Explicit Solutions to Single Scattering of SH Waves with a Radially Gradient Interphase Layer

Jiading Bao,1 Jun Zhang,2,3 and Longhai Zeng2

1School of Mechanical and Electrical Engineering, Guilin University of Electronic Technology, Guilin 541004, China
2College of Aerospace Engineering, Chongqing University, Chongqing 400044, China
3Chongqing Key Laboratory of Heterogeneous Material Mechanics, Chongqing University, Chongqing 400044, China

Correspondence should be addressed to Jun Zhang; mejzhang@cqu.edu.cn

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In this work, analytical solutions to the single scattering of horizontally polarized shear waves (SH) by cylindrical fibers with two specific radially gradient interphase layers are presented. In the first case, the shear modulus \( \mu(r) = e^{2\beta r} \) and the square of wave number \( k^2 \) is a linear function of \( 1/r \); in the second case \( \mu(r) = e^{-\beta r^2} \) and \( k^2 \) is a linear function of \( r^2 \). As an example, we solve the single scattering of SH waves by a SiC fiber with the two interphase layers in an aluminum matrix. The calculated scattering cross sections are compared to values obtained by an approximate method (dividing the continuous varying layer into multiple homogeneous sublayers). The results indicate the current approach gives excellent performance.

1. Introduction

The multiple scattering of waves occurs throughout our daily lives, for example, the scattering of sound by water drops in fog and the scattering of light by fine particles in the air. Study on the multiple scattering of waves has a lengthy history by virtue of its ubiquity [1]. To date, multiple scattering of elastic waves in composites is still an active topic since more and more new composites appear. Even if the composite’s matrix and inclusions are both elastic, elastic waves in such a composite are still attenuated due to the multiple scattering effects; this is called attenuation caused by scattering. There have been many theoretical models [2] proposed to evaluate this attenuation because its accurate evaluation facilitates the dynamic characterization and nondestructive assessment [3] of composite materials.

The extant research can be roughly split into two groups. The first group consists of methods based on the wave function expansion that employ rational averaging techniques, which was pioneered by Foldy [4–8]. In these methods, the exciting waves and scattered waves of each inclusion in the composite were expressed as a series of wave functions. The expansion coefficients of the scattered waves by each inclusion are related through a matrix \( T \) to those of the exciting waves. The components of the matrix \( T \) are determined by the boundary conditions of the corresponding single scattering problem [9]. Note that the matrix \( T \) becomes a scalar for SH waves since no mode conversion happens. After a sequence of mathematical operations, a homogeneous system for the ensemble-averaged expansion coefficients of the exciting waves or scattered waves can be obtained. Solving this homogeneous system yields the attenuation coefficient. The second group consists of studies of the self-consistent method [10–13], in which each inclusion is assumed to behave as an isolated inclusion in a medium with the effective properties of the composite. In addition, the wave acting on each inclusion is a coherent wave. The attenuation coefficient was then obtained through various consistency conditions; for example, the mean wave field equals the coherent wave field [13] or the total scattering by the inclusions embedded in the effective medium effectually vanishes [12].

Both types of theoretical models necessitate solving a corresponding single scattering problem [9], that is, the scattering of a plane incident wave by a single inclusion embedded in an infinite matrix. In the theoretical models
proposed by Waterman and Truell [4] and Norris and Conoir [14], for example, the single scattering problem was solved to obtain the far-field scattering amplitude, $f(\theta)$, which was further used to evaluate the scattering cross section $\sigma$. In most of the existing theoretical models, the fibers/particulates were usually assumed to be perfectly bonded to the matrix where the interphase layer between the fibers and matrix was neglected. The interphase layer should be accounted for, however, because it features changes in gradient properties between the fibers and matrix that are generated via chemical reaction and atom diffusion during the manufacturing process (or created artificially to improve compatibility between the fibers and the matrix).

Until now the effect of the functional gradient interphase layer on the attenuation of elastic waves in composite materials has been meagerly covered in the literatures [15–19]. In order to solve the corresponding single scattering problem in these studies, the inhomogeneous interphase layer was usually divided into multilayers, each with a homogenous property to approximate a continuously varying layer. The wave field in the fibers, intermediate layers, and matrix was then still expressed as a series of wave functions. The continuity conditions of displacements and stresses at all interfaces were listed, and the resulting coupled linear equations for the expansion coefficients were solved. When the intermediate layers are sufficiently thin, this yielded exact solutions [16]. This treatment is straightforward and can be used to deal with any type of radial gradient profiles, but the coefficient matrix of the resulting linear system is sometimes ill-conditioned. Additionally, for composites with a high contrast of properties between the fibers and matrix, the coefficient matrix of the resulting linear system is sometimes ill-conditioned. Additionally, for composites with a high contrast of properties between the fibers and matrix, the interphase layer must be divided into a large number of sublayers to obtain convergent results. This increases the computational cost as well as the condition number of the coefficient matrix [20].

The transfer matrix method has also been applied to the gradient interphase problem [19]. In this method the displacements and stresses at the inner surface are related through a transfer matrix $M$ to the corresponding values at the outer interface for each intermediate sublayer, the components of which are functions of the material properties and geometries, uniquely defined by the sublayer. By using the transfer matrix for each intermediate sublayer, the displacements and stresses on the outer surface of the interphase layer are related to the inner surface. Assuming the fiber, interphase layer, and matrix are perfectly bonded, a linear system for the expansion coefficients of waves in the fiber and matrix is established and solved. For SH waves, the transfer matrix $M$ is of size $2 \times 2$. Therefore, the final linear system to be solved is of size 2, which is much smaller than that in the method described above. Similarly, the final coefficient matrix of the linear system can be ill-conditioned if the sublayers are thin.

Though several approximate methods to solve the single scattering problem with a radially gradient interphase layer have been proposed, analytical solutions still remain elusive. In this work, analytical solutions to the single scattering with a radially gradient interphase layer of several specific profiles were provided for SH waves. The derivation process followed the work of Martin [21], in which general solutions to the single scattering of acoustic waves by an inhomogeneous sphere with spherically symmetrical properties were investigated. The two transformations used in our derivation process differed from Martin’s in that the governing equations for distinct waves are different. We demonstrated that the proposed method of how to make the transformations is also applicable to other waves. Additionally, the detailed expressions of solutions were presented and a specific example was calculated.

The remainder of this paper was organized as follows: Section 2 derived the governing equations for SH waves in a radially gradient medium. The general solutions for two specific radially gradient materials were then derived in Section 3. Section 4 presented analytical solutions for the single scattering problem of the SH wave by a cylindrical fiber with the two specific interphase layers and a detailed example. Section 5 provided a brief summary and conclusion.

2. Description of the Problem and Governing Equations

In this work, as shown in Figure 1, the single scattering of SH waves by a fiber with a radially gradient interphase layer was considered [17]. Here $a$ denotes the radius of the fiber and $b$ the outer radius of the interphase layer. The incident wave is a plane SH wave propagating in the positive $x$ direction with a unit magnitude. Throughout the work, $a = 71 \mu m$ and $b = 1.1a$.

For SH waves, only the displacement in the out-of-plane has value and can be expressed as $w = w(x,t)$. The corresponding stress components can be expressed by

$$
\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0, \\
\sigma_{13} = \mu(x) \frac{\partial w}{\partial x}, \\
\sigma_{23} = \mu(x) \frac{\partial w}{\partial y},
$$

where $\mu(x)$ is the shear modulus, which is a function of position. Substitution of the stress components to the governing equations of elastodynamics yields

$$
\frac{\partial}{\partial x} \left( \mu(x) \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu(x) \frac{\partial w}{\partial y} \right) = \rho(x) \frac{\partial^2 w}{\partial t^2},
$$

where $\rho(x)$ is the mass density. After a simple mathematical operation, equation (2) can be rewritten as

$$
\frac{1}{\mu(x)} \nabla \mu(x) \cdot \nabla w + \nabla^2 w = \frac{1}{c^2(x)} \frac{\partial^2 w}{\partial t^2},
$$

where $\nabla$ and $\nabla^2$ are the 2D gradient and Laplace operator, respectively. $c(x) = \sqrt{\mu(x)/\rho(x)}$ is the wave speed, which is also a function of position.

Define $u(x,t) = \mu'(x)u(x,t)$ [21], where $\lambda$ is a constant which forces $u(x,t)$ into the form given by
\[ \nabla^2 u + Ku = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}. \]  

(4)

For SH waves, \( \lambda \) can be set to \(-1/2\) and then \( K = 1/4 \mu^2 |\nabla \mu|^2 - 1/2 \mu \nabla |\nabla \mu|^2 \). Note that value of \( \lambda \) should be different for other kinds of waves. In this work, only the radial inhomogeneity has been considered; that is, \( \mu(x) \) and \( \rho(x) \) are functions only of the radial coordinate \( r \), so \( K \) can be simplified to form:

\[ K = \frac{1}{2} \left( \frac{\partial \mu}{\partial r} \right)^2 - \frac{1}{2 \mu} \frac{\partial \mu}{\partial r} + \frac{1}{4} \left( \frac{\partial \mu}{\partial r} \right)^2. \]  

(5)

For time-harmonic problems, \( u(x,t) = \tilde{u}(x)e^{i \omega t} \), hence \( \tilde{u}(x) \) satisfies

\[ \nabla^2 \tilde{u} + \left[ k^2(r) + K \right] \tilde{u} = 0, \]  

(6)

where \( k(r) = \omega/c(r) \). Next, we seek the solution to equation (6) in the form of the following equation:

\[ \tilde{u}(x) = u_n(r) Y_n(\theta), \]  

(7)

where \( n \) is an integer and \( Y_n(\theta) \) is a cylindrical harmonic function, such as the function \( e^{i n \theta} \).

We thus have

\[ \nabla^2 (u_n Y_n) = u_n \nabla^2 (Y_n) + 2 (\nabla u_n) \cdot (\nabla Y_n) + Y_n \nabla^2 (u_n), \]  

(8)

since \( u_n \) is a function of \( r \) and \( Y_n \) is a function of variable \( \theta \), so \( (\nabla u_n) \cdot (\nabla Y_n) = 0 \). We also know that in the polar coordinate system \( r \partial Y_n(\theta) \) is a separate solution to the Laplace equation as given by

\[ \nabla^2 Y_n = -\frac{n^2}{r^2} Y_n, \quad n = (-\infty, +\infty). \]  

(9)

By substituting equations (7) and (9) into equation (6), it is clear that \( u_n(r) \) should satisfy the following equation:

\[ \frac{d^2 u_n}{dr^2} + \frac{1}{r} \frac{du_n}{dr} + \left[ k^2(r) + K(r) - \frac{n^2}{r^2} \right] u_n = 0, \]  

(10)

which is a second-order differential equation for \( u_n(r) \).

### 3. Two Specific Radially Gradient Interphase Layers

Two kinds of radially gradient interphase layers were considered here.

**Case 1.** In this case the shear modulus is an exponential function of \( r \) and \( k^2(r) \) is a linear function of \( r^{-1} \) as given by

\[ \mu(r) = \mu_1 e^{\beta r}, \]  

(11)

\[ k^2(r) = k_1^2 + 2 \alpha/ r, \]  

where \( \mu_1, \beta, k_1^2 \) and \( \alpha \) are constants that are specified according to the following equation:

\[ \begin{cases} 
\mu(r)|_{r=a} = \mu_f, \\
\mu(r)|_{r=b} = \mu_m, \\
k^2(r)|_{r=a} = k_1^2, \\
k^2(r)|_{r=b} = k_m^2,
\end{cases} \]  

(12)

where \( \mu_f, \mu_m \) and \( k_1, k_m \) are the shear modulus and wave number of the fiber and the matrix, respectively. After a mathematical operation, the expression for \( K(r) \) in this case can be expressed by

\[ K(r) = -\beta^2 \frac{\beta}{r}. \]  

(13)

Then equation (10) becomes

\[ \frac{d^2 u_n}{dr^2} + \frac{1}{r} \frac{du_n}{dr} + \left[ k_1^2 - \beta^2 + \frac{2\alpha - \beta}{r} - \frac{n^2}{r^2} \right] u_n = 0. \]  

(14)

To solve this equation, we make a substitution according to the following equation:

\[ u_n(r) = r^d w_n(x) \text{ with } x = \delta r^d, \]  

(15)

where \( c \) and \( d \) are two constants. Here, \( c \) was set to \(-1/2\) and \( d = 1 \). The general process to specify these two constants is provided in Appendix. \( \delta \) is a parameter that can be selected at the operator’s disposal. Equation (14) can be transformed into

\[ \frac{d^2 w_n}{dx^2} + \left[ k_1^2 - \beta^2 + 2\frac{\alpha - \beta}{x} - \frac{n^2 - 1/4}{x^2} \right] w_n(x) = 0. \]  

(16)

According to the relative values of \( k_1^2 - \beta^2 \), solutions to equation (16) can be classified into three types.

**Case 1(a).** \( k_1^2 - \beta^2 < 0 \)

Here, we choose \( k_1^2 - \beta^2/\delta^2 = -1/4 \) and set \( \kappa = (2\alpha - \beta)/\delta \) so that equation (16) becomes
\[
\frac{d^2 w_n}{dx^2} + \left[ \frac{1}{4} + \kappa x^{-1} + \left( \frac{1}{4} - n^2 \right) x^{-2} \right] w_n(x) = 0,
\]

which is known as Whittaker’s equation [21]. The general solution is given by

\[
w_n(x) = A_n M_{\kappa,n}(x) + B_n W_{\kappa,n}(x),
\]

where \( M_{\kappa,n}(x) \) and \( W_{\kappa,n}(x) \) are Whittaker functions.

**Case 1(b).** \((k_1^2 - \beta^2 = 0)\)

We choose \(2\alpha - \beta = 4\delta\) and set \(v = n - 1/2\) so that equation (16) becomes

\[
\frac{d^2 w_n}{dx^2} + \left[ \frac{1}{4} - \nu (v + 1)x^{-2} \right] w_n(x) = 0,
\]

which is related to Bessel’s equation. The general solution of equation (19) is given by

\[
w_n(x) = \sqrt{x} \left[ A_n J_{2n}(\sqrt{x}) + B_n Y_{2n}(\sqrt{x}) \right],
\]

where \( J_n() \) and \( Y_n() \) are Bessel functions of the first and second type, respectively.

**Case 1(c).** \((k_1^2 - \beta^2 > 0)\)

We choose \(k_1^2 - \beta^2/\delta^2 = 1\) and set \(\eta = -2\alpha - \beta/2\delta\) and \(v = n - 1/2\) so that equation (16) becomes

\[
\frac{d^2 w_n}{dx^2} + \left[ 1 - 2\eta x^{-1} - \nu (v + 1)x^{-2} \right] w_n(x) = 0,
\]

which is the Coulomb wave equation. Its general solution is given by

\[
w_n(x) = A_n F_{n-1/2}(\eta, x) + B_n G_{n-1/2}(\eta, x),
\]

where \( F_n, G_n \) is the regular Coulomb wave function and \( G_n \) is the irregular Coulomb wave function.

**Case 2.** In this case the shear modulus is a Gaussian function of \(r\), and \(k^2(r)\) is a linear function of \(r^2\) as given by

\[
\begin{align*}
\mu(r) &= \mu_1 e^{-r^2}, \\
k^2(r) &= k_1^2 + \gamma r^2,
\end{align*}
\]

where \(\mu_1, \beta, k_1^2, \gamma\) are constants that were determined in the same manner as Case 1. The expression for \(K(r)\) is given by

\[
K(r) = 2\beta - \beta^2 r^2.
\]

So equation (10) becomes

\[
\frac{d^2 w_n}{dr^2} + \frac{1}{r} \frac{dw_n}{dr} + \left[ k_1^2 + 2\beta + \left( \gamma - \beta^2 \right) r^2 - \frac{n^2}{r^2} \right] w_n = 0.
\]

We made a similar substitution as was done in Case 1 to solve this problem to get the following equation:

\[
u_n(r) = r^{-1} w_n(x) \text{ with } x = \delta r^2.
\]

Similarly, \(\delta\) is a parameter that can be selected at will. Equation (25) was transformed into

\[
\frac{d^2 w_n}{dx^2} + \left[ \frac{\gamma - \beta^2}{4\delta^2} + \frac{k_1^2 + 2\beta}{4\delta} - \frac{n^2}{4\delta} \right] x \right] w_n(x) = 0.
\]

As before, according to the relative values of \((\gamma - \beta^2)\), solutions to equation (27) can be also classified into three types.

**Case 2(a).** \((\gamma - \beta^2 < 0)\)

We choose \(\gamma - \beta^2/\delta^2 = -1\) and set \(k = k_1^2 + 2\beta/4\delta\) so that equation (27) becomes

\[
\frac{d^2 w_n}{dx^2} + \left[ \frac{1}{4} + \kappa x^{-1} + \left( \frac{1}{4} - n^2 \right) x^{-2} \right] w_n(x) = 0,
\]

with a general solution given by

\[
w_n(x) = A_n M_{\kappa,n/2}(x) + B_n W_{\kappa,n/2}(x).
\]

**Case 2(b).** \((\gamma - \beta^2 = 0)\)

We choose \(k_1^2 + 2\beta = \delta\) and set \(v = n - 1/2\) so that equation (27) becomes equation (19), for which the general solution is

\[
w_n(x) = \sqrt{x} \left[ A_n J_n(\sqrt{x}) + B_n Y_n(\sqrt{x}) \right].
\]

**Case 2(c).** \((\gamma - \beta^2 > 0)\)

We choose \(\gamma - \beta^2/4\delta^2 = 1\) and set \(\eta = -k_1^2 + 2\beta/8\delta\) and \(v = n - 1/2\) so that equation (27) becomes equation (21). The general solution of equation (27) is thus the following equation:

\[
w_n(x) = A_n F_{n-1/2}(\eta, x) + B_n G_{n-1/2}(\eta, x).
\]

### 4. Analytical Solutions for the above Two Cases

In this section, the single scattering of SH waves by a cylindrical fiber with the above two interphase layers was solved. The schematic of such a problem has been plotted in Figure 1. Under the excitation of an incident wave, the total wave in the matrix, interphase layer, and the fiber can be expressed by

\[
\begin{align*}
\sum_{n=0}^{\infty} [\varepsilon_n \varepsilon_i^*]_n (k_{m} r) + A_n H^{(1)}_{m,n}(k_{m} r) \cos(n\theta), & \quad r > b, \\
\sum_{n=0}^{\infty} [C_n F_1(r) + D_n F_2(r)] \cos(n\theta), & \quad a < r < b, \\
\sum_{n=0}^{\infty} B_n J_m(k_{m} r) \cos(n\theta), & \quad r < a,
\end{align*}
\]

where \(\varepsilon_i = 1\) and \(\varepsilon_n = 2(n \neq 0)\), \(H^{(1)}_{m,n}(\cdot)\) is the \(n\)th order Hankel function of the first type; \(F_1(r)\) and \(F_2(r)\) are two functions whose detailed expressions are determined by the interphase layer. In Case 1, \(F_1(r)\) and \(F_2(r)\) can be expressed by
\[
F_{1,2}(r) = \frac{1}{\sqrt{\nu(r)}} \left[ M_{h,n}(\delta r) \pm iW_{h,n}(\delta r) \right], \quad k^2 - \beta^2 < 0,
\]
\[
F_{1,2}(r) = \frac{1}{\sqrt{\nu(r)}} \sqrt{\delta r} H_n^{(1,2)}(\sqrt{\delta r}), \quad k^2 - \beta^2 = 0,
\]
\[
F_{1,2}(r) = \frac{1}{\sqrt{\nu(r)}} \left[ F_{h,1/2}(\eta, \delta r) \pm iG_{h,1/2}(\eta, \delta r) \right], \quad k^2 - \beta^2 > 0.
\]

In Case 2, \( F_1(r) \) and \( F_2(r) \) are given by
\[
\begin{align*}
F_{1,2}(r) &= \frac{1}{r \sqrt{\rho(r)}} \left[ M_{h,n/2}(\delta r^2) \pm iW_{h,n/2}(\delta r^2) \right], \quad y^2 < 0, \\
F_{1,2}(r) &= \frac{1}{r \sqrt{\rho(r)}} \sqrt{\delta r} H_n^{(1,2)}(\sqrt{\delta r}), \quad y^2 = 0, \\
F_{1,2}(r) &= \frac{1}{r \sqrt{\rho(r)}} \left[ F_{h,1/2}(\eta, \delta r^2) \pm iG_{h,1/2}(\eta, \delta r^2) \right], \quad y^2 > 0.
\end{align*}
\]

\( A_n, B_n, C_n, \) and \( D_n \) in equation (32) are the expansion coefficients to be determined by the continuity conditions of displacements and stresses across the interfaces at \( r = a \) and \( b \), which are given by
\[
\begin{align*}
\left. w(r) \right|_{r=a^{-}} &= \left. w(r) \right|_{r=a^{+}}, \\
\left. \frac{\partial w}{\partial r} \right|_{r=a^{-}} &= \left. \frac{\partial w}{\partial r} \right|_{r=a^{+}}.
\end{align*}
\]

As the expansion coefficients \( A_n, B_n, C_n, \) and \( D_n (n = 1, 2, \ldots, n_{\text{max}}) \) have been obtained, the far-field scattering magnitude [14] and the scattering cross section [9] could be calculated using
\[
\begin{align*}
f(\theta) &= \sum_{n=0}^{\infty} \epsilon_n \frac{A_n}{k} \cos(n\theta), \\
\sigma_s &= \frac{2}{k} \sum_{n=0}^{\infty} \epsilon_n |A_n|^2.
\end{align*}
\]

For illustrative purposes, we solved the single scattering of SH waves by a SiC fiber embedded in an aluminum matrix with the above two radially gradient interphase layers. The material properties are listed in Table 1. Figure 2 plots the calculated \( \sigma_s \) changes with \( k_{\text{mol}} \) for the two interphase types (Case 1 and Case 2). Results of the approximate approach (i.e., dividing the interphase layer into multiple homogeneous sublayers) were also plotted for comparison. Both approaches yielded similar results. In addition, the exact results for the case without interphase layers were also illustrated.

5. Conclusions

We reported the analytic solutions to the single scattering of SH waves by cylindrical fibers with two specific radially gradient interphase layers. In the first case, the shear modulus is an exponential function of \( r \) and \( k^2 \) is a linear function of \( 1/r \); in the second case, the shear modulus is a Gaussian function of \( r \) and \( k^2 \) is a linear function of \( r^2 \). An example of the single scattering of SH waves by a SiC fiber in an aluminum matrix was calculated, and the scattering cross sections were compared against the values obtained by

<table>
<thead>
<tr>
<th>Table 1: Material properties of SiC and Al.</th>
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<td>SiC</td>
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<td>Al</td>
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\( k_{\text{mol}} \) is the wave number in the matrix. The scattering cross section \( \sigma_s \) is defined as the ratio of the scattered energy to the incident energy.
dividing the continuously varying interphase layer into multiple homogeneous sublayers. The two approaches yielded similar results.

**Appendix**

**General Approach to Make the Second Transformation**

Define $u_n(r) = r^n w_n(x)$ with $x = \delta r^d$, $c$ and $d$ are two constants to be determined. The first and second derivatives of $u_n(r)$ with regard to $r$ can then be expressed as

$$
\begin{align*}
\frac{u_n(r)}{r} &= c r^{-1} w_n(x) + r^{2+d-1} w_n'(x) \delta d, \\
\frac{u_n'(r)}{r} &= c (c - 1) r^{c-2} w_n(x) + \delta d (2c + d - 1) r^{c+d-2} w_n'(x) \\
&\quad + \delta^2 d^2 r^{c+2d-2} w_n''(x).
\end{align*}
$$

(A.1)

By substituting the expressions $u_n(r)$ and $u_n'(r)$ into equation (14), we obtain

$$
\begin{align*}
c^2 r^{c-2} w_n(x) + \delta d (2c + d) r^{c+d-2} w_n'(x) + \delta^2 d^2 r^{c+2d-2} w_n''(x) \\
+ \ldots + \left( k^2_1 - \beta^2 + \frac{2\alpha - \beta}{r} - \frac{n_x}{r^2} \right) r^c w_n(x) &= 0.
\end{align*}
$$

(A.2)

Therefore, if equation (A.2) is required to have the form of the confluent hypergeometric equation, the coefficients $c$ and $d$ should satisfy the following equation:

$$
2c + d = 0. \tag{A.3}
$$

The term of the first derivative of $w_n'(x)$ disappears. Then, equation (A.2) becomes

$$
\delta^2 d^2 r^{2d-2} w_n''(x) + \left( k^2_1 - \beta^2 + \frac{2\alpha - \beta}{r} - \frac{n_x}{r^2} \right) w_n(x) = 0. \tag{A.4}
$$

Then, $d$ can be set to $1$, and $c = -1/2$. Then, equation (A.4) is simplified to

$$
w_n''(x) + \left( k^2_1 - \beta^2 + \frac{1}{\delta} \frac{2\alpha - \beta}{x} - \frac{n^2 - 1/4}{x^2} \right) w_n(x) = 0. \tag{A.5}
$$

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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