Research Article

Dynamic Modeling and Development of Symbolic Calculation Software for N -DOF Flexible-Link Manipulators Incorporating Lumped Mass

Feng Yu ^{1,2} and Xinyuan Chen ^{1,2,3}

¹Key Laboratory of Metallurgical Equipment and Control Technology (Wuhan University of Science and Technology), Ministry of Education, Wuhan, China

²Hubei Key Laboratory of Mechanical Transmission and Manufacturing Engineering, Wuhan University of Science and Technology, Wuhan, China

³Wuhan University of Science and Technology, Peace Avenue, Qingshan District, Wuhan, Hubei 430081, China

Correspondence should be addressed to Xinyuan Chen; chenxinyuan@wust.edu.cn

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Considering the complex dynamic modeling of multi-DOF planar flexible manipulators, a general-purpose method for the rigid-flexible coupling dynamic modeling of N -DOF flexible manipulators is proposed in this paper, and symbolic calculation software is developed. The modeling method is based on the Lagrange equation and assumed mode method (AMM). First, the N -DOF flexible manipulator is divided into two parts, which are assumed to be rigid and flexible. On this basis, the rigid part and the flexible part are coupled, and the calculation process of the model is further simplified. Then, the simplest general symbolic expression of the dynamic model of the N -DOF flexible manipulator is obtained with the induction method. According to the modeling method, “symbolic expression computation software for dynamic equations of N -DOF flexible manipulators” is developed using the symbolic calculation software Mathematica. Finally, the dynamic modeling method and the symbolic calculation software are verified by a trajectory tracking experiment with a PD control applied to a 2-DOF flexible manipulator. Compared with the traditional modeling method, the calculation time can be reduced by more than 90% using the modeling method proposed in this paper, which significantly reduces the complexity of the modeling process.

1. Introduction

Robotic manipulators are widely used in environments that are hazardous to humans and in situations where there are substantial amounts of repetitive work to reduce the burden on humans and to assist in accomplishing special tasks. To ensure sufficient positioning accuracy at the end of a manipulator, the links of the manipulator can generally be stiffened by a bulky structure to prevent the links from vibrating and to avoid elastic deflection during operation [1–4]. However, heavy links limit the speed of the manipulator’s movement and consume excessive amounts of energy. In some cases, thickening the structure of the links to ensure the positioning accuracy of the end of the

manipulator may not be practical. For example, the weight of a manipulator operating on the space station must be as light as possible because of launch costs [5]. In addition, to ensure that the manipulator can perform in a greater workspace, the boom of a concrete pump truck structure should be more slender [6]. Robotic manipulators are also used in nuclear maintenance [7, 8], for spraying large aircraft [9], etc. Under these circumstances, these robotic manipulators have to be designed as flexible-link manipulators (FLMs) with a slender structure.

FLMs have been found in extensive applications because of their advantages in terms of being lightweight and having high load ratios (the ratio of load mass to manipulator mass), high movement speeds, and low energy consumption

[10–12]. Moreover, in human-robot interactions, a lightweight FLM is safer to humans and has smaller workload requirements on joint drivers, sensors [13, 14], etc. However, such an FLM experiences vibration and elastic deformation during movement, which will greatly affect the stability of the entire system and the positioning accuracy of the end [15, 16]. Therefore, it is important to study the effects of the dynamic characteristics of FLMs, in terms of the vibration suppression and elastic deformation compensation of the flexible links. For example, the dynamic model of a single-joint flexible manipulator has been developed to suppress the vibration of the flexible manipulator [3, 13].

Because multijoint FLMs are highly nonlinear, strongly coupled, multi-degree-of-freedom (DOF) systems, the dynamic modeling is complex. In addition, considering the payload mass at the end of the manipulator and the lumped mass (for example, the mass of the actuators) on each link, multi-DOF rigid-flexible coupling dynamic modeling is more complicated. These challenging topics have attracted a large number of excellent researchers, who have conducted a considerable amount of research on the dynamic modeling of FLMs. In Giorgio and Del Vescovo's work [17], the lumped parameter method is used to model the dynamics of flexible links, and although this modeling method is simple, the number of elements must be as large as possible to obtain an adequate model accuracy. The modeling and control of a special class of single-link, an arbitrary number of lumped masses, flexible arms is presented in [18], in which the arm consists of massless flexible structures. Subudhi and Morris adopted the Lagrange equation and the assumed mode method (AMM) to establish the dynamic model of a manipulator with flexible links and joints [19]. A dynamic model of 3-PRR parallel manipulator with flexible links based on AMM was proposed in [20]. Wei et al. provided a global mode method for dynamic modeling of a flexible arm flexible joint manipulator with a tip mass [21]. Some other modeling methods, such as the finite element method (FEM), are also commonly used. For example, the dynamic modeling of the flexible links using FEM is presented in [22]. The FEM is used to derive the dynamic equations for finding the maximum allowable dynamic load of flexible-link mobile manipulators [23]. In addition, the Newton–Euler method is also used to dynamically model the flexible arms [24]. Nonlinear dynamics of motion are developed for flexible manipulator links consisting of rotary joints that connect pairs of flexible links [25]. Bascetta et al. developed a closed-form dynamic model of flexible manipulators based on the Newton–Euler equations [26]. However, the dynamic modeling in the above studies only considers flexible arms with a certain number of DOFs or flexible links without payload mass, and the expressions of the dynamic model are still very computationally expensive in practice.

Considering the complexity of the dynamic modeling of multi-DOF FLMs with a lumped mass on the links, a general-purpose dynamic modeling method for N -DOF FLMs based on the Lagrange equation is presented in this paper, and the symbolic expression computation software for the dynamic equations of N -DOF flexible manipulators is developed. In a more general situation, where the lumped

mass is located not only on the end of the links but also at other positions of the links (for example, the flexible arms composed of functionally graded materials (FGM) in the fields of spacecraft, nuclear industries, robotics, etc., the flexibility of the arms can be increased or changed, or changing the vibration frequency of the flexible arm structure by adding concentrated mass to the intermediate position of flexible arms, see [27]) is also considered in this paper. The modeling method is as follows. First, the flexible arms are divided into imaginary rigid and flexible parts and modeled separately. Then, the rigid and flexible parts are coupled, and the dynamic model is further simplified. Finally, the simplest general expression of the dynamic model of the flexible arm with N -DOFs is obtained by the induction method, and “symbolic expression computation software for dynamic equations of N -DOF flexible manipulators” is developed using the symbolic calculation software Mathematica. The test results indicate that compared with the traditional modeling method, the proposed modeling method can reduce the computation time by more than 90% and the software can quickly calculate and generate the dynamic model of the flexible manipulator, thereby greatly reducing the difficulty of the dynamic modeling of the flexible manipulator. Finally, the position tracking control experiment of a 2-DOF flexible manipulator is taken as an example and the dynamic model is generated automatically by the program. Meanwhile, a physical model is also constructed. In an experiment on the end effector of the manipulator tracking a trajectory of a circle, the numerical simulation results of the dynamic model are compared with the experimental results. The dynamic model and the dynamic modeling method are both verified to be correct.

The remainder of this paper is organized as follows. In Section 2, the motion of N -DOF FLM systems is described. Then, in Section 3, the dynamic equations of N -DOF FLMs are derived and established. “The symbolic expression computation software for dynamic equations of N -DOF flexible manipulators” is developed in Section 4, in which the dynamic equations of two-DOF FLMs are automatically generated by the program, and the experiments are conducted. Finally, conclusions are drawn in Section 5.

2. Motion of Flexible Links

To study the general situation of dynamic modeling for serial multijoint FLMs and to simplify the derivation process, fixed-base and small-deformation flexible manipulators moving in the vertical plane are the focus of this paper. Before we derive and establish the dynamic model of the multi-DOF FLM, the appropriate float frame for each flexible link should be assigned, as shown in Figure 1.

The fixed base of the manipulator is typically treated as the global inertial reference frame and denoted $\{0\}$, and the i -th flexible link is associated with the float frame $\{i\}$. The x -axis of the float frame $\{i\}$ passes through the start and end points of the i -th flexible link, and its origin coincides with the end point of the i -th flexible link. Assuming that the length of the i -th flexible link without deformation is L_i , the angle between the i -th and $i - 1$ -th links along the x -axis is θ_i .

FIGURE 1: Deformation of link i represented in the float frame $\{i\}$ and inertial reference frame $\{0\}$.

Thus, according to reference [28], the homogeneous transformation matrix from frame $\{i\}$ to the coordinate system $\{i-1\}$ is derived as

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & L_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i & L_i \sin \theta_i \\ 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

In equation (1), the superscript $i-1$ of the homogeneous transformation matrix $\mathbf{A}_i^{i-1}$ represents the reference frame $\{i-1\}$ and the subscript i represents the current frame $\{i\}$. Thus, the position vector of an arbitrary point P_i on link i can be expressed in the global inertial frame $\{0\}$ as

$$\mathbf{r}_i^0 = \mathbf{A}_i^0 \mathbf{r}_i^i, \quad (2)$$

where $\mathbf{A}_i^0 = \mathbf{A}_1^0 \mathbf{A}_2^1 \cdots \mathbf{A}_i^{i-1}$, $\mathbf{r}_i^i$ is the position vector of point P_i on link i in float frame $\{i\}$, and $\mathbf{r}_i^i = [x_i \ y_i \ 1]^T$.

According to the aforementioned assumption, the flexible link here is considered to be an Euler beam with a constant cross section. For a small deformation of the link, because the deflection of the flexible links in the length direction is very small, we can ignore it and only consider the transverse deformation. Because the exact mathematical description and calculation of the flexible link deformation are complex given that the flexible link deformation is infinite dimensional, a finite-dimensional deformation is generally introduced to approximate the link deformation. The assumed mode method (AMM) is adopted in this paper. In the AMM, the elastic displacements y_i are generally described by a truncated modal series function and time-dependent mode amplitudes (for example, see [29]) as

$$y_i(x_i, t) = \sum_{k=1}^s \varphi_{ik}(x_i) f_{ik}(t), \quad (3)$$

where φ_{ik} is a k -th-order mode shape function, f_{ik} are generalized coordinates with respect to the φ_{ik} of link i , and s is the number of modes used to describe the deflection of link i .

Generally, this method has sufficient approximation accuracy for small deformations of flexible links when we

truncate y_i at $s = 2$. Thus, in the following, we only consider the first second-order mode shape function and its generalized coordinates. For the Euler beam with clamped boundary conditions at both ends of the link, we choose the mode shape function $\varphi_{ik} = \sin(k\pi x_i/L_i)$, $k = 1, 2$. Thus, equation (3) can be rewritten as

$$y_i = \sin\left(\frac{\pi x_i}{L_i}\right) f_{i1} + \sin\left(\frac{2\pi x_i}{L_i}\right) f_{i2}. \quad (4)$$

The velocity vector of P_i in frame $\{0\}$ can be expressed as

$$\begin{aligned} \dot{\mathbf{r}}_i^0 &= \frac{d}{dt}(\mathbf{r}_i^0) = \frac{d}{dt}(\mathbf{A}_i^0 \mathbf{r}_i^i) \\ &= \left(\dot{\mathbf{A}}_1^0 \mathbf{A}_2^1 \cdots \mathbf{A}_i^{i-1}\right) \mathbf{r}_i^i + \left(\mathbf{A}_1^0 \dot{\mathbf{A}}_2^1 \cdots \mathbf{A}_i^{i-1}\right) \mathbf{r}_i^i + \cdots \\ &\quad + \left(\mathbf{A}_1^0 \cdots \dot{\mathbf{A}}_i^{i-1}\right) \mathbf{r}_i^i + \mathbf{A}_i^0 \dot{\mathbf{r}}_i^i \\ &= \left(\sum_{j=1}^i \frac{\partial \mathbf{A}_i^0}{\partial \theta_j} \dot{\theta}_j\right) \mathbf{r}_i^i + \mathbf{A}_i^0 \dot{\mathbf{r}}_i^i. \end{aligned} \quad (5)$$

In general, the elements in matrix $\mathbf{A}_i^0$ are a function of $(\theta_1 \ \theta_2 \ \cdots \ \theta_i)$ and the elements in vector $\mathbf{r}_i^i$ are a function of $(f_{i1} \ f_{i2})$.

Note that the partial derivative of $\mathbf{A}_i^0$ with respect to θ_i can be calculated as

$$\frac{\partial \mathbf{A}_i^{i-1}}{\partial \theta_i} = \mathbf{Q}_i \mathbf{A}_i^{i-1}, \quad (6)$$

where, for a revolute joint, the matrix $\mathbf{Q}_i$ is defined as

$$\mathbf{Q}_i = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (7)$$

The advantage of using $\mathbf{Q}_i$ is that we can transform the differential operation of $\partial \mathbf{A}_i^{i-1} / \partial \theta_i$ into a constant matrix $\mathbf{Q}_i$ multiplying the $\mathbf{A}_i^{i-1}$ matrix, which can still be used; then, we can obtain

$$\frac{\partial \mathbf{A}_i^0}{\partial \theta_j} = \begin{cases} \mathbf{A}_1^0 \mathbf{A}_2^1 \cdots \mathbf{A}_{j-1}^{j-2} \mathbf{Q}_j \mathbf{A}_j^{j-1} \cdots \mathbf{A}_i^{i-1}, & j \leq i, \\ 0, & j > i. \end{cases} \quad (8)$$

To simplify the derivation of dynamic equations of multi-DOF FLMS, we can divide the flexible link system into two parts: an imaginary rigid part and an imaginary flexible part. The expressions include the variables θ_i , representing the rigid part, and the variables f_{i1} or f_{i2} , representing the flexible part. Now, we imagine that the x -axis of frame $\{i\}$ from the start point to the end point of flexible link i is a virtual rigid link i , which can be regarded as a rigid link of the flexible link i without deformation; thus, equation (8) can be interpreted as the effect of the motion of joint j on all the points on virtual rigid link i . To simplify the notations, we define

$$\mathbf{U}_{ij} = \frac{\partial \mathbf{A}_i^0}{\partial \theta_j} = \begin{cases} \mathbf{A}_{j-1}^0 \mathbf{Q}_j \mathbf{A}_i^{j-1}, & j \leq i, \\ 0, & j > i. \end{cases} \quad (9)$$

Thus, equation (5) can be rewritten as

$$\dot{\mathbf{r}}_i^0 = \left(\sum_{j=1}^i \mathbf{U}_{ij} \dot{\theta}_j \right) \mathbf{r}_i^i + \mathbf{A}_i^0 \dot{\mathbf{r}}_i^i. \quad (10)$$

For multi-DOF manipulators, the dynamic equations are a strongly coupled system; thus, considering the interaction effects between virtual rigid links, we can obtain

$$\mathbf{U}_{ijk} = \frac{\partial \mathbf{U}_{ij}}{\partial \theta_k} = \begin{cases} \mathbf{A}_{j-1}^0 \mathbf{Q}_j \mathbf{A}_{k-1}^{j-1} \mathbf{Q}_k \mathbf{A}_i^{k-1}, & j \leq k \leq i, \\ \mathbf{A}_{k-1}^0 \mathbf{Q}_k \mathbf{A}_{j-1}^{k-1} \mathbf{Q}_j \mathbf{A}_i^{j-1}, & k \leq j \leq i, \\ 0, & i < j \cup i < k. \end{cases} \quad (11)$$

Equation (11) can be interpreted as the interaction effects of the motions of joint j and joint k on all points on virtual link i .

3. Derivation of Dynamic Equations for Flexible Links with Lumped Mass

3.1. Calculation of Kinetic Energy. After obtaining the velocity of points on link i , we can calculate the kinetic energy of link i . Assuming that the mass of flexible link i is m_i , let T_i^m be the kinetic energy of link i and let dT_i^m be the kinetic energy of a particle with differential mass dm on link i ; then, we can obtain that

$$\begin{aligned} dT_i^m &= \frac{1}{2} \text{Tr} \left[\dot{\mathbf{r}}_i^0 (\dot{\mathbf{r}}_i^0)^T \right] dm \\ &= \frac{1}{2} \text{Tr} \left\{ \left[\left(\sum_{j=1}^i \mathbf{U}_{ij} \dot{\theta}_j \right) \mathbf{r}_i^i + \mathbf{A}_i^0 \dot{\mathbf{r}}_i^i \right] \right. \\ &\quad \times \left. \left[\left(\sum_{j=1}^i \mathbf{U}_{ij} \dot{\theta}_j \right) \mathbf{r}_i^i + \mathbf{A}_i^0 \dot{\mathbf{r}}_i^i \right]^T \right\} dm \\ &= \frac{1}{2} \sum_{p=1}^i \sum_{r=1}^i \text{Tr} \left\{ \mathbf{U}_{ip} \left[\mathbf{r}_i^i (\mathbf{r}_i^i)^T dm \right] \mathbf{U}_{ir}^T \right\} \dot{\theta}_p \dot{\theta}_r \\ &\quad + \sum_{j=1}^i \text{Tr} \left\{ \mathbf{U}_{ij} \left[\mathbf{r}_i^i (\mathbf{r}_i^i)^T dm \right] (\mathbf{A}_i^0)^T \right\} \dot{\theta}_j \\ &\quad + \frac{1}{2} \text{Tr} \left\{ \mathbf{A}_i^0 \left[\dot{\mathbf{r}}_i^i (\dot{\mathbf{r}}_i^i)^T dm \right] (\mathbf{A}_i^0)^T \right\}, \end{aligned} \quad (12)$$

where Tr is a trace operator and $\text{Tr}(\mathbf{X}) \triangleq \sum_{i=1}^n x_{ii}$. Now, let us compute the kinetic energy of lumped mass T_i^M on flexible

link i . We assume that the lumped mass can locate at any position on flexible link i . Similarly, we use the same method as equations (10) and (12) and obtain

$$\begin{aligned} T_i^M &= \frac{1}{2} \text{Tr} \left[\dot{\mathbf{R}}_i^0 (\dot{\mathbf{R}}_i^0)^T \right] M_i \\ &= \frac{1}{2} \sum_{p=1}^i \sum_{r=1}^i \text{Tr} \left\{ \mathbf{U}_{ip} \left[\mathbf{R}_i^i (\mathbf{R}_i^i)^T M_i \right] \mathbf{U}_{ir}^T \right\} \dot{\theta}_p \dot{\theta}_r \\ &\quad + \sum_{j=1}^i \text{Tr} \left\{ \mathbf{U}_{ij} \left[\mathbf{R}_i^i (\mathbf{R}_i^i)^T M_i \right] (\mathbf{A}_i^0)^T \right\} \dot{\theta}_j \\ &\quad + \frac{1}{2} \text{Tr} \left\{ \mathbf{A}_i^0 \left[\dot{\mathbf{R}}_i^i (\dot{\mathbf{R}}_i^i)^T M_i \right] (\mathbf{A}_i^0)^T \right\}, \end{aligned} \quad (13)$$

where M_i is the lumped mass and $\mathbf{R}_i^0$ and $\dot{\mathbf{R}}_i^0$ are the position vector and velocity vector of the lumped mass M_i in the global inertial reference frame $\{0\}$, respectively. $\mathbf{R}_i^i$ and $\dot{\mathbf{R}}_i^i$ are the position vector and velocity vector of lumped mass M_i in float frame $\{i\}$, respectively. Thus, we let $\mathbf{R}_i^i = [X_i \ Y_i \ 1]^T$, as shown in Figure 2.

Because we ignore the deformation of the flexible link in the longitudinal direction, the length of flexible link i is always L_i . To simplify the derivation, we define $X_i = \mu_i L_i$, $\mu_i \in [-1, 0]$. When $\mu_i = 0$, lumped mass M_i locates on the end of flexible link i if $\mu_i = -(1/2)$ represents M_i at the middle of flexible link i . Thus, we can obtain $\mathbf{R}_i^i = [\mu_i L_i \ Y_i \ 1]^T$, and according to equation (4), we can obtain

$$Y_i = \sin(\pi \mu_i) f_{i1} + \sin(2\pi \mu_i) f_{i2} = \sum_{k=1}^2 f_{ik} S_{ki}. \quad (14)$$

To simplify the notation, we define $S_{ki} = \sin(k\pi \mu_i)$, $k = 1, 2$. The total kinetic energy of flexible link i and lumped mass M_i can be computed as

$$\begin{aligned} T_i &= \int_{-L_i}^0 dT_i^m + T_i^M \\ &= \frac{1}{2} \sum_{p=1}^i \sum_{r=1}^i \text{Tr} \left[\mathbf{U}_{ip} \mathbf{J}_{ui} \mathbf{U}_{ir}^T \right] \dot{\theta}_p \dot{\theta}_r \\ &\quad + \sum_{j=1}^i \text{Tr} \left[\mathbf{U}_{ij} \mathbf{J}_{vi} (\mathbf{A}_i^0)^T \right] \dot{\theta}_j + \frac{1}{2} \text{Tr} \left[\mathbf{A}_i^0 \mathbf{J}_{wi} (\mathbf{A}_i^0)^T \right], \end{aligned} \quad (15)$$

where $\mathbf{J}_{ui} = \mathbf{J}_{ui}^m + \mathbf{J}_{ui}^M$, $\mathbf{J}_{vi} = \mathbf{J}_{vi}^m + \mathbf{J}_{vi}^M$, and $\mathbf{J}_{wi} = \mathbf{J}_{wi}^m + \mathbf{J}_{wi}^M$. Additionally, $\mathbf{J}_{ui}^m$ is the pseudoinertia matrix of flexible link i , which, for a homogeneous Euler beam, can be calculated as

FIGURE 2: Position of lumped mass M_i with different value of μ_i .

$$\begin{aligned}
 \mathbf{J}_{ui}^m &= \int_{-L_i}^0 \mathbf{r}_i^i(\mathbf{r}_i^i)^T dm = \int_{-L_i}^0 \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \begin{bmatrix} x_i & y_i & 1 \end{bmatrix} dm \\
 &= \int_{-L_i}^0 \begin{bmatrix} x_i^2 & x_i y_i & x_i \\ x_i y_i & y_i^2 & y_i \\ x_i & y_i & 1 \end{bmatrix} dm \\
 &= \begin{bmatrix} \frac{L_i^2}{3} & \frac{L_i(2f_{i1} - f_{i2})}{2\pi} & -\left(\frac{L_i}{2}\right) \\ \frac{L_i(2f_{i1} - f_{i2})}{2\pi} & \frac{1}{2}(f_{i1}^2 + f_{i2}^2) & -\left(\frac{2f_{i1}}{\pi}\right) \\ -\left(\frac{L_i}{2}\right) & -\left(\frac{2f_{i1}}{\pi}\right) & 1 \end{bmatrix} m_i.
 \end{aligned} \tag{16}$$

As shown, the pseudoinertia matrix $\mathbf{J}_{ui}^m$ is a function of generalized coordinates $\mathbf{f}$. The variation in the generalized coordinates $\mathbf{f}$ can reflect the deformation of the flexible link. Similarly, $\mathbf{J}_{ui}^M$ is calculated as

$$\begin{aligned}
 \mathbf{J}_{ui}^M &= \mathbf{R}_i^i(\mathbf{R}_i^i)^T M_i \\
 &= \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix} \begin{bmatrix} X_i & Y_i & 1 \end{bmatrix} M_i = \begin{bmatrix} X_i^2 & X_i Y_i & X_i \\ X_i Y_i & (Y_i)^2 & Y_i \\ X_i & Y_i & 1 \end{bmatrix} M_i \\
 &= \begin{bmatrix} L_i^2 \mu_i^2 & L_i \mu_i \sum_{k=1}^2 f_{ik} S_{ki} & L_i \mu_i \\ L_i \mu_i \sum_{k=1}^2 f_{ik} S_{ki} & \left(\sum_{k=1}^2 f_{ik} S_{ki} \right)^2 & \sum_{k=1}^2 f_{ik} S_{ki} \\ L_i \mu_i & \sum_{k=1}^2 f_{ik} S_{ki} & 1 \end{bmatrix} M_i.
 \end{aligned} \tag{17}$$

Note that $\dot{x}_i = 0$, and we can easily obtain $\mathbf{J}_{vi}^m$ and $\mathbf{J}_{wi}^m$:

$$\mathbf{J}_{vi}^m = \int_{-L_i}^0 \mathbf{r}_i^i(\mathbf{r}_i^i)^T dm = \begin{bmatrix} 0 & \frac{L_i(2\dot{f}_{i1} - \dot{f}_{i2})}{2\pi} & 0 \\ 0 & \frac{1}{2} \left(\sum_{k=1}^2 f_{ik} \dot{f}_{ik} \right) & 0 \\ 0 & -\frac{2\dot{f}_{i1}}{\pi} & 0 \end{bmatrix} m_i, \tag{18}$$

$$\mathbf{J}_{wi}^m = \int_{-L_i}^0 \dot{\mathbf{r}}_i^i(\dot{\mathbf{r}}_i^i)^T dm = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(\dot{f}_{i1}^2 + \dot{f}_{i2}^2) & 0 \\ 0 & 0 & 0 \end{bmatrix} m_i.$$

$\mathbf{J}_{vi}^M$ and $\mathbf{J}_{wi}^M$ are computed in the same manner:

$$\begin{aligned}
 \mathbf{J}_{vi}^M &= \mathbf{R}_i^i(\mathbf{R}_i^i)^T M_i = \begin{bmatrix} 0 & L_i \mu_i \sum_{k=1}^2 f_{ik} S_{ki} & 0 \\ 0 & \left(\sum_{k=1}^2 f_{ik} S_{ki} \right) \left(\sum_{k=1}^2 \dot{f}_{ik} S_{ki} \right) & 0 \\ 0 & \sum_{k=1}^2 \dot{f}_{ik} S_{ki} & 0 \end{bmatrix} M_i, \\
 \mathbf{J}_{wi}^M &= \dot{\mathbf{R}}_i^i(\dot{\mathbf{R}}_i^i)^T M_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(\sum_{k=1}^2 \dot{f}_{ik} S_{ki} \right)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} M_i.
 \end{aligned} \tag{19}$$

3.2. Calculation of Potential Energy. Let the gravitational potential energy of flexible link i with lumped mass M_i be $\bar{P}_i$; thus,

$$\bar{P}_i = -(m_i \mathbf{g} \bar{\mathbf{r}}_i^0 + M_i \mathbf{g} \mathbf{R}_i^0) = -(m_i \mathbf{g} \mathbf{A}_i^0 \bar{\mathbf{r}}_i^i + M_i \mathbf{g} \mathbf{A}_i^0 \mathbf{R}_i^i), \tag{20}$$

where $\mathbf{g} = (g_x \ g_y \ 0) = (0 \ -|g| \ 0)$ is a gravity vector expressed in the inertial reference frame $\{0\}$ and g is the gravitational constant. Because of the small deformation of the flexible link, the variation in the gravitational potential energy of the flexible link caused by the flexible link's deformation can be neglected and is small compared with the total gravitational potential energy of the flexible link. Let $\bar{\mathbf{r}}_i^0$ and $\bar{\mathbf{r}}_i^i$ be the mass center vector of flexible link i without deformation in inertial frame $\{0\}$ and float frame $\{i\}$, respectively.

Meanwhile, the elastic potential energy of flexible link i caused by deformation can be expressed as

$$K_i = \frac{E_i I_i}{2} \int_{-L_i}^0 \left(\frac{\partial^2 y_i}{\partial x_i^2} \right)^2 dx_i = \frac{\pi^4 E_i I_i (f_{i1}^2 + 16f_{i2}^2)}{4L_i^3}, \quad (21)$$

where E_i and I_i are Young's modulus and the second moment of the area of flexible link i , respectively. Now, from equations (12) to (21), we define the Lagrangian function of FLMs with N -DOFs as

$$\begin{aligned} L &= \sum_{i=1}^n (T_i - K_i - \bar{P}_i) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{p=1}^i \sum_{r=1}^i \text{Tr}[\mathbf{U}_{ip} \mathbf{J}_{ui} \mathbf{U}_{ir}^T] \dot{\theta}_p \dot{\theta}_r \\ &\quad + \sum_{i=1}^n \sum_{j=1}^i \text{Tr}[\mathbf{U}_{ij} \mathbf{J}_{vi} (\mathbf{A}_i^0)^T] \dot{\theta}_j + \frac{1}{2} \sum_{i=1}^n \text{Tr}[\mathbf{A}_i^0 \mathbf{J}_{wi} (\mathbf{A}_i^0)^T] \\ &\quad + \sum_{i=1}^n (m_i \mathbf{g} \mathbf{A}_i^0 \bar{\mathbf{r}}_i + M_i \mathbf{g} \mathbf{A}_i^0 \mathbf{R}_i) - \frac{1}{2} \sum_{i=1}^n \frac{\pi^4 E_i I_i (f_{i1}^2 + 16f_{i2}^2)}{2L_i^3}. \end{aligned} \quad (22)$$

We now apply the Lagrange–Euler formulation to the Lagrangian function of the N -DOF FLM equation (22) to obtain the necessary generalized torque to drive the manipulators:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\tau}, \quad (23)$$

where $\mathbf{q}$ and $\dot{\mathbf{q}}$ are a vector of generalized coordinates and its derivative with respect to time, respectively. We define generalized coordinates $\mathbf{q} = (\boldsymbol{\theta} \ \mathbf{f})^T$, $\boldsymbol{\theta} = (\theta_1 \ \dots \ \theta_n)$, and $\mathbf{f} = (f_{11} \ f_{12} \ \dots \ f_{n1} \ f_{n2})$, and $\boldsymbol{\tau}$ is a $3n \times 1$ generalized torque vector.

Before we apply the Lagrange–Euler formulation to the Lagrangian function of equation (22), we should note that the transformation matrix $\mathbf{A}_i^0$ and the matrix $\mathbf{U}_{ij}$ are only functions of generalized coordinates θ_i , $i = 1, 2, \dots, n$; thus, the derivation of $\mathbf{A}_i^0$ or $\mathbf{U}_{ij}$ with respect to $f_{i\lambda}$, $i = 1, 2, \dots, n$, $\lambda = 1, 2$ will be zero. The matrices $\mathbf{J}_{ui}$, $\mathbf{J}_{vi}$, and $\mathbf{J}_{wi}$ are only functions of the generalized coordinates $f_{i\lambda}$, $\lambda = 1, 2$ and generalized velocity $\dot{f}_{i\lambda}$, $\lambda = 1, 2$; thus, the derivatives of $\mathbf{J}_{ui}$, $\mathbf{J}_{vi}$, and $\mathbf{J}_{wi}$ with respect to θ_i , $i = 1, 2, \dots, n$ will be zero.

First, we only consider the generalized coordinate $\boldsymbol{\theta}$; thus, take $(\theta_1 \ \dots \ \theta_n)^T$ as $\mathbf{q}$ and substitute equation (22)

into equation (23) to obtain the rigid generalized torque vector $\mathbf{F}_\theta$:

$$\begin{aligned} F_{\theta i} &= \sum_{j=i}^n \sum_{k=1}^2 \text{Tr}[\mathbf{U}_{jk} \mathbf{J}_{uj} \mathbf{U}_{ji}^T] \ddot{\theta}_k \\ &\quad + \sum_{j=i}^n \sum_{\lambda=1}^2 \text{Tr}[\mathbf{U}_{ji} \mathbf{J}_{v\lambda} (\mathbf{A}_j^0)^T] \ddot{f}_{j\lambda} \\ &\quad + \sum_{j=i}^n \sum_{k=1}^j \sum_{m=1}^j \text{Tr}[\mathbf{U}_{jkm} \mathbf{J}_{uj} \mathbf{U}_{ji}^T] \dot{\theta}_k \dot{\theta}_m \\ &\quad + \sum_{j=i}^n \sum_{k=1}^j \sum_{\lambda=1}^2 \text{Tr}[\mathbf{U}_{jk} \mathbf{J}_{uj} \mathbf{U}_{ji}^T] \dot{\theta}_k \dot{f}_{j\lambda} - \sum_{j=i}^n m_j \mathbf{g} \mathbf{U}_{ji} \bar{\mathbf{r}}_j^j \\ &\quad - \sum_{j=i}^n M_j \mathbf{g} \mathbf{U}_{ji} \mathbf{R}_j^j, \end{aligned} \quad (24)$$

where $i = 1, 2, \dots, n$. Then, we derive the flexible generalized torque vector $\mathbf{F}_f$ by substituting equation (22) into equation (23) when considering $(f_{11} \ f_{12} \ \dots \ f_{n1} \ f_{n2})^T$ as $\mathbf{q}$

$$\begin{aligned} F_{fi\lambda} &= \sum_{j=1}^i \text{Tr}[\mathbf{U}_{ij} \mathbf{J}_{vi\lambda} (\mathbf{A}_i^0)^T] \ddot{\theta}_j + \frac{1}{2} m_i \ddot{f}_{i\lambda} \\ &\quad + \sum_{j=1}^i \sum_{k=1}^i \text{Tr}[\mathbf{U}_{ijk} \mathbf{J}_{vi\lambda} (\mathbf{A}_i^0)^T + \mathbf{U}_{ij} (\mathbf{J}_{vi\lambda} - \frac{1}{2} \mathbf{J}_{ui\lambda}) \mathbf{U}_{ik}^T] \dot{\theta}_j \dot{\theta}_k \\ &\quad + \frac{\pi^4 E_i I_i [(2-\lambda)f_{i\lambda} + 16(\lambda-1)f_{i\lambda}]}{2L_i^3} - M_i \mathbf{g} \mathbf{A}_i^0 \mathbf{R}_{i\lambda}. \end{aligned} \quad (25)$$

For $i = 1, 2, \dots, n$, $\lambda = 1, 2$, where $\mathbf{R}_{i\lambda}$ is a partial derivative of $\mathbf{R}_i^j$ with respect to the generalized coordinate $f_{i\lambda}$ and $\mathbf{R}_{i\lambda} = [0 \ \sin(\lambda\pi\mu_i) \ 0]^T$.

Equations (24) and (25) are the dynamic equations of FLMs with N -DOFs. Because the entire process of deriving the dynamic equations from equations (23) to (25) is very complex, we divide the equations into two parts: a rigid part with respect to the generalized coordinate $\boldsymbol{\theta}$ and a flexible part with respect to the generalized coordinate $\mathbf{f}$. Therefore, the complexity is greatly reduced. Note that in equations (24) and (25), we should calculate the derivative of the matrices $\mathbf{J}_{ui}$, $\mathbf{J}_{vi}$, and $\mathbf{J}_{wi}$ with respect to time t because

$$\frac{d(\mathbf{J}_{xj})}{dt} = \frac{\partial(\mathbf{J}_{xj})}{\partial f_{j\lambda}} \dot{f}_{j\lambda}, \quad (26)$$

where $x = u, v, w$ and $j = 1, 2, \dots, n$ and $\lambda = 1, 2$ because the $\partial(\mathbf{J}_{wj})/\partial f_{j\lambda}$ is canceled during the calculation, and thus, we define

$$\begin{aligned} \mathbf{J}_{uj\lambda} &= \frac{\partial(\mathbf{J}_{uj})}{\partial f_{j\lambda}} = \frac{\partial \mathbf{J}_{uj}^m}{\partial f_{j\lambda}} + \frac{\partial \mathbf{J}_{uj}^M}{\partial f_{j\lambda}}, \\ \mathbf{J}_{vj\lambda} &= \frac{\partial(\mathbf{J}_{vj})}{\partial f_{j\lambda}} = \frac{\partial \mathbf{J}_{vj}^m}{\partial f_{j\lambda}} + \frac{\partial \mathbf{J}_{vj}^M}{\partial f_{j\lambda}}. \end{aligned} \quad (27)$$

It is easy to calculate $\mathbf{J}_{uj\lambda}$ and $\mathbf{J}_{vj\lambda}$ according to the expressions of $\mathbf{J}_{uj}$ and $\mathbf{J}_{vj}$ for $j = 1, 2, \dots, n$, $\lambda = 1, 2$:

$$\begin{aligned} \mathbf{J}_{uj\lambda} &= (2-\lambda) \begin{bmatrix} 0 & \frac{L_j}{\pi} & 0 \\ \frac{L_j}{\pi} & f_{j\lambda} & -\frac{2}{\pi} \\ 0 & -\frac{2}{\pi} & 0 \end{bmatrix} \mathbf{m}_j + (2-\lambda) \begin{bmatrix} 0 & L_j \mu_j & 0 \\ L_j \mu_j & 2 \left(\sum_{k=1}^2 f_{jk} S_{kj} \right) & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{S}_{1j} \\ &+ (\lambda-1) \begin{bmatrix} 0 & -\frac{L_j}{2\pi} & 0 \\ -\frac{L_j}{2\pi} & f_{j\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{m}_j \\ &+ (\lambda-1) \begin{bmatrix} 0 & L_j \mu_j & 0 \\ L_j \mu_j & 2 \left(\sum_{k=1}^2 f_{jk} S_{kj} \right) & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{S}_{2j}, \\ \mathbf{J}_{vj\lambda} &= (2-\lambda) \begin{bmatrix} 0 & \frac{L_j}{\pi} & 0 \\ 0 & \frac{f_{j\lambda}}{2} & 0 \\ 0 & -\frac{2}{\pi} & 0 \end{bmatrix} \mathbf{m}_j + (2-\lambda) \begin{bmatrix} 0 & L_j \mu_j & 0 \\ 0 & \left(\sum_{k=1}^2 f_{jk} S_{kj} \right) & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{S}_{1j} \\ &+ (\lambda-1) \begin{bmatrix} 0 & -\frac{L_j}{2\pi} & 0 \\ 0 & \frac{f_{j\lambda}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{m}_j + (\lambda-1) \begin{bmatrix} 0 & L_j \mu_j & 0 \\ 0 & \left(\sum_{k=1}^2 f_{jk} S_{kj} \right) & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{S}_{2j}. \end{aligned} \quad (28)$$

From equations (24) and (25), we find that there are not only rigid motion terms with $\dot{\theta}_i$ but also flexible motion terms with $\dot{f}_{i\lambda}$, in addition to the coupling terms between rigid motion $\dot{\theta}_i \dot{\theta}_j$ and rigid motion and flexible link deformation $\dot{\theta}_i \dot{f}_{i\lambda}$. Notably, there are no flexible deformation coupling terms, which should have been $\dot{f}_{i\lambda} \dot{f}_{j\lambda}$. This makes solving the dynamic equations easier, and this is an important reason for establishing such float frames to describe the FLMs. In essence, the two equations also reflect the vibration of the FLMs during movement.

3.3. Relationship between the Generalized Forces and Actual Driving Forces. We continue to discuss the relationship

between the generalized forces corresponding to the generalized coordinates and the actual forces applied to the manipulator's joints. As shown in Figure 3, $\tau_{i,i-1}$ is the actual force applied to joint i , where the first subscript i denotes the link that the force acts on and the second subscript $i-1$ denotes the link that the force reacts on. According to the principle of virtual work, the virtual work acting on link i is

$$\delta W_i = \tau_{i,i-1} (\delta \theta_i + \delta \alpha_i) + \tau_{i+1,i} \delta \beta_i, \quad (29)$$

where $\alpha_i = (\partial y_i / \partial x_i)|_{x_i=0}$ and $\beta_i = (\partial y_i / \partial x_i)|_{x_i=L_i}$; thus, the total virtual work on all joints is

$$\begin{aligned} \delta W &= \sum_{i=1}^n \tau_{i,i-1} \delta \theta_i + \sum_{\lambda=1}^2 [\tau_{n,n-1} \phi'_{i\lambda}(0)] \delta f_{n\lambda} \\ &+ \sum_{i=1}^{n-1} \sum_{\lambda=1}^2 [\tau_{i,i-1} \phi'_{i\lambda}(0) + \tau_{i+1,i} \phi'_{i\lambda}(L_i)] \delta f_{i\lambda}, \end{aligned} \quad (30)$$

where $\phi'_{i\lambda}(0) = (d\phi_{i\lambda}/dx_i)|_{x_i=0}$ and $\phi'_{i\lambda}(L_i) = (d\phi_{i\lambda}/dx_i)|_{x_i=L_i}$, and the generalized forces corresponding to the generalized coordinates θ_i are

$$F_{\theta i} = \tau_{i,i-1}, \quad i = 1, 2, \dots, n. \quad (31)$$

The generalized forces corresponding to the generalized coordinates $f_{i\lambda}$ are

$$\begin{aligned} F_{f_{i\lambda}} &= \tau_{i,i-1} \phi'_{i\lambda}(0) + \tau_{i+1,i} \phi'_{i\lambda}(L_i), \\ F_{f_{n\lambda}} &= \tau_{n,n-1} \phi'_{n\lambda}(0), \end{aligned} \quad (32)$$

where $i = 1, 2, \dots, n-1$, $\lambda = 1, 2$.

3.4. Matrix Form of Dynamic Equations for FLM. To make the dynamic equations easy to understand, we rewrite the equations from equations (24) to (31) in matrix form as

$$\begin{bmatrix} \mathbf{M}_{\theta\theta} & \mathbf{M}_{\theta f} \\ \mathbf{M}_{f\theta} & \mathbf{M}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\theta}} \\ \ddot{\mathbf{f}} \end{bmatrix} + \begin{bmatrix} \mathbf{H}_{\theta\theta} & \mathbf{H}_{\theta f} \\ \mathbf{H}_f & \mathbf{0}_{2n \times 2n} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \dot{\mathbf{f}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_r \\ \mathbf{Kf} + \mathbf{C}_f \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\theta \\ \mathbf{F}_f \end{bmatrix}, \quad (33)$$

where $\mathbf{M}_{\theta\theta} \in \mathcal{R}^{n \times n}$, $\mathbf{M}_{\theta f} \in \mathcal{R}^{n \times 2n}$, $\mathbf{M}_{f\theta} \in \mathcal{R}^{2n \times n}$, and $\mathbf{M}_{ff} \in \mathcal{R}^{2n \times 2n}$ are mass matrices and $\mathbf{H}_{\theta\theta} \in \mathcal{R}^{n \times n}$, $\mathbf{H}_{\theta f} \in \mathcal{R}^{n \times 2n}$, and $\mathbf{H}_f \in \mathcal{R}^{2n \times n}$ are Coriolis and centrifugal matrices. $\mathbf{C}_r \in \mathcal{R}^{n \times 1}$ and $\mathbf{C}_f \in \mathcal{R}^{2n \times 1}$ are gravity vectors. $\mathbf{K} \in \mathcal{R}^{2n \times 2n}$ is a diagonal stiffness matrix. The detailed expressions are as follows:

$$\mathbf{M}_{\theta\theta}(i, k) = \sum_{j=\max(i,k)}^n \text{Tr}[\mathbf{U}_{jk} \mathbf{J}_{uj} \mathbf{U}_{ji}^T], \quad (34)$$

where $i = 1, 2, \dots, n$ and $k = 1, 2, \dots, n$ and

$$\mathbf{M}_{\theta f}(i, j, \lambda) = \text{Tr}[\mathbf{U}_{ji} \mathbf{J}_{vj\lambda} (\mathbf{A}_j^0)^T], \quad (35)$$

where $i = j = 1, 2, \dots, n$, $\lambda = 1, 2$. Because we only take the first two mode shape functions and generalized coordinates, we can regard every two-column vector as one group in the matrix $\mathbf{M}_{\theta f} \in \mathcal{R}^{n \times 2n}$, and “ λ ” means the λ -th column vector in the j -th group. Hence, $\mathbf{M}_{\theta f}(i, j, \lambda)$ represents the i -th row, j -th group, and λ -th element in the matrix $\mathbf{M}_{\theta f}$:

FIGURE 3: Torques applied at link i .

$$\mathbf{M}_{f\theta}(i, j, \lambda) = \text{Tr}[\mathbf{U}_{ij} \mathbf{J}_{vi\lambda} (\mathbf{A}_i^0)^T], \quad (36)$$

where $i = j = 1, 2, \dots, n$, $\lambda = 1, 2$. From the expression, we find that the matrix $\mathbf{M}_{f\theta}$ is the transpose of $\mathbf{M}_{\theta f}$:

$$\mathbf{M}_{ff}(i, i) = \frac{1}{2} m_j, \quad (37)$$

where $i = 1, 2, \dots, 2n$, $j = (2i + 1 + (-1)^{i+1})/4$, and $\mathbf{m}_{ff} \in \mathbb{R}^{2n \times 2n}$ is a diagonal matrix.

$$\mathbf{H}_{\theta\theta}(i, k) = \sum_{m=1}^n \mathbf{h}_{\theta\theta}(i, k, m) \dot{\theta}_m, \quad (38)$$

where $i = 1, 2, \dots, n$, $k = 1, 2, \dots, n$, with

$$\mathbf{h}_{\theta\theta}(i, k, m) = \sum_{j=\max(i,k,m)}^n \text{Tr}[\mathbf{U}_{jkm} \mathbf{J}_{uj} \mathbf{U}_{ji}^T], \quad (39)$$

$$\mathbf{H}_{\theta f}(i, j, \lambda) = \sum_{k=1}^j \text{Tr}[\mathbf{U}_{jk} \mathbf{J}_{u\lambda} \mathbf{U}_{ji}^T] \dot{\theta}_k,$$

where $i = j = 1, 2, \dots, n$, $\lambda = 1, 2$. The computation of the matrix $\mathbf{H}_{\theta f}$ is the same as that of $\mathbf{M}_{\theta f}$:

$$\mathbf{H}_f(i, j, \lambda) = \sum_{k=1}^i \text{Tr}[\mathbf{U}_{ijk} \mathbf{J}_{vi\lambda} (\mathbf{A}_i^0)^T + \mathbf{U}_{ij} (\mathbf{J}_{vi\lambda} - \frac{1}{2} \mathbf{J}_{ui\lambda}) \mathbf{U}_{ik}^T] \dot{\theta}_k, \quad (40)$$

where $i = j = 1, 2, \dots, n$, $\lambda = 1, 2$. The calculation of the matrix $\mathbf{H}_f$ is the same as that of the matrix $\mathbf{M}_{f\theta}$:

$$\mathbf{C}_r(i) = - \sum_{j=i}^n m_j \mathbf{g} \mathbf{U}_{ji} \bar{\mathbf{r}}_j^i - \sum_{j=i}^n M_j \mathbf{g} \mathbf{U}_{ji} \mathbf{R}_j^j, \quad (41)$$

where $i = 1, 2, \dots, n$.

$$\mathbf{K}(i, i) = \text{diag} \left\{ \frac{\pi^4 E_i I_i}{2L_i^3}, \frac{16\pi^4 E_i I_i}{2L_i^3} \right\}, \quad (42)$$

where $i = 1, 2, \dots, n$. $\mathbf{K} \in \mathbb{R}^{2n \times 2n}$ is a diagonal matrix, and the element $\mathbf{K}(i, i)$ is also a 2×2 diagonal matrix.

$$\mathbf{C}_f(i, \lambda) = -M_i \mathbf{g} \mathbf{A}_i^0 \mathbf{R}_{i\lambda}, \quad (43)$$

where $i = 1, 2, \dots, n$, $\lambda = 1, 2$. Here, we also regard every two-row vector as one group, similar to equation (35). Thus, $\mathbf{C}_f(i, \lambda)$ represents the i -th group, λ -th element of the matrix $\mathbf{C}_f$.

We now rewrite equation (33) in a more compact and general matrix form as

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{N}(\mathbf{q}) = \mathbf{F}, \quad (44)$$

where $\mathbf{M} \in \mathbb{R}^{3n \times 3n}$ is the mass matrix, the matrix $\mathbf{H} \in \mathbb{R}^{3n \times 3n}$ consists of the centrifugal force and Coriolis force, the column vector $\mathbf{N} \in \mathbb{R}^{3n \times 1}$ denotes the gravitational and elastic force, and $\mathbf{F} \in \mathbb{R}^{3n \times 1}$ is a generalized force.

3.5. Symbolic Expression Calculation Procedure. The complete symbolic expression of the rigid-flexible coupling dynamic equation of the N -DOF flexible manipulator with the lumped mass on an arbitrary location of links has been obtained. According to the whole derivation process of this paper and from equations (33) to (44), software that can automatically generate the rigid-flexible coupling dynamic equation of an N -DOF flexible manipulator is developed by using Wolfram Mathematica symbolic calculation software. The operating interface of the software is shown in Figure 4.

In the program interface, only the number of N -DOFs, the arm length, the section inertia moment, the elastic modulus, the mass of the flexible arm, the symbolic parameters of the end mass, and the location of the lumped mass are input. According to the program, the symbolic expression of the dynamic equation of the flexible manipulator with N -DOFs can be obtained conveniently and quickly. Here, we rearrange the symbolic expression computation procedures as follows:

(i) *Step 0.* Initialization of the algorithm

The number of DOFs and the parameters of the FLM, such as $N, m_i, M_i, L_i, E_i, I_i, \theta_i, \mu_i$, are predefined

Computation of

$\mathbf{J}_{ui}, \mathbf{J}_{vi}, \mathbf{J}_{wi}, \mathbf{J}_{ui\lambda}, \mathbf{J}_{ui\lambda}, \mathbf{Q}, \mathbf{g}$ are predefined

$\mathbf{A}_i^{i-1}$ and $\mathbf{A}_i^0$ from (1); $\mathbf{r}_i^0$ from (2)

$\mathbf{U}_{ij}$ from (9); $\mathbf{U}_{ijk}$ from (11)

(ii) *Step 1.* Computation of the global mass matrix

Computation of

$\mathbf{M}_{\theta\theta}$ from (34); $\mathbf{M}_{\theta f}$ from (35)

$\mathbf{M}_{f\theta}$ from (36); $\mathbf{M}_{ff}$ from (37)

(iii) *Step 2.* Computation of the global matrices of the centrifugal and Coriolis forces

Computation of

$\mathbf{H}_{\theta\theta}$ from (38); $\mathbf{H}_{\theta f}$ from (39)

$\mathbf{H}_f$ from (40)

(iv) *Step 3.* Computation of the global vector of the gravity, stiffness matrix, and generalized force

Computation of

$\mathbf{C}_r$ from (41); $\mathbf{C}_f$ from (43)

$\mathbf{K}$ from (42)

$\mathbf{F}_\theta$ from (31); $\mathbf{F}_f$ from (32)

(v) *Step 4.* Postprocessing of the result

Simplicity of the dynamic model: some terms can be ignored for special situations

FIGURE 4: Operating interface of symbolic expressions calculation software.

Output of the symbolic expression for the model.

The flowchart of the symbolic expression computation procedures of the dynamic equation of the flexible manipulator with N-DOFs is shown in Figure 5.

3.6. Comparison with Traditional Modeling Methods. The traditional dynamic modeling methods of flexible manipulators include the Lagrange method, Newton–Euler method, and Kane method, which have their advantages and disadvantages. In this paper, the traditional modeling method is compared with the proposed method, which is also based on the Lagrange equation. The traditional modeling method is as shown in [19].

The traditional modeling method is relatively simple, but it is quite complex in terms of the actual calculation process for solving the dynamic model expression and there is no general formula for flexible manipulators with different DOFs. Therefore, it often takes substantial time to calculate the dynamic model of various types of flexible manipulators in practice.

Here, the program developed in Section 3.5 is tested. Additionally, according to the modeling method in [19], the calculation programs for flexible manipulators with different DOFs developed using Mathematica are also tested. The time required for the two modeling methods to run the computational program (on a PC with a i3-7100 3.90 GHz CPU with 8 GB of RAM) was tested when $N=1, 2, \dots, 5$. The comparison results are shown in Table 1.

As shown in Table 1, the modeling method proposed in this paper can reduce the modeling time by more than 90% compared with the traditional modeling method. This result is mainly because in the traditional modeling process, the Lagrange energy function of the manipulator system requires a large number of integral operations; then, in the derivation of the generalized coordinates, there are many differential operations. Moreover, there are many repeated intermediate calculations in the differential calculations. The method proposed in this paper eliminates the repetitive intermediate calculation process theoretically and directly

derives the simplest expression of the dynamic model equation.

Note that to explain the modeling method proposed in this paper, the simplest planar flexible manipulator is selected as the research object. However, from the entire modeling process shown above, the flexibility characteristics of rigid-flexible coupling manipulators are related only to the matrices $\mathbf{J}_{ui}$, $\mathbf{J}_{vi}$, and $\mathbf{J}_{wi}$ and their derivatives. Thus, for the dynamic modeling of complex spatial flexible manipulators, large-deformation flexible arms, or even irregularly shaped flexible links, we only need to modify the matrices $\mathbf{J}_{ui}$, $\mathbf{J}_{vi}$, and $\mathbf{J}_{wi}$ and their derivatives. The structure of the symbolic expressions remains unchanged; thus, this dynamic modeling approach also has good portability. It can be predicted that to satisfy the increasingly complex working conditions, the structure of the flexible arm will be more complex, and it will be a difficult problem to establish a sufficiently accurate dynamic model. Therefore, the modeling method proposed in this paper will have substantial advantages in terms of modeling convenience and computational efficiency.

4. Experimental Study

To verify the correctness of the modeling method proposed in this paper, the dynamic model of a 2-DOF flexible manipulator is automatically generated using the software. Simultaneously, a flexible manipulator experimental platform is set up, as shown in Figure 6. Figure 7 is a schematic diagram of the assembly structure of the flexible arm device.

The expressions of the dynamic equations for the 2-DOF flexible manipulator are shown in the appendix. The symbols in the formula are the same as those in the derivation process.

In this experiment, the two arms of the flexible manipulator are controlled to follow the given trajectories. The torque τ_{ref} output from the controller and the angles $\mathbf{q}$ of the joints of the flexible manipulator measured by the encoders are recorded. Simultaneously, the angles are transmitted to the dynamics simulation model, and the estimations of the

FIGURE 5: Flowchart of symbolic expressions calculation procedures.

TABLE 1: Required computation time of two modeling methods for different DOF models.

DOF	Proposed method (s)	Traditional method (s)	Time saved (%)
$N = 1$	$t = 0.36$	$t = 41.63$	99.12
$N = 2$	$t = 0.92$	$t = 106.28$	99.17
$N = 3$	$t = 4.21$	$t = 212.57$	98.01
$N = 4$	$t = 21.3$	$t = 365.28$	94.14
$N = 5$	$t = 114.6$	$t = 1316.45$	91.29

control torques τ_{est} are calculated for comparison with the actual situation. The whole experimental scheme and recorded data are shown in Figure 8. Figure 9 is a schematic diagram of the experimental signal acquisition and control.

According to the scheme shown in Figure 8, the Simulink control model is established and compiled into a “so file,” the input and output (IO) interface between the control model and the real manipulator is configured using VeriStand, and finally, all the related control model files are downloaded to the real-time cRIO-9035 controller. The

output torque calculated by the controller is converted into a PWM signal by dividing the motor torque constant and the gear ratio of the reducer. A 40 MHz FPGA chip is mounted onto the back of the control slot to acquire the high-speed pulse signal produced by the encoders through the digital IO interface. The motors, gears, and drivers are manufactured by Times Brilliant in China. The type of brushless DC motor 1 with the power of 150 W and brushless DC motor 2 with the power of 100 W is 57BL95S15-230TF9B and 57BL75S10-230TF9, respectively. The type of driver 1 and driver 2 is ZM-6610M. Reduction ratios of PX series planetary gears matched with motors are 64 and 32, respectively. The other specific parameters of the components in the experimental device are listed in Table 2.

4.1. Case 1: Free Vibration of the End. In the experiment, the initial working positions of the flexible manipulator are selected as $\theta_1 = 45^\circ$ and $\theta_2 = -45^\circ$. At the beginning, the external support makes the manipulator as free as possible from gravity and the second flexible manipulator keeps the

FIGURE 6: Physical model of 2-DOF flexible manipulators.

FIGURE 7: Schematic diagram of the assembly structure of 2-DOF flexible manipulators.

FIGURE 8: Experimental scheme and data recording.

horizontal position. Then, the flexible manipulator system is allowed to vibrate freely under the force of gravity, and the acceleration of the end of the flexible manipulator under free vibration is observed and recorded. In addition, the accelerometer type is WT901C485, the acquisition time and sampling time are 10 s and 20 ms, respectively, and the simulation interval of Simulink is 0.001 s. The measured

FIGURE 9: Schematic diagram of the experimental signal acquisition and control.

acceleration graph is compared with the simulation graph, as shown in Figure 10(a). It is apparent that there are very good correlations between the measured values and simulated values and the amplitude of simulation is smaller, which can be due to model approximation and initial attitude error between the simulation model and the physical model. The spectrum of the end acceleration curve of the simulation model is obtained by fast Fourier transform, as shown in Figure 10(b). It can be seen that the first-order vibration frequency is $\omega_1 = 12.246$ rad/s and the second-order vibration frequency is $\omega_2 = 40.82$ rad/s. From the experiment, it is obvious that the simulation curves are basically consistent with the experimental curves, which verifies the correctness

TABLE 2: Parameters of components in the experimental devices.

Parameters	Values
Length of arms (L_1, L_2 (m))	0.475, 0.435
Mass of arms (m_1, m_2 (kg))	0.374, 0.29
Young's modulus of arms (E_1, E_2 (GPa))	2.06×10^{11}
Second moment of area of arms (I_1 (m ⁴))	16×10^{-11}
Second moment of area of arms (I_2 (m ⁴))	6.75×10^{-11}
Lumped mass of joints (M_1, M_2 (Kg))	2.94, 0.343
Torque values of motors (τ_1, τ_2 (Nm))	0.5, 0.32
Gear reduction ratio (r_1, r_2)	64, 32
Coulomb friction of actuators (k_{c1}, k_{c2} (Nm))	0.0345, 0.0226
Viscous friction of actuators (k_{v1}, k_{v2} (Nms/rad))	0.0037, 0.0023

FIGURE 10: (a) Acceleration of flexible link's tip. (b) The spectrum of the flexible link's tip acceleration.

of the proposed modeling method to some extent. The similar results of vibration characteristics of flexible manipulators can be referred to [30], which can be used to compare with these experimental results.

4.2. Case 2: End Effector Tracking a Trajectory of Circle.

In this experiment, the initial working positions of the flexible manipulator are selected as $\theta_1 = 45^\circ$ and $\theta_2 = -45^\circ$. A circle with a radius of 0.1 m in the XOY plane is selected as the end track of the flexible arm, as shown in Figure 11. The equations of the circle are $x = x_0 + 0.1 \cos(0.2\pi t)$, $y = y_0 + 0.1 \sin(0.2\pi t)$, whereas the center coordinate of the circle is (x_0, y_0) , $x_0 = 0.67\text{m}$, $y_0 = 0.34\text{m}$. The end of the flexible arm is substituted into the inverse equation of rigid kinematics, and the desired joint angles are obtained by real-time operation. In addition, the simple PD control is adopted in the cRIO-9035 controller to make the joints track the desired angles, in which k_p and k_d were debugged through repeated simulation experiments and finally determined to be $k_p = 125$ and $k_d = 25$, and the calculation interval is 0.001 s. Figures 12(a) and 12(b) are the angle

FIGURE 11: Reference trajectory in the workspace.

FIGURE 12: (a) Angle of joint 1. (b) Angle of joint 2.

FIGURE 13: (a) Torque of joint 1. (b) Torque of joint 2.

curves of joint 1 and joint 2. It can be seen from the figures that when the flexible arm starts to move, the experimental curve and the simulation curve have some errors, and then the errors gradually decrease and the curves tracking effect is better. The input torque curves of the joint 1 and the joint 2 are as shown in Figures 13(a) and 13(b), and the vertical acceleration curves of the flexible arm 1 and the flexible arm 2 are as shown in Figures 14(a) and 14(b). It can be seen that the torque simulation curves and the experimental curves follow the same trend, and there are small tracking errors, due to the inevitable vibration of the flexible mechanical arm during the tracking process, the acceleration will cause frequent fluctuations, but the simulation curve and the experimental curve can also achieve a better

tracking effect on the whole. The main cause of some vibration of the manipulator during movement is that, to simplify the control model, the distance between the output shafts of two motors is regarded as the arm length of the flexible arms, the mass of the motor reducers is treated as the lumped mass, and the unavoidable clearance of the motor reducers makes the simulation model and the actual model have certain errors, but these errors are within acceptable ranges.

5. Conclusions

A general modeling method for the dynamic equations of an N -DOF flexible manipulator based on the Lagrange equation

FIGURE 14: (a) Acceleration of tip of link 1. (b) Acceleration of tip of link 2.

and the small deformation assumption of an Euler beam is proposed in this paper. According to this modeling method, symbolic calculation software for the dynamic equations of N -DOF flexible manipulators is developed, greatly reducing the computational complexity. Compared with the traditional modeling method, the modeling time can be reduced by more than 90%. Finally, the correctness of the modeling method is verified by experiments.

The modeling method proposed in this paper takes the planar flexible manipulator as the research object, but according to the modeling process, it can be applied to the more complex rigid-flexible coupling dynamic modeling of flexible manipulators only through modifying the corresponding matrices of the flexible part. This is also future work that needs to be studied. Simultaneously, the modeling method proposed in this paper is also helpful for the dynamic modeling of N -DOF spatial flexible manipulators, large deformation flexible manipulators, etc.

Appendix

The dynamic equations for the 2-DOF flexible manipulator are shown as follows:

$$\mathbf{M}_{\theta\theta} = \begin{bmatrix} 2\bar{A} + \bar{B} + \bar{C} & \bar{A} + \bar{B} \\ \bar{A} + \bar{B} & \bar{B} \end{bmatrix}, \quad (\text{A.1})$$

$$\mathbf{M}_{\theta f} = - \begin{bmatrix} \frac{L_1 m_1}{\pi} & \frac{L_1 m_1}{2\pi} & \frac{m_2 (L_2 + 2L_1 \cos \theta_2)}{\pi} & \frac{L_2 m_2}{2\pi} \\ 0 & 0 & \frac{L_2 m_2}{\pi} & \frac{L_2 m_2}{2\pi} \end{bmatrix}, \quad (\text{A.2})$$

$$\mathbf{M}_{ff} = \text{diag}(m_1, m_1, m_2, m_2), \quad (\text{A.3})$$

$$\mathbf{H}_{\theta\theta} = \bar{D} \begin{bmatrix} -\frac{1}{2\pi} L_1 \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) \\ \frac{1}{2\pi} L_1 \dot{\theta}_1^2 \end{bmatrix}, \quad (\text{A.4})$$

$$\mathbf{H}_{\theta f} = \begin{bmatrix} m_1 \dot{\theta}_1 \left(\sum_{i=1}^2 \dot{f}_{1i} f_{1i} \right) + \bar{E} + \bar{F} (2\dot{\theta}_1 + \dot{\theta}_2) \\ \bar{E} + \bar{F} \dot{\theta}_1 \end{bmatrix}, \quad (\text{A.5})$$

$$\mathbf{H}_f = \begin{bmatrix} -\frac{m_1 \dot{\theta}_1^2 f_{11}}{2} \\ -\frac{m_1 \dot{\theta}_1^2 f_{12}}{2} \\ -\frac{m_2}{2\pi} \left(4L_1 \sin \theta_2 \dot{\theta}_1^2 + \pi (\dot{\theta}_1 + \dot{\theta}_2)^2 f_{21} \right) \\ -\frac{m_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 f_{22}}{2} \end{bmatrix}, \quad (\text{A.6})$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{2} \mathbf{g} (\mathbf{L}_1 (\mathbf{m}_1 + 2\mathbf{m}_2 + 2\mathbf{M}_1 + 2\mathbf{M}_2) \cos \theta_1) + \bar{G} \\ \bar{G} \end{bmatrix}, \quad (\text{A.7})$$

$$\mathbf{K} = \text{diag} \left(\frac{\pi^4 E_1 I_1}{2L_1^3}, \frac{8\pi^4 E_1 I_1}{L_1^3}, \frac{\pi^4 E_2 I_2}{2L_2^3}, \frac{8\pi^4 E_2 I_2}{L_2^3} \right), \quad (\text{A.8})$$

$$\mathbf{F} = \left[\tau_1, \tau_2, \frac{\pi(\tau_1 - \tau_2)}{L_1}, \frac{2\pi(\tau_1 + \tau_2)}{L_1}, \frac{\pi\tau_2}{L_2}, \frac{2\pi\tau_2}{L_2} \right]^T. \quad (\text{A.9})$$

The matrix $\mathbf{M}_{f\theta}$ is omitted in the above expressions because $\mathbf{M}_{f\theta} = \mathbf{M}_{\theta f}^T$, which yields

$$\begin{aligned}
\bar{A} &= \frac{(m_2 + 2M_2)L_1L_2\pi \cos \theta_2}{2} + \frac{2L_1m_2 \sin \theta_2 f_{21}}{\pi}, \\
\bar{B} &= \frac{(m_2 + 3M_2)L_2^2}{3} + \frac{m_2(f_{21}^2 + f_{22}^2)}{2}, \\
\bar{C} &= \left(\frac{m_1}{3} + m_2 + M_1 + M_2\right)L_1^2 + \frac{1}{2}m_1(f_{11}^2 + f_{12}^2), \\
\bar{D} &= (m_2 + 2M_2)L_2\pi \sin \theta_2 - 4m_2 \cos \theta_2 f_{21}, \\
\bar{E} &= m_2(\dot{\theta}_1 + \dot{\theta}_2) \left(\sum_{i=1}^2 \dot{f}_{2i} f_{2i} \right), \\
\bar{F} &= \frac{2m_2L_1\dot{f}_{21}\dot{\theta}_1 \sin \theta_2}{\pi}, \\
\bar{G} &= \frac{1}{2}gL_2(m_2 + 2M_2)\cos(\theta_1 + \theta_2).
\end{aligned} \tag{A.10}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] M. Sayahkarajy, Z. Mohamed, and A. A. Mohd Faudzi, "Review of modelling and control of flexible-link manipulators," *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, vol. 230, no. 8, pp. 861–873, 2016.
- [2] J. Lou, Y. Wei, G. Li, Y. Yang, and F. Xie, "Optimal trajectory planning and linear velocity feedback control of a flexible piezoelectric manipulator for vibration suppression," *Shock and Vibration*, vol. 2015, Article ID 952708, 11 pages, 2015.
- [3] S. Mahto, "Shape optimization of revolute-jointed single link flexible manipulator for vibration suppression," *Mechanism and Machine Theory*, vol. 75, pp. 150–160, 2014.
- [4] S.-M. Kim, H. Kim, and K. Boo, "Use of a simple mechanical analogy to analytically tune the PD controller of a flexible manipulator system," *Shock and Vibration*, vol. 2018, Article ID 4073963, 15 pages, 2018.
- [5] D. Meng, Y. She, W. Xu, W. Lu, and B. Liang, "Dynamic modeling and vibration characteristics analysis of flexible-link and flexible-joint space manipulator," *Multibody System Dynamics*, vol. 43, no. 4, pp. 321–347, 2018.
- [6] H. Tang and W. Ren, "Research on rigid-flexible coupling dynamic characteristics of boom system in concrete pump truck," *Advances in Mechanical Engineering*, vol. 7, no. 3, article 1687814015573854, 2015.
- [7] H. N. Rahimi and M. Nazemizadeh, "Dynamic analysis and intelligent control techniques for flexible manipulators: a review," *Advanced Robotics*, vol. 28, no. 2, pp. 63–76, 2014.
- [8] S. K. Dwivedy and P. Eberhard, "Dynamic analysis of flexible manipulators, a literature review," *Mechanism and Machine Theory*, vol. 41, no. 7, pp. 749–777, 2006.
- [9] M. Tahmasebi, R. A. Rahman, M. Mailah, and M. Gohari, "Roll movement control of a spray boom structure using active force control with artificial neural network strategy," *Journal of Low Frequency Noise, Vibration and Active Control*, vol. 32, no. 3, pp. 189–201, 2013.
- [10] M. Benosman and G. Le Vey, "Control of flexible manipulators: a survey," *Robotica*, vol. 22, no. 5, pp. 533–545, 2004.
- [11] E. Lu, W. Li, X. Yang, M. Fan, and Y. Liu, "Modelling and composite control of single flexible manipulators with piezoelectric actuators," *Shock and Vibration*, vol. 2016, Article ID 2689178, 14 pages, 2016.
- [12] X. Zhang and Y.-Q. Yu, "A new spatial rotor beam element for modeling spatial manipulators with joint and link flexibility," *Mechanism and Machine Theory*, vol. 35, no. 3, pp. 403–421, 2000.
- [13] V. Feliu, E. Pereira, and I. M. Díaz, "Passivity-based control of single-link flexible manipulators using a linear strain feedback," *Mechanism and Machine Theory*, vol. 71, pp. 191–208, 2014.
- [14] J. Malzahn, A. S. Phung, F. Hoffmann, and T. Bertram, "Vibration control of a multi-flexible-link robot arm under gravity," in *Proceedings of the IEEE International Conference on Robotics and Biomimetics (ROBIO 2011)*, pp. 1249–1254, Karon Beach, Thailand, December 2011.
- [15] H. Karagülle, L. Malgaca, M. Dirilmiş, M. Akdağ, and Ş. Yavuz, "Vibration control of a two-link flexible manipulator," *Journal of Vibration and Control*, vol. 23, no. 12, pp. 2023–2034, 2017.
- [16] D. Liang, Y. Song, T. Sun, and X. Jin, "Rigid-flexible coupling dynamic modeling and investigation of a redundantly actuated parallel manipulator with multiple actuation modes," *Journal of Sound and Vibration*, vol. 403, pp. 129–151, 2017.
- [17] I. Giorgio and D. Del Vescovo, "Non-linear lumped-parameter modeling of planar multi-link manipulators with highly flexible arms," *Robotics*, vol. 7, no. 4, p. 60, 2018.
- [18] V. Feliu, K. S. Rattan, and H. B. Brown, "Modeling and control of single-link flexible arms with lumped masses," *Journal of Dynamic Systems, Measurement, and Control*, vol. 114, no. 1, pp. 59–69, 1992.
- [19] B. Subudhi and A. S. Morris, "Dynamic modelling, simulation and control of a manipulator with flexible links and joints," *Robotics and Autonomous Systems*, vol. 41, no. 4, pp. 257–270, 2002.
- [20] X. Zhang, J. K. Mills, and W. L. Cleghorn, "Dynamic modeling and experimental validation of a 3-PRR parallel manipulator with flexible intermediate links," *Journal of Intelligent and Robotic Systems*, vol. 50, no. 4, pp. 323–340, 2007.
- [21] J. Wei, D. Cao, L. Liu, and W. Huang, "Global mode method for dynamic modeling of a flexible-link flexible-joint manipulator with tip mass," *Applied Mathematical Modelling*, vol. 48, pp. 787–805, 2017.
- [22] X. Wang and J. K. Mills, "FEM dynamic model for active vibration control of flexible linkages and its application to a

- planar parallel manipulator,” *Applied Acoustics*, vol. 66, no. 10, pp. 1151–1161, 2005.
- [23] M. Korayem, A. Heidari, and A. Nikoobin, “Maximum allowable dynamic load of flexible mobile manipulators using finite element approach,” *International Journal of Advanced Manufacturing Technology*, vol. 36, no. 5-6, pp. 606–617, 2008.
 - [24] S. Briot and W. Khalil, “Recursive and symbolic calculation of the elastodynamic model of flexible parallel robots,” *International Journal of Robotics Research*, vol. 33, no. 3, pp. 469–483, 2014.
 - [25] W. J. Book, “Recursive Lagrangian dynamics of flexible manipulator arms,” *International Journal of Robotics Research*, vol. 3, no. 3, pp. 87–101, 1984.
 - [26] L. Bascetta, G. Ferretti, and B. Scaglioni, “Closed form Newton-Euler dynamic model of flexible manipulators,” *Robotica*, vol. 35, no. 5, pp. 1006–1030, 2017.
 - [27] L. Li, W.-H. Liao, D. Zhang, and Y. Zhang, “Vibration control and analysis of a rotating flexible FGM beam with a lumped mass in temperature field,” *Composite Structures*, vol. 208, pp. 244–260, 2019.
 - [28] X. Zhang, R. Sørensen, M. R. Iversen, and H. Li, “Computationally efficient dynamic modeling of robot manipulators with multiple flexible-links using acceleration-based discrete time transfer matrix method,” *Robotics and Computer-Integrated Manufacturing*, vol. 49, pp. 181–193, 2018.
 - [29] M. H. Korayem, A. Nikoobin, and V. Azimirad, “Trajectory optimization of flexible link manipulators in point-to-point motion,” *Robotica*, vol. 27, no. 6, pp. 825–840, 2009.
 - [30] J. A. Somolinos, V. Feliu, and L. Sánchez, “Design, dynamic modelling and experimental validation of a new three-degree-of-freedom flexible arm,” *Mechatronics*, vol. 12, no. 7, pp. 919–948, 2002.