alpha \\ + (P_a - P_0) \sin \alpha. \quad (36)$$

Substituting (10), (30), (31), and (36) into (30) yields

$$F_2 = 2\pi \left\{ \left[\frac{2mg}{\pi \left[(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2 \right]} + 2(C_1 + C_2) \left(\left(\frac{r + \varepsilon_2}{r} \right)^2 - \left(\frac{r}{r + \varepsilon_2} \right)^2 \right) \right] \right. \\ \left. \cdot \left[\frac{\pi d_0 r (R + r)}{4(r + \varepsilon_2)} - \frac{2d_0 r^2}{3(r + \varepsilon_2)} \right] \right. \\ \left. + \frac{2mg}{\pi \left[(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2 \right]} \left[\left(\frac{d_0 r (R + r)}{(r + \varepsilon_2)} \right) - \frac{d_0 r^2}{3(r + \varepsilon_2)} \right] \right\}. \quad (37)$$

FIGURE 7: Force analysis of part "2" of the elastomeric diaphragm.

FIGURE 8: Force analysis of part "3" of the elastomeric diaphragm.

To obtain a function of the stiffness K_2 of part "2" with respect to variable ε_2 , the following equation can be obtained by differentiating (37) with respect to ε_2 :

$$K_2 = \frac{dF_2}{d\varepsilon_2}. \quad (38)$$

The complete equation (38) is extremely long and is provided in Appendix B.

2.4.3. Part "3" Stiffness of the Elastomeric Diaphragm. As shown in Figure 8, when an elastomeric diaphragm is

stretched, the tension F_3 acting on part "3" can be expressed as

$$F_3 = \delta_{13} s_{13} \quad (39)$$

The analysis of part "3" is similar to that of part "1". Therefore, based on the Mooney-Rivlin model, the vertical stress δ_{13} can be expressed as

$$\delta_{13} = P_a + 2 [C_1 \lambda_{13}^2 - C_2 \lambda_{13}^{-2}] - 2 [C_1 \lambda_{33}^2 - C_2 \lambda_{33}^{-2}]. \quad (40)$$

where the stretch ratio can be expressed as $\lambda_{13} = (h_3 + \varepsilon_3)/h_3$ and ε_3 is the variable length in the vertical stretch of part "3".

FIGURE 9: Cross-section of one unit of part “3”.

Additionally, s_{t3} is the cross-sectional area of part “3”, as shown in Figure 4. Herein, s_{t3} can be expressed as

$$s_{t3} = \pi \left[(R + 2d_0 + 2r)^2 - (R + 2d_0 + 2r - d_0\lambda_{33})^2 \right]. \quad (41)$$

Substituting (40) and (41) into (39) yields

$$F_3 = \pi \left\{ P_a + 2 \left[C_1 \lambda_{13}^2 - C_2 \lambda_{13}^{-2} \right] - 2 \left[C_1 \lambda_{33}^2 - C_2 \lambda_{33}^{-2} \right] \right\} \cdot \left\{ (R + 2d_0 + 2r)^2 - (R + 2d_0 + 2r - d_0\lambda_{33})^2 \right\}. \quad (42)$$

The force on the unit in the cross-section is shown in Figure 9. The force of the piston acting on the unit of part “3” can be expressed as

$$F_{N3} = F_{P3} - 2F_{23} \cdot \sin\left(\frac{\gamma_3}{2}\right) \quad (43)$$

where F_{P3} is the third principal stress surface of part “3” subjected to the force of the gas chamber, F_{23} is the second principal stress surface of part “3” subjected to the force, and γ_3 is the angle of the unit in cross-section of the dual-chamber pneumatic spring.

F_{P3} and F_{23} are written as

$$F_{P3} = P_a \cdot h_3 \lambda_{13} \cdot b_3 \lambda_{23} \quad (44)$$

$$F_{23} = \delta_{23} \cdot h_3 \lambda_{13} \cdot d_0 \lambda_{33} \quad (45)$$

where $h_3 \lambda_{13} \cdot b_3 \lambda_{23}$ is the area of the third principal stress surface of part “3”, δ_{23} is the stress of the second principal stress surface of part “3”, and $h_3 \lambda_{13} \cdot d_0 \lambda_{33}$ is the area of the second principal stress surface of part “3”, as shown in Figure 8(b).

Substituting (44) and (45) into (43) yields

$$F_{N3} = P_a \cdot h_3 \lambda_{13} \cdot b_3 \lambda_{23} - 2\delta_{23} \cdot h_3 \lambda_{13} \cdot d_0 \lambda_{33} \cdot \sin\left(\frac{\gamma_3}{2}\right) \quad (46)$$

where $\sin(\gamma_3/2) = b_3/2(R + 2r + 2d_0 - d_0\lambda_{33})$ and b_3 is the arc length of the unit of part “3”, which is smaller than $2(R + 2r + 2d_0 - d_0\lambda_{33})$. Thus, the first polynomial is larger than the second polynomial of (46); i.e., $F_{N3} > 0$. Therefore, part “3” remains in contact with the fixed ring, and the size of part “3” does not change when it is stretched in the circumferential direction; i.e., $\lambda_{23}=1$. However, the incompressibility of rubber leads to $\lambda_{13}\lambda_{23}\lambda_{33}=1$. Thus, the following equation can be obtained:

$$\lambda_{33} = \lambda_{13}^{-1} = \left(\frac{h_3 + \varepsilon_3}{h_3} \right)^{-1}. \quad (47)$$

Substituting (10) and (47) into (42) yields

$$F_3 = \pi \left\{ \frac{2mg}{\pi \left[(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2 \right]} + P_0 + 2(C_1 + C_2) \left[\left(\frac{h_3 + \varepsilon_3}{h_3} \right)^2 - \left(\frac{h_3}{h_3 + \varepsilon_3} \right)^2 \right] \right\} \left[(R + 2d_0 + 2r)^2 - \left(R + 2d_0 + 2r - \frac{d_0 h_3}{h_3 + \varepsilon_3} \right)^2 \right]. \quad (48)$$

To obtain a function of the stiffness K_3 of part “3” with respect to variable ε_3 , the following equation can be obtained by differentiating (48) with respect to ε_3 :

$$K_3 = \frac{dF_3}{d\varepsilon_3}. \quad (49)$$

The complete equation (49) is extremely long and is provided in Appendix B.

2.4.4. Part “4” Stiffness of the Elastomeric Diaphragm. As shown in Figure 10, the tension F_4 acting on part “4” can be expressed as

FIGURE 10: Force analysis of part “4” of the elastomeric diaphragm.

$$F_4 = \int_0^{\pi/2} \int_0^{2\pi} \delta_4 ds_{t4} \cdot d\beta \quad (50)$$

where F_4 originates from the pressure difference of the top chamber and the ambient environment in the vertical direction.

For the projected area of the unit, ds_{t4} can be written as

$$ds_{t4} = (R + r(1 + \cos \beta)) d\theta \cdot d_0 \lambda_{34} \cdot \cos \beta \quad (51)$$

where $R + r(1 + \cos \beta)$ is the length l_4 from the analysis point to the center axis, $(R + r(1 + \cos \beta))d\theta$ is the side length d_4 of the unit in part “4”, $d_0 \lambda_{34}$ is the thickness of the unit in part “4”, and θ is shown in Figure 10(a).

The vertical stress δ_4 of the partial unit in part “4” can be expressed as

$$\delta_4 = \delta_{14} \cos \beta + \delta_{34} \sin \beta \quad (52)$$

and δ_{14} and δ_{34} can be rewritten as

$$\delta_{14} = 2 [C_1 \lambda_{14}^2 - C_2 \lambda_{14}^{-2}] + P_4 \quad (53)$$

$$\delta_{34} = 2 [C_1 \lambda_{34}^2 - C_2 \lambda_{34}^{-2}] + P_4 = P_a - P_0 \quad (54)$$

where δ_{14} and δ_{34} are the stresses corresponding to the principal stress surface and P_4 is the unknown pressure. The stretch ratio can be expressed as $\lambda_{14} = (r + \varepsilon_4)/r$, where ε_4 is the variable length in the vertical stretch of part “4”. The size of part “4” does not change when it is stretched in the circumferential direction; i.e., $\lambda_{24} = 1$. The incompressibility of rubber leads to $\lambda_{14} \lambda_{24} \lambda_{34} = 1$. Thus, the following equation can be obtained:

$$\lambda_{34} = \lambda_{14}^{-1} = \left(\frac{r + \varepsilon_4}{r} \right)^{-1}. \quad (55)$$

The following equation can be obtained by transferring (52) to (55):

$$\begin{aligned} \delta_4 = & \left\{ P_a - P_0 \right. \\ & + 2 (C_1 + C_2) \left[\left(\frac{r + \varepsilon_4}{r} \right)^2 - \left(\frac{r + \varepsilon_4}{r} \right)^{-2} \right] \left. \right\} \cos \beta \\ & + (P_a - P_0) \sin \beta. \end{aligned} \quad (56)$$

Substituting (10), (51), (55), and (56) into (50) yields

$$F_4 = 2\pi \left\{ \left[\frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2]} + 2 (C_1 + C_2) \left(\left(\frac{r + \varepsilon_4}{r} \right)^2 - \left(\frac{r + \varepsilon_4}{r} \right)^{-2} \right) \right] \right\}$$

$$\cdot \left[\frac{\pi d_0 r (R + r)}{4 (r + \varepsilon_4)} + \frac{2 d_0 r^2}{3 (r + \varepsilon_4)} \right] + \frac{2mg}{\pi \left[(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2 \right]} \left[\frac{1}{2} d_0 \left(\frac{r + \varepsilon_4}{r} \right)^{-1} (R + r) + \frac{1}{3} d_0 \left(\frac{r + \varepsilon_4}{r} \right)^{-1} r \right] \Bigg\}. \quad (57)$$

To obtain a function of the stiffness K_4 of part “4” with respect to variable ε_4 , the following equation can be obtained by differentiating (57) with respect to ε_4 :

$$K_4 = \frac{dF_4}{d\varepsilon_4}. \quad (58)$$

The entire equation (58) is extremely long and is provided in Appendix B.

2.4.5. Total Stiffness of the Elastomeric Diaphragm. As shown in Figure 11, part “1” and part “2” are connected in series such that the equivalent stiffness K_{12} can be expressed as

$$\frac{1}{K_{12}} = \frac{1}{K_1} + \frac{1}{K_2}. \quad (59)$$

Part “3” and part “4” are connected in series such that the equivalent stiffness K_{34} can be expressed as

$$\frac{1}{K_{34}} = \frac{1}{K_3} + \frac{1}{K_4}. \quad (60)$$

The total stiffness can be expressed as

$$K_m = K_{12} + K_{34} = \frac{K_1 K_2}{K_1 + K_2} + \frac{K_3 K_4}{K_3 + K_4}. \quad (61)$$

2.5. Numerical Solution of the Vertical Stiffness for the Elastomeric Diaphragm. When the load of the dual-chamber pneumatic spring is m , each part of the elastomeric diaphragm is stretched to a certain length. The stretched lengths of each corresponding part are defined as ε_{10} , ε_{20} , ε_{30} , and ε_{40} . By analyzing (3) and (11), each component of the force can be expressed as

$$F_1 = F'_1 = F_2 = F_3 = F'_3 = F_4 = \frac{F_P}{2} = \frac{mg \left[(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 - (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2 \right]}{(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2}. \quad (62)$$

The analysis process to obtain the stretched length of each corresponding part with respect to the other variables is as follows. Substituting $\varepsilon_1 = \varepsilon_{10}$ into (28) and noting (62), we can obtain (B.5) of the stretched length ε_{10} . (B.6), (B.7), and (B.8) can also be obtained in a similar manner. Equations (B.5) to (B.8) are rather long and are provided in Appendix B.

By solving (B.5) to (B.8) in the numerical root of ε_{10} , ε_{20} , ε_{30} , and ε_{40} and substituting $\varepsilon_1 = \varepsilon_{10}$, $\varepsilon_2 = \varepsilon_{20}$, $\varepsilon_3 = \varepsilon_{30}$, and $\varepsilon_4 = \varepsilon_{40}$ into (29), (38), (49), and (58), we obtain the numerical solutions of stiffness K_1 , K_2 , K_3 , and K_4 , respectively. As a result, the total stiffness K_m can be obtained from (61).

3. Experimental Validation

To verify the theoretical analysis, we test the performance and calculate the stiffness of the specific dual-chamber pneumatic spring.

The following text describes the method for obtaining $2(C_1 + C_2)$, where C_1 and C_2 are the Mooney-Rivlin material parameters. Classical rubber theory yields the relationship between the stiffness values of different types of rubbers and their Young's elastic moduli [15]. The elastomeric diaphragms commonly used in design of the pneumatic spring structure

consist of chloroprene rubber, nitrile butadiene rubber, butyl rubber, natural rubber, and other materials. A moderate stiffness is typically used to obtain a better overall performance. Therefore, the widely applied nitrile butadiene rubber with hardness $(55 \pm 5) H_A$ is used in the next two experiments. The Young's elastic modulus E changes from 2.465 MPa to 3.619 MPa via the relationship between the hardness and Young's elastic modulus. According to the incompressibility of the rubber material, we can obtain $E \approx 3Gs$. Hence, G_s ranges from 0.82 MPa to 1.20 MPa, where G_s is the shear modulus and $G_s = 2(C_1 + C_2)$. Finally, $2(C_1 + C_2)$ also ranges from 0.82 MPa to 1.20 MPa.

The vertical total stiffness K_{total} of the dual-chamber pneumatic isolator can be calculated according to the parameters in Table 2, where the stiffness of the gas is obtained based on Erin's study. Two experiments are shown in Figures 12 and 13. The main components of the isolation platforms are composed of four dual-chamber pneumatic isolators. The microvibration of the ground can be attenuated by the dual-chamber pneumatic isolators. Thus, we can obtain the transmissibility curve using the testing instrument HANDYSAM [17]. The input and output sensors used in this study are the ultralow frequency vibration pickup sensor 991B produced

TABLE 2: Parameters of the dual-chamber pneumatic in the two experiments.

Symbol	Description	Value 1	Value 2	Unit
$2(C_1 + C_2)$	C_1 and C_2 are the Mooney-Rivlin material parameters	0.82-1.20	0.82-1.20	MPa
h_1	Part "1" vertical length	0.009	0.015	m
r	Part "2" or "4" projection length	0.0034	0.0050	m
h_3	Part "3" vertical length	0.014	0.020	m
d_0	Diaphragm thickness	0.0006	0.002	m
p_0	Atmospheric pressure	101325	101325	MPa
g	Gravitational acceleration	9.81	9.81	m/s ²
M	Payload mass	20	375	kg
n	Polytropic exponent	1.4	1.4	-
L	Flow restrictor length	4e-4	8e-3	m
D	Flow restrictor diameter	1e-3	1e-3	m
A_b	Effective piston area	0.0015	0.016	m ²
μ	Air viscosity	1.824e-5	1.824e-5	Pa·s
C_r	Flow restriction constant	3.3640e-06	1.6019e-07	m ³ /(s·Pa)
V_b	Lower chamber volume	3.0000e-04	1.8170e-04	m ³
V_t	Upper chamber volume	2.7298e-05	2.3000e-03	m ³

FIGURE 11: Total stiffness of the elastomeric diaphragm.

FIGURE 13: Photograph of experiment 2.

FIGURE 12: Photograph of experiment 1.

by the Institute of Engineering Mechanics, China Earthquake Administration. The sensitivity of the sensors is 4×10^{-8} m/s. The load in experiment 1 is 50 kg, and the load in experiment 2 is 200 kg. Ground excitation used in the two experiments is just ambient vibration, whose signal in vertical direction measured by the input sensors is shown in Figure 14.

The theoretical expression of the frequency response functions has been inferred without considering the elastomeric diaphragm [10]. In this paper, the theoretical transmissibility curves are obtained as shown in Figures 15 and 16 by substituting the actual parameters of the experimental pneumatic vibration absorber shown in Table 2 into Erin's analytical model. The first curve is Erin's result without consideration of the elastomeric diaphragm. The second and third curves are our results with consideration of the elastomeric diaphragm. The last curve is the experimental result with the parameters shown in Table 2. Our model yields two curves, namely, the upper and lower bounds for the inherent frequencies, which correspond to the maximum and minimum values of $2(C_1 + C_2)$.

The results of experiment 1 are shown in Figure 15, where $f_{a1}=1.89$ Hz, $f_{C1}=6.03-7.70$ Hz, and $f_{E1}=6.75$ Hz are the inherent frequencies of the double-chamber pneumatic springs based on Erin's study and the proposed theory in this paper that considers the elastomeric diaphragm stiffness and experiment results, respectively. The results of experiment 2 are shown in Figure 16, where $f_{a2}=1.31$ Hz, $f_{C2}=3.38-4.30$ Hz, and $f_{E2}=3.75$ Hz are the inherent frequencies based on

TABLE 3: Stiffnesses from the calculations and experiments.

Symbol	Description	Value 1	Value 2	Unit
ε_{10}	Part “1” length stretched	15.0e-04 to 9.19e-04	22.0e-04 to 14.0e-04	m
ε_{20}	Part “2” length stretched	8.53e-04 to 5.16e-04	9.90e-04 to 5.85e-04	m
ε_{30}	Part “3” length stretched	13.0e-04 to 8.55e-04	18.0e-04 to 11.0e-04	m
ε_{40}	Part “4” length stretched	6.26e-04 to 3.88e-04	8.23e-04 to 4.94e-04	m
K_1	Part “1” stiffness	1.87e+04 to 3.24e+04	9.97e+04 to 1.75e+05	N/m
K_2	Part “2” stiffness	3.54e+04 to 6.32e+04	2.34e+05 to 4.12e+05	N/m
K_3	Part “3” stiffness	1.90e+04 to 3.10e+04	1.04e+05 to 1.73e+05	N/m
K_4	Part “4” stiffness	4.84e+04 to 8.42e+04	2.52e+05 to 4.54e+05	N/m
K_{12}	Part “1” and “2” equivalent stiffness	1.23e+04 to 2.14e+04	6.99e+04 to 1.23e+05	N/m
K_{34}	Part “3” and “4” equivalent stiffness	1.36e+04 to 2.27e+04	7.35e+04 to 1.25e+05	N/m
K_m	Rubber diaphragm stiffness	2.59e+04 to 4.41e+04	1.43e+05 to 2.48e+05	N/m
K_a	Air stiffness	0.2850e+04	2.5406e+04	N/m
K_{total}	Total stiffness	2.87e+04 to 4.90e+04	1.69e+05 to 2.74e+05	N/m
K_E	Experimental stiffness	3.5975e+04	2.0819e+05	N/m
f_C	Calculated inherent frequency	6.0332 to 7.7096	3.3769 to 4.2931	Hz
f_E	Experimental inherent frequency	6.625	3.5	Hz

FIGURE 14: Ground vibration velocity curve in vertical direction versus time.

Erin's study and the proposed theoretical and experiment results, respectively. The value of the damping in our two experiments is related to both the length and diameter of the flow restrictor. In experiment 1, the flow restrictor length is 0.0004, and the flow restrictor diameter is 0.001; thus, the value of the damping ratio is 0.005. In experiment 2, the flow restrictor length is 0.008, and the flow restrictor diameter is 0.016; thus, the value of the damping ratio is 0.1.

Figures 15 and 16 illustrate that f_{E1} and f_{E2} are within the ranges of f_{C1} and f_{C2} , respectively, demonstrating the effectiveness of the vertical stiffness model for the elastomeric diaphragm.

The stiffness values of the experiments can be obtained from the relationship of the inherent frequency, the stiffness, and the payload; i.e.,

$$K_{E1} = m\omega_{E1}^2 = m(2\pi f_{E1})^2, \quad (63a)$$

$$K_{E2} = m\omega_{E2}^2 = m(2\pi f_{E2})^2. \quad (63b)$$

FIGURE 15: Transmissibility curves of experiment 1.

FIGURE 16: Transmissibility curves of experiment 2.

The calculation results of the two structures with different parameters of the dual-chamber pneumatic spring are shown in Table 3. The vertical stiffness, K_{E1} and K_{E2} , in the experimental tests is within the corresponding ranges of total

stiffness K_{total} obtained via calculations, demonstrating the effectiveness of the vertical stiffness model for the elastomeric diaphragm.

4. Conclusions

This work developed an analytical model of the elastomeric diaphragm using the Mooney-Rivlin modeling method and elastomeric theory. The stiffness of the elastomeric diaphragm was ignored in the previous model of pneumatic vibration isolators. Specifically, each segmented elastomeric diaphragm can be simplified as uniaxial stretching, and the Mooney-Rivlin model can be adopted to establish the analytical expression of the vertical stiffness. The analytical model of the elastomeric diaphragm is built by combining all segments and can be resolved by numerical solution. The experimental results demonstrated that the analytically predicted behavior

is highly similar to the practically observed behavior in terms of the resonant frequency of the isolator after the elastomeric diaphragm model is added to the standard pneumatic cylinder model. Differing from the prior converse correction and experience design, this paper supplies a direct design method that is helpful for optimizing and designing the pneumatic spring parameters in practice.

Appendix

A.

See Table 4.

B. Selected Complex Equations in This Paper

$$K_1 = \frac{dF_1}{d\varepsilon_1} = \left\{ \frac{4mgh_1d_0 [d_0h_1 + R(h_1 + \varepsilon_1)]}{(h_1 + \varepsilon_1)^3 [(R + 2d_0 + 2r - d_0h_3/(h_3 + \varepsilon_3))^2 + (h_1d_0/(h_1 + \varepsilon_1) + R)^2]^2} + 4\pi(C_1 + C_2) \left(\frac{h_1 + \varepsilon_1}{h_1^2} + \frac{h_1^2}{(h_1 + \varepsilon_1)^3} \right) \right\} \left[\left(R + \frac{h_1d_0}{h_1 + \varepsilon_1} \right)^2 - R^2 \right] - 2\pi \left\{ \frac{mg}{\pi [(R + 2d_0 + 2r - d_0\lambda_{33})^2 + (d_0\lambda_{31} + R)^2]} + P_0 + 2(C_1 + C_2) \right. \quad (B.1)$$

$$\cdot \left[\left(\frac{h_1 + \varepsilon_1}{h_1} \right)^2 - \left(\frac{h_1}{h_1 + \varepsilon_1} \right)^2 \right] \left\{ \frac{h_1d_0 [d_0h_1 + R(h_1 + \varepsilon_1)]}{(h_1 + \varepsilon_1)^3} \right\}$$

$$K_2 = \frac{dF_2}{d\varepsilon_2} = 2\pi \left\{ 4(C_1 + C_2) \left(\frac{r + \varepsilon_2}{r^2} + \frac{r^2}{(r + \varepsilon_2)^3} \right) \left[\frac{\pi}{4} d_0 \left(\frac{r + \varepsilon_2}{r} \right)^{-1} (R + r) - \frac{2}{3} d_0 \left(\frac{r + \varepsilon_2}{r} \right)^{-1} r \right] \right.$$

$$- \left[\frac{2mg}{\pi [(R + 2d_0 + 2r - d_0h_3/(h_3 + \varepsilon_3))^2 + (d_0h_1/(h_1 + \varepsilon_1) + R)^2]} + 2(C_1 + C_2) \left(\left(\frac{r + \varepsilon_2}{r} \right)^2 - \left(\frac{r + \varepsilon_2}{r} \right)^{-2} \right) \right]$$

$$\cdot \left[\frac{\pi}{4} d_0 (R + r) - \frac{2}{3} d_0 r \right] \frac{r}{(r + \varepsilon_2)^2} - \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0h_3/(h_3 + \varepsilon_3))^2 + (d_0h_1/(h_1 + \varepsilon_1) + R)^2]} \left[\frac{1}{2} d_0 (R + r) \right.$$

$$\left. - \frac{1}{3} d_0 r \right] \frac{r}{(r + \varepsilon_2)^2} \left\{ \right.$$

$$K_3 = \frac{dF_3}{d\varepsilon_3} = \left\{ 4\pi(C_1 + C_2) \left(\frac{h_3 + \varepsilon_3}{h_3^2} + \frac{h_3^2}{(h_3 + \varepsilon_3)^3} \right) \right.$$

$$- \frac{4mgh_3d_0 [(R + 2d_0 + 2r)(h_3 + \varepsilon_3) - d_0h_3]}{(h_3 + \varepsilon_3)^3 [(R + 2d_0 + 2r - d_0h_3/(h_3 + \varepsilon_3))^2 + (d_0h_1/(h_1 + \varepsilon_1) + R)^2]^2} \left\{ \left[(R + 2d_0 + 2r)^2 - \left(R + 2d_0 + 2r \right. \right. \right.$$

$$\left. - \frac{h_3d_0}{h_3 + \varepsilon_3} \right)^2 \right] - 2\pi \left\{ P_0 + \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0h_3/(h_3 + \varepsilon_3))^2 + (d_0h_1/(h_1 + \varepsilon_1) + R)^2]} + 2(C_1 + C_2) \right.$$

$$\cdot \left[\left(\frac{h_3 + \varepsilon_3}{h_3} \right)^2 - \left(\frac{h_3}{h_3 + \varepsilon_3} \right)^2 \right] \left\{ \left[R + 2d_0 + 2r - \frac{h_3d_0}{h_3 + \varepsilon_3} \right] \frac{(h_3d_0)}{(h_3 + \varepsilon_3)^2} \right.$$

TABLE 4: All parameter descriptions in this paper.

Sym	Description	Sym	Description
K_{total}	Total stiffness of the pneumatic spring	δ_2	Vertical stress of the unit of part "2"
K_a	Vertical stiffness of the air	δ_4	Vertical stress of the unit of part "4"
K_m	Vertical stiffness of the elastomeric diaphragm	λ_{ij}	Corresponding stretch ratio representing the corresponding principal stress surface
R	Radius of the piston	δ_{ij}	Corresponding stress of the surface
r	Radius of the arc of the elastomeric diaphragm	i	Corresponding principal stress surface ($i = 1, 2, 3$)
h_1	Height of the inner circle of the elastomeric diaphragm	j	Corresponding part of the elastomeric diaphragm ($j = 1, 2, 3, 4$)
h_3	Height of the outer circle	s_{i1}	Sectional area of part "1"
d_0	Thickness of the elastomeric diaphragm	P_1	Unknown pressure
"1"	Inner circle of the elastomeric diaphragm	F_{N1}	Force of piston acting on the unit of "1"
"2"	Inner arc of the elastomeric diaphragm	F_{P1}	Third principal stress surface of "1" subjected to the force of the gas chamber
"3"	Outer circle of the elastomeric diaphragm	F_{Z1}	Second principal stress surface of "1" subjected to the force of the gas chamber
"4"	Outer arc of the elastomeric diaphragm	γ_1	Angle of the unit of "1" in the cross-section of the dual-chamber pneumatic spring
F_1	Pulling force of "2" acting on "1"	F_2	Tension acting on part "2"
F_3	Pulling force of "4" acting on "3"	ds_{i2}	Sectional area of the unit of "1"
F'_1	Reaction forces of F_1	l_2	Length from the unit of "2" to the center axis
F'_3	Reaction forces of F_3	d_2	Side length of the unit of "2"
G	Gravity of the load	θ	Angle
P_0	Atmospheric pressure	α	Angle
P_a	Air pressure in the chamber	P_2	Unknown pressure
m	Payload mass	s_{i3}	Sectional area of part "3"
g	Gravitational acceleration	F_{N3}	Force of piston acting on the unit of "3"
F_p	Force acting on the arc of the elastomeric diaphragm	F_{P3}	Third principal stress surface of "3" subjected to the force of the gas chamber
s_t	Projection area of the arc of the elastomeric diaphragm	F_{Z3}	Second principal stress surface of "3" subjected to the force of the gas chamber
W	Rubber energy density function	γ_3	Angle of the unit of "1" in the cross-section of the dual-chamber pneumatic spring
J_1	First strain invariant	F_4	Tension acting on part "4"
J_2	Second strain invariant	ds_{i4}	Sectional area of the unit of "4"
J_3	Third strain invariant	l_4	Length from the unit of "4" to the center axis
λ_1	Stretch ratios in the X direction	d_4	Side length of the unit of "4"
λ_2	Stretch ratios in the Y direction	α	Angle
λ_3	Stretch ratios in the Z direction	P_4	Unknown pressure
C_1	Material parameter	K_1	Stiffness of part "1"
C_2	Material parameter	K_2	Stiffness of part "2"
ϵ_1	Length variable of part "1"	K_3	Stiffness of part "3"
ϵ_2	Length variable of part "2"	K_4	Stiffness of part "4"
ϵ_3	Length variable of part "3"	K_{12}	Series stiffness of part "1" and "2"
ϵ_4	Length variable of part "4"	K_{34}	Series stiffness of part "3" and "4"

$$\begin{aligned}
K_4 = \frac{dF_4}{d\varepsilon_4} = 2\pi & \left\{ 4(C_1 + C_2) \left(\frac{r + \varepsilon_4}{r^2} + \frac{r^2}{(r + \varepsilon_4)^3} \right) \cdot \left[\frac{\pi}{4} d_0 \left(\frac{r + \varepsilon_4}{r} \right)^{-1} (R + r) + \frac{2}{3} d_0 \left(\frac{r + \varepsilon_4}{r} \right)^{-1} r \right] \right. \\
& - \left[\frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2]} + 2(C_1 + C_2) \left(\left(\frac{r + \varepsilon_4}{r} \right)^2 - \left(\frac{r + \varepsilon_4}{r} \right)^{-2} \right) \right] \\
& \cdot \left[\frac{\pi}{4} d_0 (R + r) + \frac{2}{3} d_0 r \right] \frac{r}{(r + \varepsilon_4)^2} - \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_3))^2 + (d_0 h_1 / (h_1 + \varepsilon_1) + R)^2]} \left[\frac{1}{2} d_0 (R + r) \right. \\
& \left. \left. + \frac{1}{3} d_0 r \right] \frac{r}{(r + \varepsilon_4)^2} \right\}
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
& \pi \left\{ \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]} + P_0 + 2(C_1 + C_2) \left[\left(\frac{h_1 + \varepsilon_{10}}{h_1} \right)^2 \right. \right. \\
& \left. \left. - \left(\frac{h_1}{h_1 + \varepsilon_{10}} \right)^2 \right] \right\} \left[\left(R + \frac{h_1 d_0}{h_1 + \varepsilon_{10}} \right)^2 - R^2 \right] = \frac{mg [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 - (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]}{(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2}
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
2\pi & \left\{ \left[\frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]} + 2(C_1 + C_2) \left(\left(\frac{r + \varepsilon_{20}}{r} \right)^2 - \left(\frac{r}{r + \varepsilon_{20}} \right)^2 \right) \right] \right. \\
& \cdot \left[\frac{\pi d_0 r (R + r)}{4(r + \varepsilon_{20})} - \frac{2d_0 r^2}{3(r + \varepsilon_{20})} \right] + \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]} \left[\left(\frac{d_0 r (R + r)}{(r + \varepsilon_{20})} \right) \right. \\
& \left. \left. - \frac{d_0 r^2}{3(r + \varepsilon_{20})} \right] \right\} = \frac{mg [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 - (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]}{(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2}
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
& \pi \left\{ \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]} + P_0 + 2(C_1 + C_2) \left[\left(\frac{h_3 + \varepsilon_{30}}{h_3} \right)^2 \right. \right. \\
& \left. \left. - \left(\frac{h_3}{h_3 + \varepsilon_{30}} \right)^2 \right] \right\} \left[(R + 2d_0 + 2r)^2 - \left(R + 2d_0 + 2r - \frac{d_0 h_3}{h_3 + \varepsilon_{30}} \right)^2 \right] \\
& = \frac{mg [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 - (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]}{(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2}
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
2\pi & \left\{ \left[\frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]} + 2(C_1 + C_2) \left(\left(\frac{r + \varepsilon_{40}}{r} \right)^2 - \left(\frac{r}{r + \varepsilon_{40}} \right)^2 \right) \right] \right. \\
& \cdot \left[\frac{\pi d_0 r (R + r)}{4(r + \varepsilon_{40})} + \frac{2d_0 r^2}{3(r + \varepsilon_{40})} \right] \\
& + \frac{2mg}{\pi [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]} \left[\frac{1}{2} d_0 \left(\frac{r + \varepsilon_{40}}{r} \right)^{-1} (R + r) \right. \\
& \left. \left. + \frac{1}{3} d_0 \left(\frac{r + \varepsilon_{40}}{r} \right)^{-1} r \right] \right\} = \frac{mg [(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 - (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2]}{(R + 2d_0 + 2r - d_0 h_3 / (h_3 + \varepsilon_{30}))^2 + (d_0 h_1 / (h_1 + \varepsilon_{10}) + R)^2}
\end{aligned} \tag{B.8}$$

Data Availability

The “Experiment 1.rar” and “Experiment 2.rar” compressed files used to support the findings of this study have been deposited in the Figshare repository (<https://figshare.com/s/86ee3484e7f439547b71>). Open the compressed files and then you can find Erin result, experimental result, experimental data, lower bound for the inherent frequency, and upper bound for the inherent frequency, respectively. You can use MATLAB software to open “.m” and “.fig” type files and use Microsoft Excel software to open “.xlsx” type files.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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