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Research Article

A Suspension Footbridge Model under Crowd-Induced Lateral Excitation

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In this paper, a plane pendulum model is proposed to investigate the lateral vibration of a suspension bridge under crowd excitation. The plane model consists of two strings and a rigid body, which represent cables and the bridge deck, respectively. The lateral force induced by crowd is expressed as a cosine function with random phase. Comparing with other existing pedestrian-footbridge interaction models, the proposed model has two features: one is that the structural characteristics of the suspension bridge are taken into account. The other is that the expression of the lateral force induced by crowd has a unified form for different lateral bridge amplitudes. By numerically analyzing the solution stability of the plane model, we exhibit the whole changing process how a suspension bridge increases its lateral amplitude from small to large. It is shown that the worst case occurs when the lateral natural frequency of the bridge is half the lateral step frequency of the pedestrians. Based on the analysis results, the plane pendulum model can be easily used to explain why the central span of the London Millennium Bridge has large lateral oscillations at about 0.48 Hz.

1. Introduction

The excessive lateral vibration problem of footbridges induced by pedestrians in recent years has drawn particular attention in design and research communities around the world [1]. So far, such kind of lateral vibration problem in bridges has never involved structural failures but caused discomfort for pedestrians. Excessive lateral vibrations have been observed in almost all types of footbridges, such as suspension bridges [2, 3], arch bridges, truss bridges [4], and so on. The mechanism of excessive lateral vibrations for footbridges has been studied theoretically and experimentally in the last decade. The key to resolving the problem involves two aspects: one is how to determine the lateral force induced by pedestrians; the other is how to describe the motion of the bridge. For the former, the expression of the lateral force generally is obtained by using an empirical approach because of complexity. For the latter, a bridge usually is viewed as a single degree-of-freedom (SDOF) system for simplicity.

According to experiments carried out on rigid surfaces [5], the ground reaction force (GRF) occurs due to acceleration (and deceleration) of the center of mass of a pedestrian's body. The GRF is a three-dimensional vector which varies in time and space due to the forward movement of the pedestrian. Early studies on GRFs from Harper et al. [6] revealed that horizontal lateral component of the force is generally very small. Chao et al. [7] measured single footstep forces from several persons and found that the lateral component of the force is about 4 – 5% of the body weight for men and little less for women. Both Andriacchi et al. [8] and Masani et al. [9] showed that the peak value of lateral force increases with the walking speed. The crowd excitation usually is estimated by adding together the GRF of every pedestrian on the bridge. A notion of "dynamics loading factor" was introduced [10] in the expression of lateral force induced by crowd suggested by Matsumoto et al. [11]. An analogous model was also given by Roberts [12].

The above force models are mainly proposed based on experimental measurements carried out on stationary platforms. Through video analysis for T-Bridge in Japan, Fujino et al. [2] concluded that pedestrians' gait increases with the lateral bridge amplitude. Kay and Warren Jr. [13] experimentally showed that the pedestrian gait cycle frequency locks to the driving frequency of a moving surface in a large range of frequencies. Comparing with the case for stationary platforms, McAndrew et al. [14] reported that the step width and the frequency increase, whereas the step length decreases. The experiments of Brady et al. [15] were carried out for the case of low frequencies (0.2–0.3 Hz) and a large amplitude (127 mm), which also indicated that the step width generally increases. McRobie et al. [16] constructed a suspended platform equipped with a treadmill having an adjustable lateral frequency in the range of 0.7 to 0.9 Hz. It was observed that people tended to spread their feet further apart and walked at the same frequency (with a constant phase) as that of the platform. In case of great lateral oscillations of the footbridges, an additional non-negligible force component is potentially generated due to the interaction between the movement of the center of mass of the pedestrian's body and that of the structure. For great lateral oscillations of the footbridge, Piccardo and Tubino [17] considered that the lateral force exerted by crowd is harmonic, and its amplitude depends on the bridge displacement. Based on such force model, the parametric excitation mechanism was proposed to explain the causes of excessive lateral vibrations of the Millennium Bridge.

It is observed that even if a footbridge is static at the beginning, excessive lateral vibrations still occur under crowd. However, we have to choose different models to analyze oscillations of the bridge in small or large amplitude. No force models mentioned above allow describing the whole process how the amplitude increases from small to large. When the lateral bridge amplitude is small, the distribution of pedestrians' walking phase is random. According to the existing experiments, a moving surface always makes pedestrians synchronize with the surface over a large frequency range. The randomness of pedestrians' walking phase reduces with the increase of the lateral bridge amplitude. Then, the randomness of pedestrians' walking phase can be used to evaluate the influence of the moving deck. In this paper, we try to consider a unified expression of the lateral force induced by pedestrians for different bridge amplitudes. The details about the lateral force model will be given in Section 3.

In many dynamic interaction models [3, 4, 18, 19], a footbridge always was simplified as a SDOF system no matter what its structural characteristics. Roberts [12] modelled the Millennium Bridge as a beam but also eventually converted it to a SDOF system. Blekherman [20] used a pendulum system with two degrees of freedom to simulate the lateral vibrations of a footbridge under crowd excitation. Although there is no evidence that the structural characteristics of a footbridge have impact on dynamic interaction between pedestrians and the footbridge, McRobie et al. [21] experimentally investigated the phenomenon of human-structure lock in by using a section model consisting of two strings and a rigid body; Zhou and Ji [22] theoretically and experimentally analyzed dynamic characteristics of a generalized suspension system developed on the basis of the section model. Their research studies showed the parameters about strings have great influence on dynamic behaviour of the suspension system. This implies that we should not ignore structural characteristics in the analysis for mechanism of excessive lateral vibrations of suspension footbridges. The other motivation of our paper is to involve structural characteristic of the suspension bridge in the proposed model.

In this paper, we attempt to use a simple model to exhibit the whole process how a suspension footbridge increases its lateral amplitude from small to large under crowd. According to the research studies of McRobie et al. [21] and Zhou and Ji [22], we study a suspension footbridge by using the section model consisting of two strings and a rigid body. The strings and the rigid body represent cables and bridge deck, respectively, as shown in Figure 1. The lateral force induced by crowd on the bridge here is expressed by a cosine function with random phase evaluating the effect of the moving bridge deck. The randomness of the phase reduces when the lateral bridge amplitude increases, because the swaying bridge always makes pedestrians synchronize with the moving deck. By referring to some criteria on stability of nonlinear and stochastic equations [23, 24, 25, 26], we will show the influence of randomness on stability of a suspension footbridge to understand the whole process that the lateral bridge amplitude increases from small to large. The most advantage of the proposed model is the unified form on description of the lateral force induced by pedestrians, which usually is discontinuously expressed for different lateral bridge amplitudes in other models. Some new explanations might be given based on the plane pendulum model to understand the mechanism of the pedestrian-footbridge interaction. However, it should be pointed out that the section model is appropriate if the distribution of the pedestrian mass is uniform along both the bridge length and the direction orthogonal bridge axis. If an uncertainty of the lateral force induced by the pedestrian occurs, the result based on the model may not be accurate. However, we focus on excessive lateral vibrations of a footbridge, and under such condition, there always are a lot of pedestrians on the bridge. Usually, pedestrians do not cluster together in any place on the bridge. Instead, they always trend to distribute uniformly on the deck. Once the pedestrian flows are steady, the distribution of pedestrians on the deck will be approximately uniform. In addition, only the first-order lateral modal vibration of the bridge is concerned in this model. Since we mainly intend to find a unified form to describe the lateral force induced by pedestrians for different amplitudes, some important parameters, such as walking speed and step frequency, have not been involved in this model. Furthermore, we consider the structural characteristics of the suspension bridge in our model, and then the analysis results are only valid for the suspension bridge.

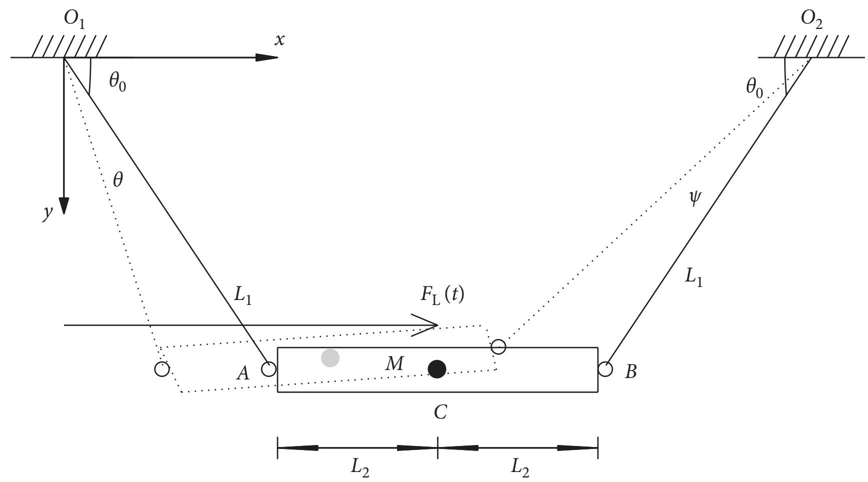

Figure 1: The section model for a suspension footbridge.

The rest of the paper is organized as follows: the details of the section model and the crowd excitation are explained in Sections 2 and 3, respectively. The governing equation of the dynamic interaction model is established in Section 4. Stability analysis is carried out to understand how a suspension bridge increases the lateral amplitude from small to large in Section 5. Finally, discussions and conclusions are given in Section 6.

2. The Section Model for a Suspension Footbridge

We investigate a suspension bridge in this paper by using a plane pendulum model shown in Figure 1. O_1A, O_2B represent the cables and AB the bridge deck. Two symmetrically inclined cables are viewed as strings with infinite axial stiffness but without mass and transverse stiffness, and the bridge deck is simplified as a rigid body. The lengths of O_1A and O_2B both are L_1; the width, mass, and lateral damping of AB are $2L_2$, M, and C_L, respectively. For simplicity, the damping force applied on the bridge is assumed to be proportional to the lateral velocity of AB. Point C denotes the center of mass of AB, and the rotary inertia around point C is I. Let a Cartesian coordinate system center at point O_1. The two inclined angles of the strings under gravity at static equilibrium state are the same and defined by θ_0; the rotations of the strings O_1A and O_2B are θ and ψ, respectively, when the bridge laterally sways. The lateral force induced by crowd is denoted by $F_L(t)$, which is assumed to act on point C.

3. The Lateral Force Model of the Crowd Excitation

When pedestrians begin to walk on a stationary bridge, pedestrians walk freely; the randomness of the walking phase of the crowd reaches to the maximum. With increase of the amplitude, pedestrians synchronize with the bridge, and then randomness of phase reduces. To describe the effect of a moving surface on lateral walking force, we introduce a random process in the force model suggested by Matsumoto et al. [11] and Bachmann [10], which is expressed as

$$F_L(t) = \lambda\alpha g m_p(x)\cos[\Omega t + \phi + \gamma w(t)], \tag{1}$$

where λ is the percentage of synchronized pedestrians, α is the so-called "dynamic loading factor" and depends on the considered load harmonic and on the load direction [10], g is the gravity acceleration, $m_p(x) = N_p m_{ps}/L$, in which N_p is the number of pedestrians on the bridge, m_{ps} the mass of a single pedestrian, and L is the footbridge span length. Additionally, Ω is the dominant walking frequency and is commonly assumed around 1 Hz in the horizontal direction; $\phi + \gamma w(t)$ is the initial phase, in which ϕ is a random variable uniformly distributed on $[0, 2\pi]$ representing the dominant walking phase corresponding to large amplitude. $w(t)$ is a standard Wiener process [27], which also is called the Brownian motion [28]. $w(t)$ represents the influence of the randomness of the phase and is independent of ϕ. γ is a deterministic parameter which describes the intensity of the random phase modulation. $\gamma \longrightarrow 0$ means that the Brownian motion disappears, which occurs for a very large lateral vibration of the bridge. With the value of γ increasing, the randomness of the phase enhances. When the value of γ is large enough, the phase of the walking frequency of the crowd varies very freely. Then the value of γ can be used to evaluate the effect of the moving deck on the lateral force induced by crowd.

4. The Governing Equation

In this section, we establish the governing equation of the section model under crowd excitation by using Lagrange method [29]. Since AB is a rigid body in Figure 1, θ and ψ are related to each other. Considering that the lateral vibrations of the suspension footbridge are small, the following relationship approximately holds (see reference [22] for the details):

$$\psi = \theta - (1 + a\cos\theta_0)\cot\theta_0\,\theta^2, \tag{2}$$

where $a = L_1/L_2$. Then, θ is chosen as the independent generalized coordinates. The kinetic energy T and potential energy U of the suspension system separately are

$$\begin{aligned} T &= \frac{1}{2}\left(ML_1^2\sin^2\theta_0 + Ia^2\cos^2\theta_0\right)\dot{\theta}^2, \\ U &= \frac{1}{2}MgL_1\left[\cos\theta_0\cot\theta_0(1 + a\cos\theta_0) + \sin\theta_0\right]\theta^2. \end{aligned} \tag{3}$$

Using the Lagrange equation:

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} + \frac{\partial U}{\partial\theta} = Q, \tag{4}$$

in which Q represents the generalized forces, one obtains the following governing equation:

$$\begin{aligned} &\left(ML_1^2\sin\theta_0^2 + Ia^2\cos\theta_0^2\right)\ddot{\theta} \\ &\quad + MgL_1\sin\theta_0\left[1 + \cot\theta_0^2(1 + a\cos\theta_0)\right]\theta \\ &\quad = F_{\mathrm{L}}(t)L_1\sin\theta_0\left[1 - (1 + a\cos\theta_0)\cot\theta_0\theta\right] \\ &\quad - \frac{1}{2}C_{\mathrm{L}}L_1^2\sin^2\theta_0\,\dot{\theta}. \end{aligned} \tag{5}$$

Substituting equation (1) into equation (5) leads to

$$\begin{aligned} &\ddot{\theta} + \eta\dot{\theta} + \left[\varepsilon_1 + \varepsilon_2\cos(\Omega t + \phi + \gamma w(t))\right]\theta \\ &\quad = \varepsilon_3\cos(\Omega t + \phi + \gamma w(t)), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \eta &= \frac{C_{\mathrm{L}}L_1^2\sin^2\theta_0}{2\left(ML_1^2\sin^2\theta_0 + Ia^2\cos^2\theta_0\right)}, \\ \varepsilon_1 &= \frac{MgL_1\sin\theta_0\left[1 + \cot^2\theta_0(1 + a\cos\theta_0)\right]}{ML_1^2\sin^2\theta_0 + Ia^2\cos^2\theta_0}, \\ \varepsilon_2 &= \frac{\lambda\alpha g m_{\mathrm{p}}(x)L_1\sin\theta_0\cot\theta_0(1 + a\cos\theta_0)}{ML_1^2\sin^2\theta_0 + Ia^2\cos^2\theta_0}, \\ \varepsilon_3 &= \frac{MgL_1\sin\theta_0\left[1 + \cot^2\theta_0(1 + a\cos\theta_0)\right]}{ML_1^2\sin^2\theta_0 + Ia^2\cos^2\theta_0}. \end{aligned} \tag{7}$$

The parameter η is called the "damping coefficient" of the bridge in the following sections. Equation (6) describes the motion of the section model under crowd excitation and is coincident to the Mathieu equation [30] with imperfect periodicity.

5. Stability Analysis for the Section Model

The lateral instability of a suspension footbridge will lead to excessive lateral vibrations. In this section, we analyze the solution stability of equation (6) to understand the mechanism of unexpected lateral vibrations in the suspension footbridge. Solutions to system (6) are said to be stable if all solutions are bounded and unstable otherwise. Since the external forces do not affect the stability of system (6), we consider the homogeneous equation of system (6), which is expressed by

$$\ddot{\theta} + \eta\dot{\theta} + \omega^2\left[1 + b\cos(\Omega t + \phi + \gamma w(t))\right]\theta = 0, \tag{8}$$

where

$$\begin{aligned} b &= \frac{\lambda\alpha m_{\mathrm{p}}(x)\cot\theta_0(1 + a\cos\theta_0)}{M\left[1 + \cot^2\theta_0(1 + a\cos\theta_0)\right]}, \\ \omega^2 &= \frac{MgL_1\sin\theta_0\left[1 + \cot^2\theta_0(1 + a\cos\theta_0)\right]}{ML_1^2\sin^2\theta_0 + Ia^2\cos^2\theta_0}. \end{aligned} \tag{9}$$

Notice that ω is just the free vibration circle frequency of the section model. To discuss the influence of parameters on stability of system (8), we need to obtain the stability charts on a parametric plane. Transition curves are boundaries on a stability chart which separate the stable and unstable regions. For comparison, we first find the transition curves of system (8) with $\gamma = 0$.

5.1. The Transition Curves of System (8) with $\gamma = 0$. Under the condition $\gamma = 0$, system (8) can be regarded to be deterministic. Rescaling time according to $\tau = \Omega t + \phi$ and assuming that the damping coefficient η is small of the same order as b: $\eta = 2b\beta_1$, system (8) can be rewritten as

$$\ddot{y} + 2b\beta_1\dot{y} + \beta_2(1 + b\cos\tau)y = 0, \tag{10}$$

where $y = y(\tau) = \theta(\Omega t + \phi)$, "$\cdot$" represents derivatives with respect to τ, and

$$\begin{aligned} b\beta_1 &= \frac{\eta}{2\Omega}, \\ \beta_2 &= \frac{\omega^2}{\Omega^2}. \end{aligned} \tag{11}$$

Expand the solution y and β_2 in power of b:

$$\begin{aligned} y &= y_0 + by_1 + b^2y_2 + \cdots, \\ \beta_2 &= \beta_{20} + b\beta_{21} + b^2\beta_{22} + \cdots. \end{aligned} \tag{12}$$

Substituting equation (12) into equation (10) and equating the coefficient of each power of b, one has

$$\begin{aligned} b^0&:\ \ddot{y}_0 + \beta_{20}y_0 = 0, \\ b^1&:\ \ddot{y}_1 + \beta_{20}y_1 = -y_0\cos\tau\beta_{20} - 2\dot{y}_0\beta_1 - y_0\beta_{21}, \\ b^2&:\ \dot{y}_2 + \beta_{20}y_2 = -y_1(\beta_{21} + \cos\tau\beta_{20}) - 2\dot{y}_1\beta_1 \\ &\qquad - y_0(\beta_{22} + \cos\tau\beta_{21}). \end{aligned} \tag{13}$$

The general solution of the first equation in equation (13) is

$$y_0 = \alpha_1\cos\left(\sqrt{\beta_{20}}\,\tau\right) + \alpha_2\sin\left(\sqrt{\beta_{20}}\,\tau\right), \tag{14}$$

where the coefficients $\alpha_{1,2}$ are the dependent initial conditions constants. It is easy to verify that $\beta_{20} = 1/4$ is the principal resonance condition. Substituting equation (14) into the second equation of equation (13) with $\beta_{20} = 1/4$ and eliminating the terms that produce a secular term in y_1 demands that

$$\beta_1^2 + \beta_{21}^2 = \frac{1}{64}. \tag{15}$$

From [17], the first-order approximation of the transition curves near $\beta_2 = 1/4$ has enough precision because of small value of b. By substituting equation (15) into the second equation in equation (12), one has the transition curves to the first-order approximation:

$$\beta_2 = \frac{1}{4} \pm b\sqrt{\frac{1}{64} - \beta_1^2}. \tag{16}$$

Next, we consider the secondary resonance condition $\beta_{20} = 1$. Substituting equation (14) into the second equation in equation (13) with $\beta_{20} = 1$ and eliminating the terms that produce a secular term in y_1 demands that

$$\beta_1^2 + \frac{1}{4}\beta_{21}^2 = 0. \tag{17}$$

Equation (17) cannot be satisfied with $\beta_1 > 0$, which means that no periodic solution exists around the secondary resonance condition $\beta_{20} = 1$. According to [31–34], the secondary resonance is possible only when the damping is smaller than the parametric excitation term. However, the case of very small structural damping or very great pedestrian mass is not of technical interest. Therefore, under the condition of $\gamma = 0$, equation (8) only has one resonance frequency $\omega = 1/2\ \Omega$ for small value of b. In next section, we will show the influence of the random phase modulation on resonance frequency.

5.2. Stability Chart of System (8) with $\gamma > 0$. Several notions of stability have been used for stochastic systems. Here the notion of mean square (asymptotic) stability is adopted since this is a structural dynamic problem. The solutions of equation (8) are said to be stable if all solutions satisfy the following equation:

$$\lim_{t \longrightarrow \infty} E\left[\theta(t)^2 + \theta(t)\dot{\theta}(t) + \dot{\theta}(t)^2\right] = 0, \tag{18}$$

where $E[\cdot]$ represents mathematical expectation.

In reference [35], the mean square stability for equation (8) is considered when the parameter b is small, in which the stability boundaries only near the principal resonance frequency are obtained by using asymptotic methods. In this paper, we employ the numerical procedure developed by Bobryk and Chrzeszczyk [26] to derive the mean square stability charts [36] for equation (8) in the case of arbitrary b and the higher-order resonance frequency ratios ω/Ω.

Rewriting equation (8) as

$$\dot{X}(t) = AX(t) + \xi(t)BX(t), \quad t > 0, \tag{19}$$

where

$$\begin{aligned}
X(t) &= \left\{\dot{\theta}(t)^2, \theta(t)\dot{\theta}(t), \theta(t)^2\right\}^T, \\
\xi(t) &= b\cos(\Omega t + \phi + \gamma w(t)), \\
A &= \begin{bmatrix} -2\eta & -2\omega^2 & 0 \\ 1 & -\eta & -\omega^2 \\ 0 & 2 & 0 \end{bmatrix}, \\
B &= \begin{bmatrix} 0 & -2\omega^2 & 0 \\ 0 & 0 & -\omega^2 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{20}$$

From Bobryk and Chrzeszczyk [26], by using the Cameron-Martin formula [37] for the density of the Wiener measure, the following infinite hierarchy of linear differential equations can be obtained for the mean $E[X(t)]$:

$$\begin{aligned}
\frac{dE[X(t)]}{dt} &= AE[X(t)] + bB(u_1(t) + v_1(t)), \\
\frac{du_1}{dt} &= \left(-\frac{\gamma^2}{2} + i\Omega\right)u_1 + Au_1 + bBu_2 + \frac{b}{4}BE[X(t)], \\
\frac{dv_1}{dt} &= \left(-\frac{\gamma^2}{2} - i\Omega\right)v_1 + Av_1 + bBv_2 + \frac{b}{4}BE[X(t)], \\
\frac{du_k}{dt} &= \left(-\frac{k^2\gamma^2}{2} + ik\Omega\right)u_k + Au_k + bBu_{k+1} + \frac{b}{4}Bu_{k-1}, \\
\frac{dv_k}{dt} &= \left(-\frac{k^2\gamma^2}{2} - ik\Omega\right)v_k + Av_k + bBv_{k+1} + \frac{b}{4}Bv_{k-1},
\end{aligned} \tag{21}$$

where $i = \sqrt{-1}$, $k = 2, 3, \cdots$, $E[X(0)] = X(0)$, $u_k(0) = v_k(0) = 0$, $k = 1, 2, 3, \cdots$ and

$$\begin{aligned}
u_k(t) &= \frac{1}{2^k}e^{ik\Omega-(k^2\gamma^2 t/2)}E\left[e^{ik\theta X(t;w(s)+ik\gamma s)}\right], \\
v_k(t) &= \frac{1}{2^k}e^{-ik\Omega-(k^2\gamma^2 t/2)}E\left[e^{-ik\theta X(t;w(s)-ik\gamma s)}\right], \quad k = 1, 2, 3, \cdots,
\end{aligned} \tag{22}$$

in which $X(t; w(s) \pm ik\gamma s)$ is the solution of equation (19) with $w(t)$ replaced by $w(t) \pm ik\gamma t$.

By this way, the problem of mean square stability for equation (8) is reduced to the asymptotic stability for this hierarchy. For computational purposes, we can obtain the closed system of linear differential equations of first order with constant coefficients by neglecting the terms u_{n+1}, v_{n+1} in the equations for u_n, v_n, where the index n is called the truncation index. From [26,38–40], the procedure is quickly convergent. For sufficiently large truncation index, the asymptotic stability or instability of the system (21) determines the mean square stability or instability for equation (8). Based on well-known Routh–Hurwitz criterion [41], the origin of system (21) is asymptotic stability if and only if the matrix of its coefficients has all eigenvalues with negative real parts.

In the following numerical simulations, we always let truncation index $n = 15$ for enough precision. We first numerically analyze the influence of η on the transition curves on the stability charts of system (8). We choose $\Omega = 1$ and different values of γ to carry out numerical simulations for system (8), the stability charts of which on the $\omega - b$ parametric plane are presented in Figure 2. Recalling that ω describes the lateral natural frequency of the bridge and b represents the amplitude of the lateral force exerted by crowd, the ranges of ω and b are limited within $[0, 2]$ and $[0, 1]$, respectively. It is obvious that η nearly does not change the shape of the transition curves for different values of γ, but changes their positions. Figure 2 indicates that the larger the value of η, the smaller the unstable region is. If the value of η is increased, the bridge will be more safe.

Next, we numerically derive the stability charts of system (8) to discuss the occurrence of the excessive lateral vibrations of a suspension footbridge. Since we have clarified the influence of η on stability charts, we choose $\eta = 0.005$ in the following simulations. Considering the lateral step frequency of pedestrians varies in the range 0.7 to 1.2 Hz, we separately choose $\Omega = 0.7$, 1.0, and 1.2 to obtain the stability charts of system (8) for different values of γ, which are depicted in Figures 3–5, respectively. The values of ω and b are still limited within the ranges of $\omega \in [0, 2]$ and $b \in [0, 1]$, respectively. The instability regions are shaded. From Figures 3–5, when $\gamma > 1$, the footbridge is less likely to lose its stability near $\omega = \Omega/2$. With reducing the value of γ, resonance points occur gradually at $\omega/\Omega = 1/2, 1, 2, \cdots$. Considering the engineering practice, only the first resonance condition $\omega = \Omega/2$ is our concern. The numerical results presented in Figures 3–5 show that the plane pendulum model can exhibit the whole changing process that the lateral amplitude of the bridge increases from small to large. When pedestrians begin to walk on a footbridge, the lateral amplitude of the bridge is very small. The phase of the walking frequency of the crowd varies very freely. In such a case, the value of γ is very large. With the lateral amplitude of the bridge increasing, pedestrians feel that it is more comfortable to synchronize with the deck. As a consequence of synchronization, the randomness of the pedestrians' walking phase reduces, which corresponds to the decrease of the value of γ. Small γ eventually results in instability of the bridge under the condition $\omega = \Omega/2$.

Additionally, we denote the minimum value of b on the transition curves in Figures 3–5 by $b_{\min}$. We present the relation between $b_{\min}$ and the value of γ for different values of Ω in Figure 6. $b_{\min}$ decreases with the increase of Ω from 0.7 to 1.2 Hz. The higher dominant step frequency results in smaller value of $b_{\min}$. Then, we conclude from Figure 6 that it is more dangerous if the pedestrians cause instability of the bridge with a higher dominant walking frequency. However, regardless of the value of Ω if $\gamma < 1$, $b_{\min}$ increases with γ increasing; while when $\gamma > 1$, $b_{\min}$ decreases as γ increases. To conform to the observations and consideration for the plane pendulum model, the case of $\gamma > 1$ should be deleted. Then, we add the limiting condition $0 < \gamma < 1$ for the plane pendulum model described by equation (6).

6. Conclusions

In this paper, we propose a plane pendulum model to discuss the mechanism of excessive lateral vibrations of a suspension bridge under crowd. In this model, the suspension bridge is simplified as a section model consisting of two strings and a rigid body separately representing cables and bridge deck. The lateral force induced by pedestrians is expressed as a cosine function with random phase. We use the randomness of the walking pedestrians' phase to evaluate the influence of the moving deck on the lateral force induced by pedestrians. In previous research studies, the suspension bridge usually was simplified as a SDOF system. And the lateral force induced by pedestrians has different forms for different lateral bridge amplitudes. Comparing with other pedestrian-bridge interaction models, the proposed model has two distinctive features: one is that the suspension characteristics of the bridge are reflected. Another is that the lateral force induced by pedestrians has a unified form for different lateral bridge amplitudes.

By numerically analyzing the solution stability of the plane pendulum model for different randomness of phase, we provide an explanation how a suspension bridge increases its lateral vibration from small to large. When the lateral bridge amplitude is small, the phase of the walking frequency of the crowd varies very freely. At such situation, the randomness of the walking phase is very large, no resonance occurs for the pedestrian-footbridge interaction. With the lateral amplitude of the bridge increasing, pedestrians feel more comfortable to synchronize with the bridge. Synchronization between bridge and pedestrians results in reduction of the randomness of the walking phase, which leads to gradual occurrence of the resonance points. From the engineering practice, the worst case corresponds to the first resonance condition that the lateral natural frequency of the bridge is half the lateral step frequency of pedestrians.

Based on the analysis results for the proposed pendulum model, we can easily explain why the London Millennium Bridge has large lateral oscillations at 0.48 Hz. The London Millennium Bridge is the best-known footbridge closed after opening due to excessive lateral vibrations. The Millennium Bridge is a shallow pedestrian suspension bridge consisting of three spans: a northern span of 81 m, a central span of 144 m, and a southern span of 108 m. The amplitude of lateral vibration of the central span was observed about 70 mm in tests, which was the largest of three amplitudes. Furthermore, two groups of frequencies were observed: 1.15, 1.54, 1.89, and 2.32 Hz for central vertical modes and 0.48 and 0.96 Hz for central lateral modes. The third vertical mode (1.89 Hz) and the second lateral mode (0.96 Hz) of the central span of the Millennium Bridge satisfied the necessary condition for internal resonance; nevertheless, the analysis of vertical forces and lateral oscillations of the bridge showed no correlation between such quantities. In addition, the first frequency of the lateral modes of the bridge (0.48 Hz) was so far below normal range of lateral step frequency of pedestrians on the bridge ($0.7 - 1.2$ Hz), which is difficult to explain by using the direct or internal resonance theory. The

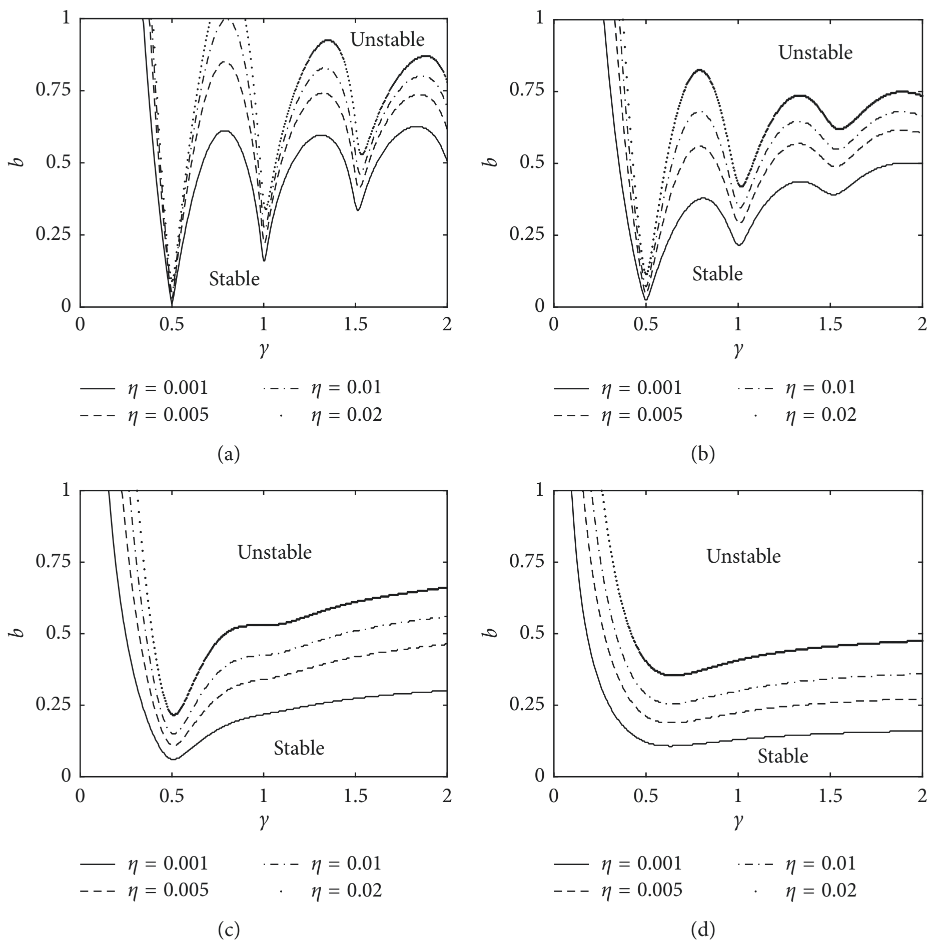

FIGURE 2: The influence of η on the transition curves on mean square stability charts of equation (8) for $\Omega = 1.0\,Hz$ and different values of γ: (a) $\gamma = 0.1$; (b) $\gamma = 0.2$; (c) $\gamma = 0.5$; (d) $\gamma = 1.0$.

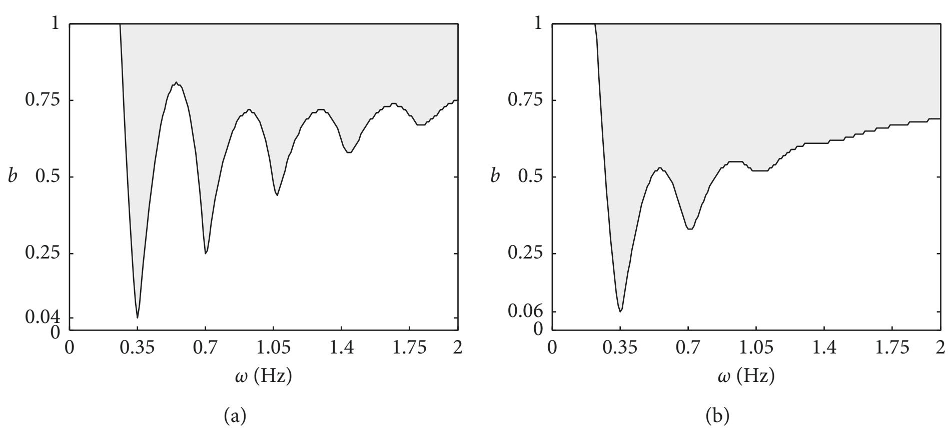

FIGURE 3: Continued.

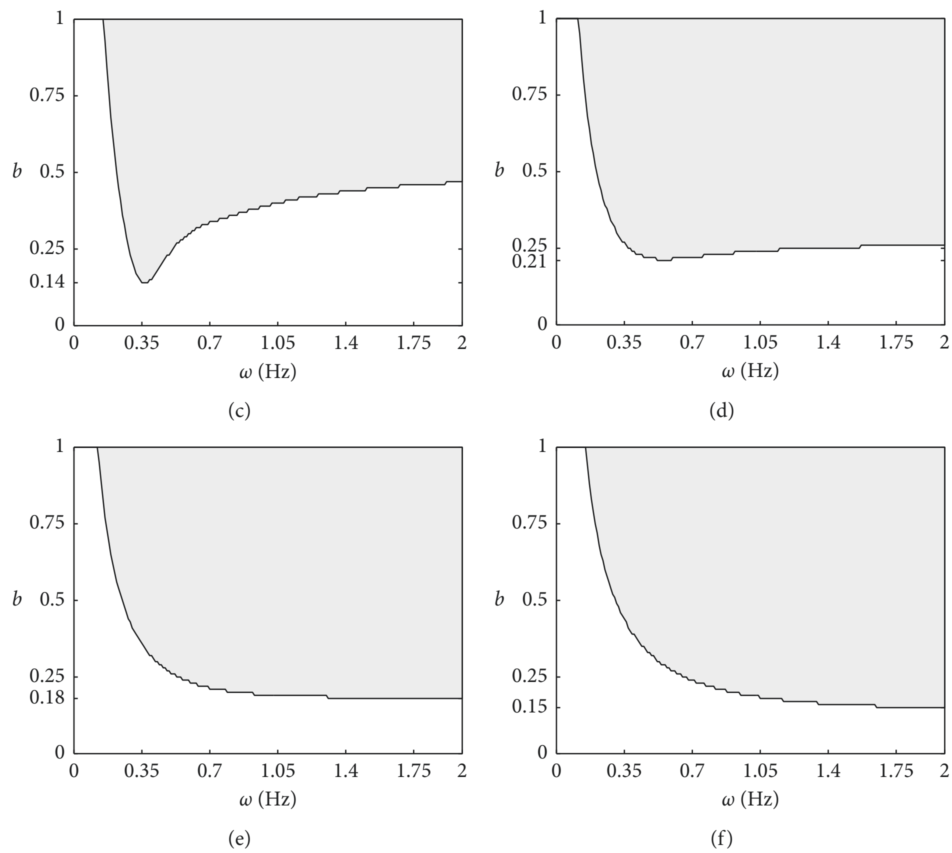

Figure 3: Mean square stability charts to equation (8) for the values $\eta = 0.005$ and $\Omega = 0.7\,Hz$. (a) $\gamma = 0.1$; (b) $\gamma = 0.2$; (c) $\gamma = 0.5$; (d) $\gamma = 1.0$; (e) $\gamma = 1.5$; (f) $\gamma = 2.0$. The instability regions are shaded.

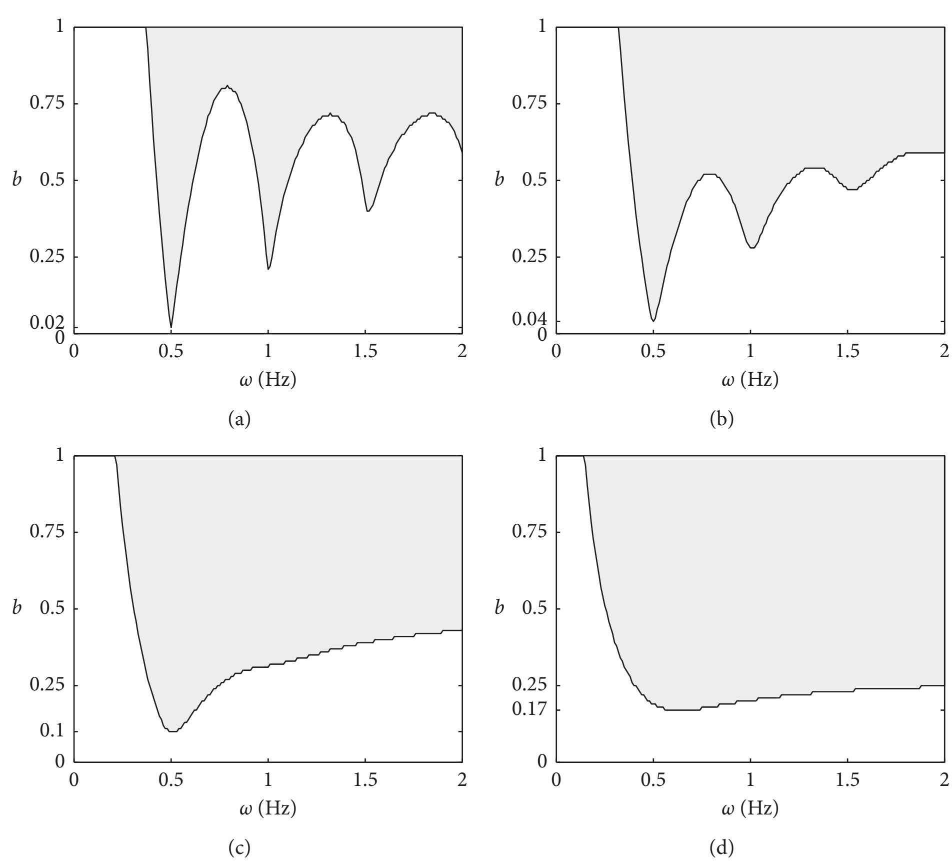

Figure 4: Continued.

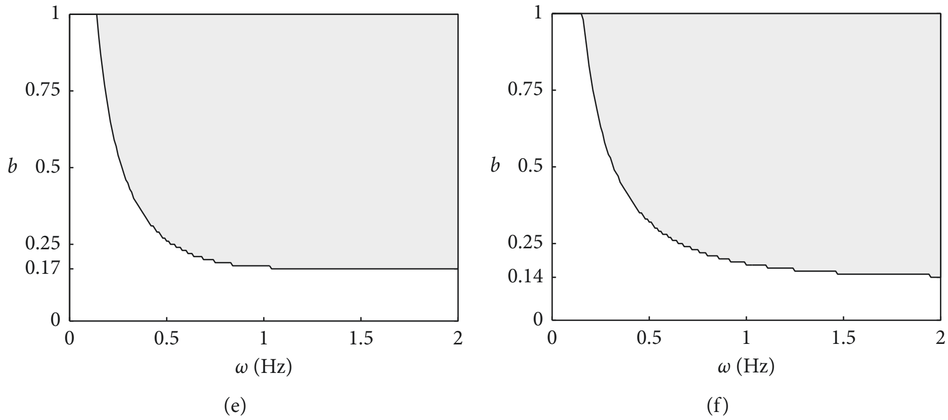

Figure 4: Mean square stability charts to equation (8) for the values $\eta = 0.005$ and $\Omega = 1.0\,Hz$. (a) $\gamma = 0.1$; (b) $\gamma = 0.2$; (c) $\gamma = 0.5$; (d) $\gamma = 1.0$; (e) $\gamma = 1.5$; (f) $\gamma = 2.0$. The instability regions are shaded.

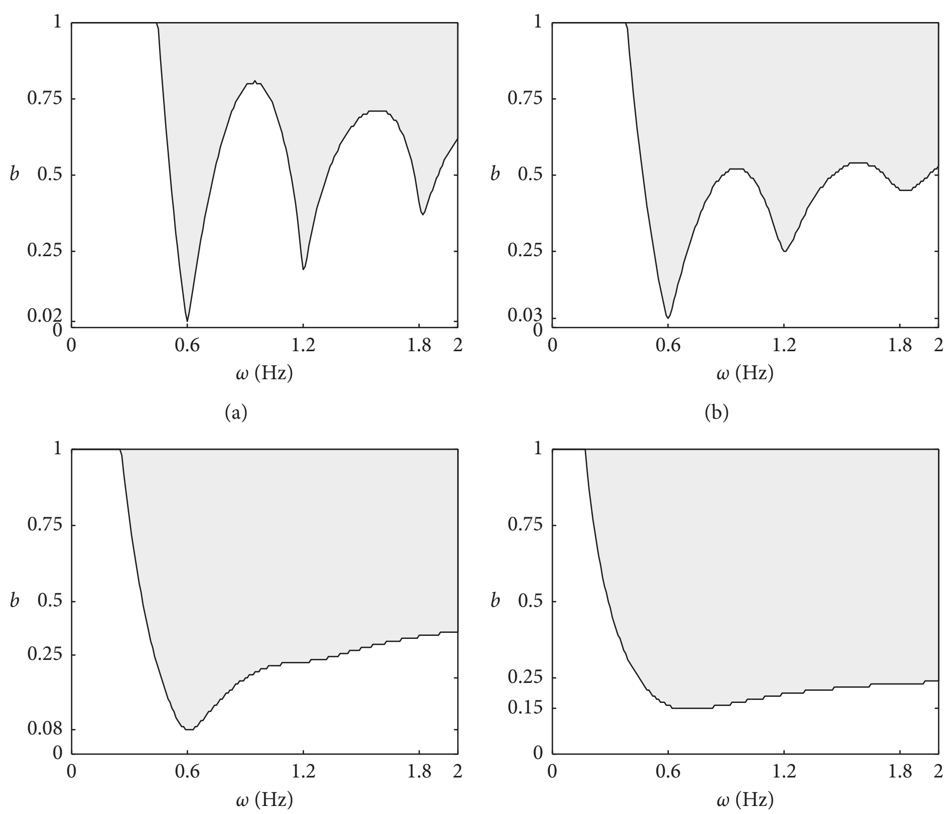

Figure 5: Continued.

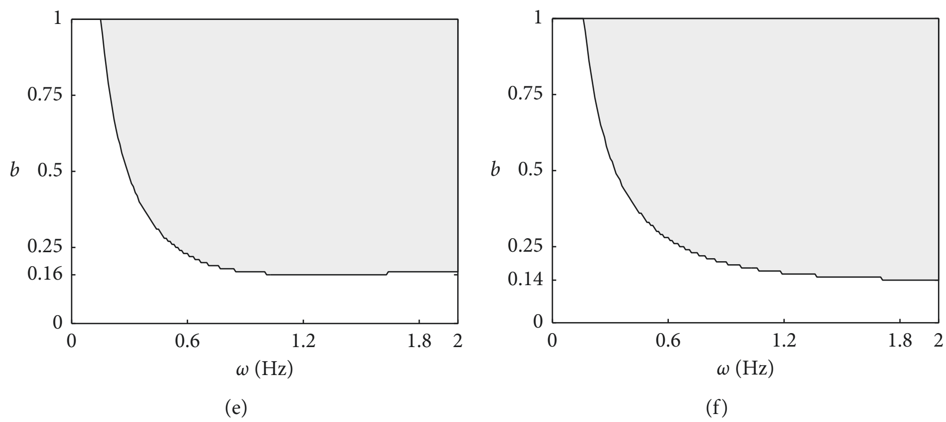

Figure 5: Mean square stability charts to equation (8) for the values $\eta = 0.005$ and $\Omega = 1.2\,Hz$. (a) $\gamma = 0.1$; (b) $\gamma = 0.2$; (c) $\gamma = 0.5$; (d) $\gamma = 1.0$; (e) $\gamma = 1.5$; (f) $\gamma = 2.0$. The instability regions are shaded.

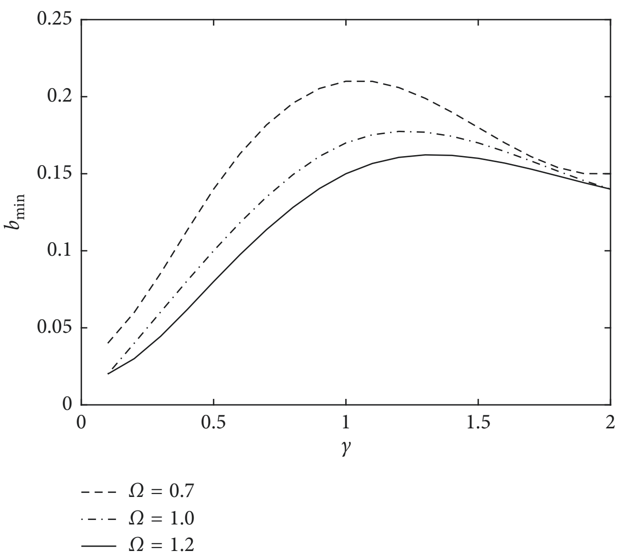

Figure 6: The curves of the critical value of b ($b_{\min}$) versus γ for different values of Ω.

stability analysis for our plane pendulum model shows that the worst case occurs when the lateral natural frequency of a suspension bridge is half the lateral step frequency of pedestrians on the bridge. Since the lateral step frequency locates in the range of $0.7-1.2$ Hz, if the lateral mode frequency of the Millennium Bridge is between 0.35 and 0.6 Hz, large lateral oscillations of the Millennium Bridge will be inevitable.

It is worth discussing that Piccardo and Tubino [17] have proposed the parametric excitation mechanism to explain why the Millennium Bridge has large lateral oscillations with 0.48 Hz. In their model, it is assumed that the lateral force exerted by pedestrians on a moving bridge is expressed by using a harmonic function with amplitude depending on the bridge displacement. However, their lateral force model is only valid for the case of large lateral oscillations of a bridge. When the lateral oscillations are small at the beginning, the lateral force is a harmonic function with constant amplitude. It is unclear how the lateral force changes its expression. The most advantage of our plane pendulum model is to provide a unified expression of the lateral force induced by pedestrians, by which we can exhibit the whole changing process how lateral oscillations of a suspension footbridge increase from small to large.

Data Availability

The calculated data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors carried out the proofreading of the paper. All authors conceived the study and participated in its design and coordination. All authors read and approved the final paper.

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