Research Article

A Two-Step Composite Time Integration Scheme for Vehicle-Track Interaction Analysis considering Contact Separation

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The two-step composite time integration scheme is combined with the linear contact relation between the wheel and rail (termed the two-step method) to solve coupled vehicle-track dynamic problems. First, two coupled vehicle-track models with different degrees-of-freedom are constructed, in which a vehicle-track system is modeled as two subsystems and the interaction between the two subsystems is implemented via kinematic constraints. Then, the two-step method is applied in the models to simulate several cases, and the accuracy and efficiency of this method are discussed in comparison with additional methods of the Newmark family. Contact separation can also be simulated using the same scheme without causing spurious oscillations for large integration steps. The results indicate that track irregularities can cause wheels to momentarily lose contact with the track, causing large impacts between the wheels and rail. Additionally, with the increase of the running velocity, contact separation will occur more frequently. The adoption of the two-step integration method can ensure the accuracy of solutions and significantly increase the computational efficiency; moreover, the matrixes representing the model information will not change during the calculation process, so the substructure data exported from commercial finite element software can be directly adopted.

1. Introduction

The accuracy and efficiency of the integration method to solve dynamic responses of the vehicle-track coupled system are very important. The track is an elastic structure with damping. Vibrations of the vehicle are transmitted to the track through wheel-rail contact, resulting in the excitation of track vibrations, which is in reverse act on the vehicle [1]. Meanwhile, the wheel flat [2] and track irregularities [3] are input to the coupled system via the wheel-rail force which significantly amplifies the dynamic interaction and generally displays nonlinear properties. The computational cost is greatly increased.

The vehicle-track coupled system can be represented as two sets of equations of motion, for the vehicle and track subsystems. The first group of methods solves the two sets of equations of motion fully coupled by combining the vehicle and track subsystems as a single integrated system that can be defined as the direct coupled method. Yang et al. [4, 5]

constrained the wheel to the bridge to eliminate the degrees-of-freedom of the vehicles at the contact point between the wheel and rail. Nguyen et al. [6] proposed an approximate linear contact model, which defines that the wheel-rail force can be multiplied by the elastic compression and the contact stiffness at the contact point. The equations of motion can be coupled by integrated wheel-rail force into the stiffness matrix [7]. Zhai et al. [1] and Sun et al. [8] used the explicit algorithm [9, 10] to solve the equations of motion with Hertzian nonlinear contact theory for simulating the wheel and rail contact.

However, the explicit integration scheme is conditionally stable. To solve the nonlinear dynamic problems with an unconditionally stable integration scheme, the other kind of method treats the vehicle-track system as two subsystems, and the interaction between the two subsystems is implemented via additional constraints at the contact point between the wheel and rail. The equations of motion of the vehicle and the track are solved separately by using iterative

procedures, so the kind of method can be defined as the iterative method. Baeza et al. [11] solved the nonlinear vehicle-track interaction problem with the Newmark method in the frequency domain. Zhang et al. [12] used a precise integration method for solving coupled vehicle-track dynamics to avoid calculation of the inversion of the stiffness matrix which can efficiently calculate the solution. Lei and Mao [13] solved two subsystems with the Newmark method by the iteration progress to meet the constraint condition at each time step. Neves et al. [14] and Montenegro et al. [15] used the Alpha method, an improvement of the Newmark method, for analyzing the nonlinear vehicle structure. Sadeghi et al. [16] used the Newmark Beta integration method together with full Newton-Raphson incremental iterative to solve the equations of motion of the vehicle-track system which can reduce computational cost. Uzzal et al. [17] used the Rayleigh-Ritz method to solve the coupled equations of the vehicle-track system by considering wheel flat. Zhang et al. [18] solved the nonlinear coupled system by the combination of the Newmark method and Newton method with consideration of contact separation and substructure surface roughness.

It can be seen that the selection of an integration method is closely related to the assumption of wheel-rail contact relation. In general, the direct coupled method has high computational efficiency, but the coupled matrix is time varying, which is inconvenient for using the substructure data obtained from finite element software. Additionally, the solution of the responses is not accurate enough especially in the case of track irregularities or wheel flat [19]. The iterative method can better solve the equations of motion of the vehicle and the track with nonlinear contact relation between the wheel and rail, but for interaction equations of higher order, several iterations are required in the numerical integration process [20]. Hence, these integration schemes are computationally costly.

In this paper, a two-step integration scheme [21–23] associated with contact separation is applied to analyze the dynamics of a vehicle-track coupled model. Furthermore, the calculation efficiency is evaluated by comparison with the methods of the Newmark family. The effects of vehicle speed and track irregularities are investigated on a multi-wheel vehicle and multilayer track-coupled system with contact separation. It is expected that the algorithm can be more widely applied to different types of models.

2. Equations of the System of Vehicle-Track Interaction

A conceptual two-dimensional model of a vehicle running on a track is illustrated in Figure 1(a). The model consists of two parts: the upper vehicle model and the lower track model. The vehicle is modeled in three degrees-of-freedom: u_1 , u_2 , and u_3 . The carbody and the bogie are connected by a spring-damper of stiffness k_2 and damping c_2 , while the bogie and the wheelset are connected by a spring-damper of stiffness k_1 and damping c_1 . The vehicle is moving along the track at a constant velocity v .

The motion equation of the vehicle system can be written in matrix form as follows:

$$\mathbf{M}^v \ddot{\mathbf{u}}^v + \mathbf{C}^v \dot{\mathbf{u}}^v + \mathbf{K}^v \mathbf{u}^v + \mathbf{L}^v \boldsymbol{\lambda} = \mathbf{P}_v, \quad (1)$$

where $\mathbf{M}^v$, $\mathbf{C}^v$, and $\mathbf{K}^v$ are the vehicle mass, damping, and stiffness matrixes, respectively; $\mathbf{L}^v$ is the vector describing the effects of wheel-rail force on the vehicle; $\mathbf{P}_v$ is the external force matrix acting on the vehicle; $\ddot{\mathbf{u}}^v$, $\dot{\mathbf{u}}^v$, and $\mathbf{u}^v$ are the acceleration, velocity, and displacement vectors, respectively; and $\boldsymbol{\lambda}$ is the wheel-rail force vector.

Using a free-body diagram with vertical displacement u_1 of car body m_c , vertical displacement u_2 of bogie m_t , and vertical displacement u_3 of bogie m_w , the motion equations of the vehicle model shown in Figure 1(a) are derived as follows:

$$\begin{aligned} & \begin{bmatrix} m_c & 0 & 0 \\ 0 & m_t & 0 \\ 0 & 0 & m_w \end{bmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{pmatrix} + \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_1 + c_2 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} \\ & + \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_1 + k_2 & -k_1 \\ 0 & -k_1 & k_1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ & + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{\lambda} = \begin{pmatrix} -m_c g \\ -m_t g \\ -m_w g \end{pmatrix}. \end{aligned} \quad (2)$$

The track is simplified as an Euler beam, and the motion equation can be written as follows:

$$\mathbf{M}^t \ddot{\mathbf{u}}^t + \mathbf{C}^t \dot{\mathbf{u}}^t + \mathbf{K}^t \mathbf{u}^t + \mathbf{L}^t(t) \boldsymbol{\lambda} = \mathbf{P}_t, \quad (3)$$

where $\mathbf{M}^t$, $\mathbf{C}^t$, and $\mathbf{K}^t$ are the track mass, damping, and stiffness matrixes, respectively; $\mathbf{P}_t$ is the external force matrix acting on the vehicle; $\ddot{\mathbf{u}}^t$, $\dot{\mathbf{u}}^t$, and $\mathbf{u}^t$ are the acceleration, velocity, and displacement vectors of the track nodes, respectively; and $\mathbf{L}^t(t)$ is the influence matrix of the wheel-rail force acting on the track.

In the finite element model of the simplified track, the nonnodal wheel-rail force λ is assigned to the nodes of track elements through the influence matrix $\mathbf{L}^t$. The influence matrix can be written as follows:

$$\mathbf{L}^t(t) = \mathbf{N}(t) \mathbf{L}_m^T, \quad (4)$$

where $\mathbf{N}(t)$ is the position transforming matrix consisting of 0 and 1 and $\mathbf{L}_m$ is the element influence matrix at the nodes due to a unit point load Q acting at the location (a, b) of the track element, as shown in Figure 1(b), and

$$\mathbf{L}_m = \left[\frac{b^2(b+3a)}{l^3} \frac{ab^2}{l^2} \frac{a^2(a+3b)}{l^3} - \frac{a^2b}{l^2} \right]^T, \quad (5)$$

where l is the length of the track element. Then, the displacement of the track at the wheel-rail contact point can be expressed as follows:

$$u^c(t) = \mathbf{L}^t(t) u^t(t). \quad (6)$$

Since track irregularities amplify the dynamic impact of the wheel to the track, the multiwheel vehicle and multilayer track model, as shown in Figure 2(a), can reflect the stability of the solution more accurately. In this case, the equation of motion of the vehicle can be written as follows:

FIGURE 1: Simplified vehicle-track model: (a) vehicle-track model; (b) equivalent nodal forces acting on a track element.

FIGURE 2: (a) Multiwheel vehicle and multilayer track-coupled model; (b) track element model.

$$\begin{bmatrix} \mathbf{M}^{vv} & 0 \\ 0 & \mathbf{M}^{ww} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}^{vv} \\ \ddot{\mathbf{u}}^{ww} \end{pmatrix} + \begin{bmatrix} \mathbf{C}^{vv} & \mathbf{C}^{vw} \\ \mathbf{C}^{wv} & \mathbf{C}^{ww} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{u}}^{vv} \\ \dot{\mathbf{u}}^{ww} \end{pmatrix} + \begin{bmatrix} \mathbf{K}^{vv} & \mathbf{K}^{vw} \\ \mathbf{K}^{wv} & \mathbf{K}^{ww} \end{bmatrix} \begin{pmatrix} \mathbf{u}^{vv} \\ \mathbf{u}^{ww} \end{pmatrix} + \begin{bmatrix} 0 \\ -\mathbf{I} \end{bmatrix} \begin{pmatrix} 0 \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{P}^{vv} \\ \mathbf{P}^{ww} \end{pmatrix}, \quad (7)$$

where $\mathbf{u}^{vv} = (u_1^v \ u_2^v \ u_3^v \ u_4^v \ u_5^v \ u_6^v)^T$ is the vertical displacement and rotation angle of the carbody and bogies (Figure 2(a)) and $\mathbf{u}^{ww} = (u_7^v \ u_8^v \ u_9^v \ u_{10}^v)^T$ is the vertical displacement of the four wheelsets.

$$\lambda = (\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4)^T,$$

$$\mathbf{M}^{vv} = \text{diag}(m_c \ I_c \ m_t \ I_t \ m_t \ I_t),$$

$$\mathbf{M}^{ww} = \text{diag}(m_w \ m_w \ m_w \ m_w),$$

$$\mathbf{P}^{vv} = (-m_c g \ 0 \ -m_t g \ 0 \ -m_t g \ 0)^T,$$

$$\mathbf{P}^{ww} = (-m_w g \ -m_w g \ -m_w g \ -m_w g)^T,$$

$$\mathbf{K}^{vv} = \begin{bmatrix} 2k_2 & 0 & -k_2 & 0 & -k_2 & 0 \\ 0 & \frac{k_2 l_v^2}{2} & \frac{k_2 l_v}{2} & 0 & -\frac{k_2 l_v}{2} & 0 \\ -k_2 & \frac{k_2 l_v}{2} & 2k_1 + k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k_1 d^2}{2} & 0 & 0 \\ -k_2 & -\frac{k_2 l_v}{2} & 0 & 0 & 2k_1 + k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{k_1 d^2}{2} \end{bmatrix}, \quad (8)$$

$$\mathbf{K}^{ww} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & k_1 \end{bmatrix},$$

$$\mathbf{K}^{wv} = (\mathbf{K}^{vw})^T = \begin{bmatrix} 0 & 0 & -k_1 & \frac{k_1 d}{2} & 0 & 0 \\ 0 & 0 & -k_1 & -\frac{k_1 d}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_1 & \frac{k_1 d}{2} \\ 0 & 0 & 0 & 0 & -k_1 & -\frac{k_1 d}{2} \end{bmatrix},$$

where $\mathbf{I}$ is the identity matrix; m_c , m_t , and m_w are the masses of the carbody, the bogie, and the wheelset, respectively; I_c

and I_t are the moments of inertia of the carbody and the bogie, respectively; c_1 and c_2 are the primary and secondary suspension dampings, respectively; k_1 and k_2 are the primary and secondary suspension stiffnesses, respectively; l_v is the longitudinal distance of the primary suspension; d is the longitudinal distance of the secondary suspension; and g is the acceleration due to gravity.

Finite elements are also used to simplify the multilayer track model, and the track between two neighboring fasteners is divided into one element; each element contains 12 degrees-of-freedom, as shown in Figure 2(b) [24]. In Figure 2(b), λ_n represents the n -th ($n = 1, 2, 3$, and 4) wheel-rail force. The stiffness matrix of the track element can be written as follows:

$$\mathbf{K}_e^v = \mathbf{K}_r^e + \mathbf{K}_s^e + \mathbf{K}_b^e + \mathbf{K}_{t1}^e + \mathbf{K}_{t2}^e + \mathbf{K}_{t3}^e, \quad (9)$$

where $\mathbf{K}_e^v$ is the track element stiffness matrix; $\mathbf{K}_r^e$, $\mathbf{K}_s^e$, and $\mathbf{K}_b^e$ are the stiffnesses of the rail, track slab, and base plate resisting elastic deformation, respectively; and $\mathbf{K}_{t1}^e$, $\mathbf{K}_{t2}^e$, and $\mathbf{K}_{t3}^e$ are the element stiffnesses of the fasteners, CA mortar, and subgrade, respectively.

The mass matrix of the track element can be written as follows:

$$\mathbf{M}_e^v = \mathbf{M}_r^e + \mathbf{M}_s^e + \mathbf{M}_b^e, \quad (10)$$

where $\mathbf{M}_e^v$ is the mass matrix of track elements and $\mathbf{M}_r^e$, $\mathbf{M}_s^e$, and $\mathbf{M}_b^e$ are the masses of the rail, track slab, and base plate elements, respectively.

The damping matrix of the track element can be written as follows:

$$\mathbf{C}_e^v = \alpha \mathbf{M}_e^v + \beta (\mathbf{K}_r^e + \mathbf{K}_s^e + \mathbf{K}_b^e) + \mathbf{C}_{t1}^e + \mathbf{C}_{t2}^e + \mathbf{C}_{t3}^e, \quad (11)$$

where $\mathbf{C}_e^v$ is the damping matrix of the track elements; α and β are the proportional damping coefficients; and $\mathbf{C}_{t1}^e$, $\mathbf{C}_{t2}^e$, and $\mathbf{C}_{t3}^e$ are the dampings of the fasteners, CA mortar, and subgrade elements, respectively.

3. Nonlinear Contact Relation with Separation

The nonlinear contact relation between the track and the vehicle is represented by a Hertz nonlinear spring. According to the Hertz theory [25], the vertical force between the wheel and rail can be expressed as

$$F = \left(\frac{\Delta u}{G} \right)^{3/2}, \quad (12)$$

where F is the wheel-rail force; G is the Hertz spring constant; and Δu is the vertical relative compressive deformation at the wheel-rail contact point.

The vertical elastic compression can be obtained by the following equation:

$$\Delta u = L^v u^v(t) + u^c(t) + \rho(t), \quad (13)$$

where $\rho(t)$ represents the vertical-track irregularities. Track irregularities have significant dynamic impact on both vehicle and track and should be considered carefully.

When separation of the vehicle and track is considered, it is crucial to check whether separation occurs between the

vehicle and track. Since only compressive stress possibly exists between the wheels and rail, the contact force F is always nonnegative. Hence, when the relative compression deformation Δu between a wheel and the rail is not less than zero, the wheel-rail force is applied on the track, and when it is less than zero, the wheel is separated from the track. In this respect, equation (12) is changed as follows:

$$F = \begin{cases} \left(\frac{\Delta u}{G}\right)^{3/2}, & \Delta u \geq 0, \\ 0, & \Delta u < 0. \end{cases} \quad (14)$$

Then, the numerical integration method is used to solve equations (1) and (3), and the responses of the system “ $(\ddot{\mathbf{u}}^v, \dot{\mathbf{u}}^v, \mathbf{u}^v)_i^n$, $(\ddot{\mathbf{u}}^t, \dot{\mathbf{u}}^t, \mathbf{u}^t)_i^n$ ” at the n time steps are obtained. The vertical elastic compression at the n time steps is expressed as follows:

$$\Delta u_i^n = (u^v)_i^n + (u^c)_i^n + \rho_i^n, \quad i = 1, \dots, m, \quad (15)$$

where m is the number of wheelsets. Once Δu_i^n is derived, the wheel-rail force F_i^n is calculated with equation (14) and then used as the initial condition at the $(n+1)$ time steps:

$$(w_i^{n+1})_s = \left[(\Delta u_i^{n+1})_s - (\Delta u_i^{n+1})_{s-1} \right]. \quad (16)$$

At each time step, the calculation process is repeated several times until the difference $(w_i^{n+1})_s$ (s is the times of iteration) between adjacent results is less than 10^{-6} .

4. Linear Contact Relation with Separation

The linear approximate contact model assumes that the wheel and rail are connected by a linear spring, representing elastic contact, and the vertical wheel-rail force is equal to the product of vertical contact stiffness and vertical elastic compression at the wheel-rail contact point [6, 25]. In this section, the relationship between the wheel and the rail is described by the linear contact relation with consideration of separation.

According to the linearized contact relation at the equilibrium point, the wheel-rail force F is obtained as

$$k\Delta u = F, \quad (17)$$

$$k = \begin{cases} k_0, & \Delta u \geq 0, \\ 0, & \Delta u < 0, \end{cases} \quad (18)$$

where k is the linear contact stiffness.

The contact force F is always nonnegative. The wheel-rail relative compression deformation Δu is used to judge whether the wheel is separated from the track.

4.1. Solving the Kinematic Equations with the Direct Coupled Method. Combining equations (1), (3), and (17), the equations of motion can be obtained:

$$\begin{bmatrix} \mathbf{M}^v & \\ & \mathbf{M}^t \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}}^v \\ \ddot{\mathbf{u}}^t \end{pmatrix} + \begin{bmatrix} \mathbf{C}^v & \\ & \mathbf{C}^t \end{bmatrix} \begin{pmatrix} \dot{\mathbf{u}}^v \\ \dot{\mathbf{u}}^t \end{pmatrix} + \left\{ \begin{bmatrix} \mathbf{K}^v & \\ & \mathbf{K}^t \end{bmatrix} + \mathbf{K}^q \right\} \begin{pmatrix} \mathbf{u}^v \\ \mathbf{u}^t \end{pmatrix} = \begin{pmatrix} \mathbf{P}^v \\ \mathbf{P}^t \end{pmatrix} + \mathbf{P}^q, \quad (19)$$

where $\mathbf{K}^q$ and $\mathbf{P}^q$ are the additional stiffness matrix and additional wheel-rail force matrix caused by the vertical wheel-rail contact force, respectively.

Taking the moving vehicle model shown in Figure 1(a) as an example, the additional matrixes $\mathbf{K}^q$ and $\mathbf{P}^q$ can be expressed as follows:

$$\mathbf{K}^q = \begin{bmatrix} k\mathbf{L}^v & -k\mathbf{L}^v(\mathbf{L}^t)^T \\ k\mathbf{L}^t & -k\mathbf{L}^t(\mathbf{L}^t)^T \end{bmatrix}, \quad (20)$$

$$\mathbf{P}^q = \begin{pmatrix} -k\rho \\ (k\rho - m_w g)\mathbf{L}^t \end{pmatrix}. \quad (21)$$

Because of the linear equilibrium condition of equation (18), the equations of motion of the vehicle and track subsystems are coupled [14] into global equation (19). It can be seen that the wheel-rail force is expressed by the additional matrixes $\mathbf{K}^q$ and $\mathbf{P}^q$; both matrixes are updated at each time step. At each time step, equation (18) is still valid. The value of the linear contact stiffness k in equations (20) and (21) at the $(n+1)$ time steps is determined by the vertical elastic compression calculated at the n time steps.

4.2. Solving the Kinematic Equations with the Two-Step Integration Method. The two-step integration scheme [21, 22] is stabilized by the assumptions of the trapezoidal rule and backward difference approximation. This integration scheme is available in some nonlinear problems of structural and solid mechanics against the trapezoidal rule and the Wilson- θ method [21]. In view of the flexibility of the track system, the inertia forces caused by the wheel mass play an important role in the interaction between the vehicle and track. To make the two-step integration scheme remain stable and to maintain the second-order accuracy of the vehicle-track interaction analysis, the wheel is no longer constrained to the track [23], and the linear equilibrium equation is used as the constraint equation.

Assume that the response at the n time steps is known; the goal is to solve the response at the $(n+1)$ time steps. Different from the majority of time integration schemes, the two-step integration scheme calculates all unknown variables not only at the $(n+1)$ time steps but also at the $(n+1/2)$ time steps at the same time. Then, according to the known responses at the n and $(n+1/2)$ time steps, the response at the $(n+1)$ time steps can be solved by using a backward difference.

The kinematic equations consist of two parts: one part is the governing equations of the motion of the vehicle equation (1) and the track equation (3), and the other part is the constraint equation (18). Hence, the coupled equations of motion can be obtained as follows:

$$\mathbf{M}^v \ddot{\mathbf{u}}^v + \mathbf{C}^v \dot{\mathbf{u}}^v + \mathbf{K}^v \mathbf{u}^v + \mathbf{L}^v \boldsymbol{\lambda} = \mathbf{P}^v, \quad (22a)$$

$$\mathbf{M}^t \ddot{\mathbf{u}}^t + \mathbf{C}^t \dot{\mathbf{u}}^t + \mathbf{K}^t \mathbf{u}^t + \mathbf{L}^t(t) \boldsymbol{\lambda} = \mathbf{P}^t, \quad (22b)$$

$$\begin{cases} \Delta u = \frac{\lambda}{k_t}, & \Delta u \geq 0, \\ \lambda = 0, & \Delta u < 0, \end{cases} \quad (22c)$$

where k_t is the contact stiffness between the wheel and rail.

According to the two-step integration scheme, in the first substep, the response at the $(n + 1/2)$ time steps is calculated using the trapezoidal rule:

$$\dot{u}_{n+(1/2)} = \dot{u}_n + \frac{\Delta t}{4} (\ddot{u}_n + \ddot{u}_{n+(1/2)}), \quad (23a)$$

$$u_{n+(1/2)} = u_n + \frac{\Delta t}{4} (\dot{u}_n + \dot{u}_{n+(1/2)}), \quad (23b)$$

where Δt is the time interval.

The coupled equations (equations (22a)–(22c)) at the $(n + 1/2)$ time steps can then be rewritten as follows:

$$\bar{\mathbf{M}}_{n+(1/2)}^v \tilde{\mathbf{u}}_{n+(1/2)}^v = \mathbf{a}_{n+(1/2)}^v, \quad (24a)$$

$$\bar{\mathbf{M}}_{n+(1/2)}^t \tilde{\mathbf{u}}_{n+(1/2)}^t = \mathbf{a}_{n+(1/2)}^t, \quad (24b)$$

$$\begin{aligned} \left(\frac{1}{k_t} + \mathbf{M}\mathbf{M}_{n+(1/2)}^v + \mathbf{M}\mathbf{M}_{n+(1/2)}^t \right) \boldsymbol{\lambda}_{n+(1/2)} &= \bar{\boldsymbol{\rho}}_{n+(1/2)}, \bar{\boldsymbol{\rho}}_{n+(1/2)} \geq 0, \\ \boldsymbol{\lambda}_{n+(1/2)} &= 0, \bar{\boldsymbol{\rho}}_{n+(1/2)} < 0, \end{aligned} \quad (24c)$$

where

$$\begin{aligned} \bar{\mathbf{M}}_{n+(1/2)} &= \frac{16}{\Delta t^2} \mathbf{M} + \frac{4}{\Delta t} \mathbf{C} + \mathbf{K}, \\ \tilde{\mathbf{u}}_{n+(1/2)}^v &= \mathbf{u}_{n+(1/2)}^v + (\bar{\mathbf{M}}_{n+(1/2)}^v)^{-1} (\mathbf{L}^v)^T \boldsymbol{\lambda}_{n+(1/2)}, \\ \tilde{\mathbf{u}}_{n+(1/2)}^t &= \mathbf{u}_{n+(1/2)}^t + (\bar{\mathbf{M}}_{n+(1/2)}^t)^{-1} (\mathbf{L}^t(\mathbf{t}_{n+(1/2)}))^T \boldsymbol{\lambda}_{n+(1/2)}, \\ \mathbf{M}\mathbf{M}_{n+(1/2)}^v &= \mathbf{L}^v (\bar{\mathbf{M}}_{n+(1/2)})^{-1} (\mathbf{L}^v)^T, \\ \mathbf{M}\mathbf{M}_{n+(1/2)}^t &= \mathbf{L}^t(\mathbf{t}_{n+(1/2)}) (\bar{\mathbf{M}}_{n+(1/2)})^{-1} (\mathbf{L}^t(\mathbf{t}_{n+(1/2)}))^T, \\ \bar{\boldsymbol{\rho}}_{n+(1/2)} &= \mathbf{L}^v \tilde{\mathbf{u}}_{n+(1/2)}^v + \mathbf{L}^t(\mathbf{t}_{n+(1/2)}) \tilde{\mathbf{u}}_{n+(1/2)}^t + \boldsymbol{\rho}(\mathbf{t}_{n+(1/2)}), \\ \mathbf{a}_{n+(1/2)} &= \mathbf{P}_{n+(1/2)} + \mathbf{C} \left(\frac{4}{\Delta t} \mathbf{u}_n + \dot{\mathbf{u}}_n \right) + \mathbf{M} \left(\frac{16}{\Delta t^2} \mathbf{u}_n + \frac{8}{\Delta t} \dot{\mathbf{u}}_n + \ddot{\mathbf{u}}_n \right). \end{aligned} \quad (25)$$

In the second substep, the $(n+1)$ time steps are calculated with the 3-point backward difference method:

$$\dot{u}_{n+1} = \frac{1}{\Delta t} u_n - \frac{4}{\Delta t} u_{n+(1/2)} + \frac{3}{\Delta t} u_{n+1}, \quad (26a)$$

$$\ddot{u}_{n+1} = \frac{1}{\Delta t} \dot{u}_n - \frac{4}{\Delta t} \dot{u}_{n+(1/2)} + \frac{3}{\Delta t} \dot{u}_{n+1}. \quad (26b)$$

Substituting equations (26a) and (26b) into equations (22a)–(22c) yields the coupled equations at the $(n + 1)$ time steps:

$$\bar{\mathbf{M}}_{n+1}^v \tilde{\mathbf{u}}_{n+1}^v = \mathbf{a}_{n+1}^v, \quad (27a)$$

$$\bar{\mathbf{M}}_{n+1}^t \tilde{\mathbf{u}}_{n+1}^t = \mathbf{a}_{n+1}^t, \quad (27b)$$

$$\begin{aligned} \left(\frac{1}{k_t} + \mathbf{M}\mathbf{M}_{n+1}^v + \mathbf{M}\mathbf{M}_{n+1}^t \right) \boldsymbol{\lambda}_{n+1} &= \bar{\boldsymbol{\rho}}_{n+1}, \bar{\boldsymbol{\rho}}_{n+1} \geq 0, \\ \boldsymbol{\lambda}_{n+1} &= 0, \bar{\boldsymbol{\rho}}_{n+1} < 0, \end{aligned} \quad (27c)$$

where

$$\bar{\mathbf{M}}_{n+1} = \frac{9}{\Delta t^2} \mathbf{M} + \frac{3}{\Delta t} \mathbf{C} + \mathbf{K},$$

$$\tilde{\mathbf{u}}_{n+1}^v = \mathbf{u}_{n+1}^v + (\bar{\mathbf{M}}_{n+1}^v)^{-1} (\mathbf{L}^v)^T \boldsymbol{\lambda}_{n+1},$$

$$\tilde{\mathbf{u}}_{n+1}^t = \mathbf{u}_{n+1}^t + (\bar{\mathbf{M}}_{n+1}^t)^{-1} (\mathbf{L}^t(\mathbf{t}_{n+1}))^T \boldsymbol{\lambda}_{n+1},$$

$$\mathbf{M}\mathbf{M}_{n+1}^v = \mathbf{L}^v (\bar{\mathbf{M}}_{n+1})^{-1} (\mathbf{L}^v)^T,$$

$$\mathbf{M}\mathbf{M}_{n+1}^t = \mathbf{L}^t(\mathbf{t}_{n+1}) (\bar{\mathbf{M}}_{n+1})^{-1} (\mathbf{L}^t(\mathbf{t}_{n+1}))^T,$$

$$\bar{\boldsymbol{\rho}}_{n+1} = \mathbf{L}^v \tilde{\mathbf{u}}_{n+1}^v + \mathbf{L}^t(\mathbf{t}_{n+1}) \tilde{\mathbf{u}}_{n+1}^t + \boldsymbol{\rho}(\mathbf{t}_{n+1}),$$

$$\begin{aligned} \mathbf{a}_{n+1} &= \mathbf{P}_{n+1} + \mathbf{C} \left(\frac{4}{\Delta t} \mathbf{u}_{n+(1/2)} - \frac{1}{\Delta t} \mathbf{u}_n \right) \\ &\quad + \mathbf{M} \left(\frac{12}{\Delta t^2} \mathbf{u}_{n+(1/2)} - \frac{3}{\Delta t^2} \mathbf{u}_n + \frac{4}{\Delta t} \dot{\mathbf{u}}_{n+(1/2)} - \frac{1}{\Delta t} \dot{\mathbf{u}}_n \right). \end{aligned} \quad (28)$$

The present algorithm can be implemented through the steps, as shown in Figure 3.

The time-varying coupled relation caused by the vehicle-track interaction is embodied in equations (24c) and (27c). The matrix $\bar{\mathbf{M}}$ is not changed in the step-by-step integration method, and iteration is not required in the solution process. Hence, the coupled equations can be solved without modifying the model formulation so that the substructure data exported from commercial finite element software can be directly used.

5. Applications regarding the Two-Step Integration Scheme

5.1. Responses of the Vehicle-Track System. The vehicle-track model shown in Figure 1(a) is considered in this section. The track is simplified as an Euler beam continuously supported by springs and dampers, and the vehicle is represented by a multibody system comprising rigid bodies. The running speed of the vehicle is constant at 117 km/h, and only the vibration is considered in the vertical direction. Suppose that the vehicle-track model is laterally symmetric along the line; therefore, a half structure is studied. The parameters of the vehicle-track model are presented in Table 1.

The maximum error of the results [26] in this section is shown, which is defined as follows:

- (1) Assemble $\bar{\mathbf{M}}_{n+(1/2)}$, $\bar{\mathbf{M}}_{n+1}$, $\mathbf{P}_v$, and $\mathbf{P}_t$;
- (2) Solve $\bar{\mathbf{u}}_{n+(1/2)}^v$ and $\bar{\mathbf{u}}_{n+(1/2)}^t$ based on equations (24a) and (24b);
- (3) Solve $\lambda_{n+(1/2)}$ based on equation (24c);
- (4) Set $\mathbf{u}_{n+(1/2)} = \bar{\mathbf{u}}_{n+(1/2)} - (\bar{\mathbf{M}}_{n+(1/2)})^{-1} (\mathbf{L}^v)^T \lambda_{n+(1/2)}$, $\dot{\mathbf{u}}_{n+(1/2)} = \frac{4}{\Delta t} (\mathbf{u}_{n+(1/2)} - \mathbf{u}_n) - \dot{\mathbf{u}}_{n+(1/2)}$, and $\ddot{\mathbf{u}}_{n+(1/2)} = \frac{4}{\Delta t} (\dot{\mathbf{u}}_{n+(1/2)} - \dot{\mathbf{u}}_n) - \ddot{\mathbf{u}}_n$;
- (5) Repeat steps (2) and (3) and replace equation (24) with equation (27) to solve $\bar{\mathbf{u}}_{n+1}^v$, $\bar{\mathbf{u}}_{n+1}^t$, and λ_{n+1} ; and
- (6) Set $\mathbf{u}_{n+(1/2)} = \bar{\mathbf{u}}_{n+1} - (\bar{\mathbf{M}}_{n+1})^{-1} (\mathbf{L}^t(t_{n+1}))^T \lambda_{n+1}$, $\dot{\mathbf{u}}_{n+1} = \frac{1}{\Delta t} \mathbf{u}_n - \frac{4}{\Delta t} \mathbf{u}_{n+(1/2)} + \frac{3}{\Delta t} \mathbf{u}_{n+1}$, and $\ddot{\mathbf{u}}_{n+1} = \frac{1}{\Delta t} \dot{\mathbf{u}}_n - \frac{4}{\Delta t} \dot{\mathbf{u}}_{n+(1/2)} + \frac{3}{\Delta t} \dot{\mathbf{u}}_{n+1}$.

FIGURE 3: Summary of the two-step integration method.

TABLE 1: Vehicle and track parameters.

Parameters (units)	Value
Static wheel load (N)	82000
Wheelset mass (kg)	500
Rail flexural rigidity, EI (N·m ²)	4864500
Rail mass (kg/m)	119
Track stiffness (N/m ²)	200
Track damping (N·s/m ²)	30
Hertz constant, G (N/m ^{3/2})	10 ¹¹
Length of the track (m)	60
Length of track element, L (m)	0.6

$$T_r = \left[\frac{u - T_s}{T_s} \right]_{\max}, \quad (29)$$

where u is the response and T_s is the corresponding static response.

If u is the displacement, T_s is defined as follows:

$$T_s = \frac{(m_c + m_t + m_w)g\mathbf{L}_c}{\mathbf{K}^t}. \quad (30)$$

If u is the acceleration, T_s is defined as follows:

$$T_s = g. \quad (31)$$

If u is the wheel-rail force, T_s is defined as follows:

$$T_s = (m_c + m_t + m_w)g. \quad (32)$$

5.1.1. Verification of the Two-Step Integration Method.

The effectiveness of the two-step integration method (referred to as “Two-step” in the following text) in solving vehicle-track coupled problems has been verified by comparison with the results simulated with the model in Figure 1(a). The responses of the vehicle-track coupled model with linear contact relation (referred to as “NML” in the following text) and nonlinear contact relation (referred to as “NMN” in the following text) between the wheel and rail are solved by the Newmark Beta method. Moreover, the effect of track irregularities on the model investigated by Newton and Clark (referred to as “NCM” in the following text) has been calibrated by field data [27]. The result extracted from the NCM model is also used as a comparison [20].

The track irregularity selected in this case is representative of wheel flat. In an attempt to overcome the difficulties in locating the positions of the flats and their impacts relative to the track system, an equivalent track irregularity is used instead. The shape of the irregularity is determined based on a large number of test results, and the wheel diameter will be adjusted to fit various vehicle types. The equivalent irregularity can be defined as follows:

$$p(x_p) = \frac{D}{2} \left(1 - \cos\left(\frac{2\pi}{l}x_p\right) \right), \quad (33)$$

$$D = \frac{l^2}{16r},$$

where x_p is the position along the track; D is the maximum depth of the equivalent irregularity, 2.15 mm; l is the length of the equivalent irregularity, 150 mm; and r is the wheel radius.

Figure 4 presents the analysis results obtained for the model with the track irregularity shown in Figure 5(a). Comparisons are made among the effects of the wheel flat on the wheel-rail force computed with the two-step method, NMN, NML, and NCM; all of the results, except for those for NML are in quite good agreement, as shown in Figure 4(a). The high-frequency modal responses of the NCM have been filtered out in Figure 4(a); therefore, compared to the wheel-rail forces from the two-step method and NMN, oscillations do not appear in the NCM result after 0.929 s. To show the accuracy of the two-step method more clearly, the accelerations of the midpoint of the track and wheel calculated by the two-step method, NMN, and NML are compared in Figures 4(b) and 4(c). As seen, the results of the two-step method and NMN are in quite good agreement, whereas those for NML become unrealistically large.

However, the Newmark Beta method with linear contact relation between the wheel and rail becomes unstable when large deformations and long-time contact separations are considered. The two-step method can remain stable as long as an appropriate time step is chosen. More details about the two-step method regarding computational cost and application will be introduced in the following sections.

FIGURE 4: Responses of the vehicle-track system with a wheel flat: (a) wheel-rail force; (b) acceleration of the track; (c) acceleration of the wheel.

FIGURE 5: Track irregularities: (a) track irregularity of a wheel flat; (b) the measured random track irregularity.

5.1.2. Vehicle-Track System without Irregularity. In this subsection, the responses of the vehicle-track model without irregularity are solved with the two-step method, and the results are compared with those obtained by the methods of

the Newmark family. Two other methods are used, that is, the Newmark Beta method (NML) and the Alpha method (referred to as “APL” in the following text) with linear contact relation between the wheel and rail. The Alpha

FIGURE 6: Responses of the vehicle-track system without irregularity: (a) acceleration of the track; (b) acceleration of the wheel; (c) the maximum error of the displacement of the wheel and the displacement at the midpoint of the track and wheel-rail force; (d) the calculation time using the NML, APL, and two-step method.

method, a member of the Newmark family, has better stability and accuracy with introduction of alpha damping [28]. The value of the alpha damping is -0.1 for the case.

As shown in Figures 6(a)–6(c), the responses of the vehicle-track system calculated with the NML, APL, and two-step method are in great agreement. However, the calculation times required by the three methods differ, as shown in Figure 6(d). The calculation time required by the two-step method is much less than that for NML and significantly less than that for APL; it is 5.6% for NML and 61.8% for APL. The integration steps of the Newmark Beta method and Alpha method are 10^{-5} s and 10^{-4} s, respectively, whereas the integration step for the two-step method is 10^{-3} s. The smaller integration step required in the computational process greatly increases the computational cost.

Figure 7 shows the acceleration of the midpoint of the track for an integration step of 10^{-3} s. Abnormal high-frequency oscillations appear in the response of the track calculated with the Newmark family methods, probably because the solution at the time step of 10^{-3} s does not

satisfy the dynamic equilibrium accurately. Calculation errors from each time step gradually accumulate to deteriorate the accuracy of the overall solution.

5.2. Multiwheel Vehicle and Multilayer Track System with Random Track Irregularities. In this section, a multiwheel vehicle and multilayer track model is considered subject to random track irregularities, as shown in Figure 2(a). The random rail irregularities selected are the measured dynamic track irregularity of track inspection car on the Beijing–Jiulong Railway with a design running at a speed of 200 km/h (Sample 1) and the Beijing–Shanghai high-speed railway with a design running speed of 350 km/h (Sample 2), as shown in Figure 5(b). When there is track irregularity, impact loads occur due to contact separation. The responses of the system are solved with the two-step method to show the algorithm stability with high-frequency vibrations induced by impact load. The parameters of the model are presented in Table 2. The length of the track is 100 m, and the integration step is 10^{-5} s.

FIGURE 7: Accelerations of the midpoint of the track without irregularity for an integration step of 10^{-3} s.

TABLE 2: Multiwheel vehicle and multilayer track model parameters.

	Parameters (units)	Value
Vehicle	Carbody mass (t)	40
	Carbody moment of inertia ($\text{t}\cdot\text{m}^2$)	1500
	Bogie mass (t)	3
	Bogie moment of inertia ($\text{t}\cdot\text{m}^2$)	8
	Wheelset mass (t)	2
	Single side stiffness of primary suspension (kN/m)	800
	Single side damping of primary suspension ($\text{kN}\cdot\text{s/m}$)	20
	Single side stiffness of secondary suspension (kN/m)	200
	Single side damping of secondary suspension ($\text{kN}\cdot\text{s/m}$)	30
	Longitudinal distance of primary suspension (m)	2.5
	Longitudinal distance of secondary suspension (m)	17.5
Track	Spacing between adjacent fasteners (m)	0.65
	Stiffness of fastener (MN/m)	60
	Damping of fastener ($\text{kN}\cdot\text{s/m}$)	47.7
	Length of the track slab (m)	6.45
	Width of the track slab (m)	2.55
	Height of the track slab (m)	0.20
	Density of the track slab (kg/m^3)	2500
	Modulus of elasticity of the track slab (MPa)	3.9×10^4
	Stiffness of CA mortar (MN/m)	0.9×10^3
	Damping of CA mortar ($\text{kN}\cdot\text{s/m}$)	83
	Length of the base plate (m)	6.45
	Width of the base plate (m)	2.95
	Height of the base plate (m)	0.3
	Density of the base plate (kg/m^3)	2500
	Modulus of elasticity of the base plate (MPa)	3.0×10^4
	Stiffness of subgrade (MN/m)	60
	Damping of subgrade ($\text{kN}\cdot\text{s/m}$)	90

5.2.1. *Responses of the Multilayer Track with Contact Separation.* The random rail irregularity of Sample 1 is used in this subsection, and the responses of the track with perfectly smooth rail (without irregularity) are used for comparison. The running speed of the vehicle is 300 km/h.

As shown in Figure 8, the calculated responses of the multilayer track with Beijing–Shanghai measured track irregularity and without irregularity have significant differences during the time history. When oscillations appear in

the displacements of the rail, track slab, and base plate in the case with irregularity, the corresponding results are still smooth in the case without irregularity, as shown in Figures 8(a)–8(c). The contact force with irregularity continuously oscillates in the range of 0–250 kN and becomes zero from 0.49 s to 0.50 s, as indicated in Figure 8(d). It shows that, in the case with irregularity, the large amplitude of wheel-rail force variation on the substructure leads to stronger dynamic interaction compared with the case without irregularity.

FIGURE 8: Responses of the track with the Beijing-Shanghai track irregularity: (a) rail displacement; (b) track slab displacement; (c) base plate displacement; (d) wheel-rail force at the 1st wheelset.

FIGURE 9: Continued.

FIGURE 9: Number of contact separations in the presence of random track irregularities: (a) 1st wheelset; (b) 2nd wheelset; (c) 3rd wheelset; (d) 4th wheelset.

5.2.2. Influence of Random Track Irregularities on Contact Separation. To obtain more information about contact separation of a multiwheel and multilayer track system with irregularity, both the track irregularities of Sample 1 and Sample 2 are considered. The numbers of separations induced by random track irregularities are shown for all the wheelsets in Figure 9. Wheel-rail contact separation is defined when the wheel-rail force is zero, and it counts only once from wheel-rail separation to recontact. As seen, the front wheelsets have more contact separations than the rear wheelsets of the same bogie during the simulation history. More importantly, the influence of velocity is very important to the number of contact separations. Greater numbers of separations, with increased impacts on the wheel and rail, will be found with increased running velocity.

6. Conclusions

In this paper, the two-step integration method with the linear contact relation is used to solve the response of a vehicle-track coupled system, and the results and the application conditions are discussed. The following conclusions are drawn.

- (i) Compared with responses from the Newmark family methods, the responses calculated with the two-step method, NML, and APL are in good accordance, but the calculation time required for the two-step method is much shorter. The adoption of a larger integration step and the avoidance of iterations significantly reduce the calculation time and increase the efficiency of the two-step method. Moreover, the situation of contact separation can be well simulated.
- (ii) The simulation results indicate that when contact separation occurs owing to random track irregularities, the impact between the wheel and rail will boost the interaction. With an increased vehicle speed, contact separation emerges more frequently.

- (iii) The high efficiency of the two-step method is also brought about by the integration of all the model information into matrix $\bar{\mathbf{M}}$, which is the usual symmetric matrix. Hence, time-varying coefficients are absent on the left side of the substructure equations, and the substructure data exported from commercial finite element software can be directly adopted.

This method has potential for wide applications in dynamic problems of coupled vehicle-track systems.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest with respect to the research, authorship, and publication of this article.

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