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Research Article

Modal Participation Estimated from the Response Correlation Matrix

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In this paper, we are considering the case of estimating the modal participation vectors from the operating response of a structure. Normally, this is done using a fitting technique either in the time domain using the correlation function matrix or in the frequency domain using the spectral density matrix. In this paper, a more simple approach is proposed based on estimating the modal participation from the correlation matrix of the operating responses. For the case of general damping, it is shown how the response correlation matrix is formed by the mode shape matrix and two transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_1$ that contain information about the modal parameters, the generalized modal masses, and the input load spectral density matrix $\mathbf{G}_x$. For the case of real mode shapes, it is shown how the response correlation matrix can be given a simple analytical form where the corresponding real modal participation vectors can be obtained in a simple way. Finally, it is shown how the real version of the modal participation vectors can be used to synthesize empirical spectral density functions.

1. Introduction

Modal participation is a classic subject in traditional experimental modal analysis where the modal parameters are obtained from an experimentally estimated frequency response function (FRF) matrix. Recent examples of FRF-based modal participation are found in [1–3].

In the case of operational modal analysis, where the modal parameters cannot be estimated based on the FRF because the excitation forces are unknown, the modal participation has to be estimated solely from the operating responses. This means using either the correlation function (CF) matrix if the modal identification is done in the time domain or using the spectral density (SD) matrix if the modal identification is done in the frequency domain.

It is a common practice in modal analysis to estimate solely the natural frequencies, damping ratios, and the corresponding mode shape vectors from the vibration measurements. Though the modal mass and modal participation factors are normally not estimated, they also contribute to describe the modal behavior of the tested structures. In [4], it is described how the model participation factor can be used to quantify the participation and, therefore, the importance of each identified mode to the total response of the tested structures.

Another fact that justifies the necessity of estimating the modal participation factors is that it is always a good idea to synthesize the CF and/or the SD function from the estimated modal parameters in order to check the accuracy of these estimates. Furthermore, the state-of-art modal parameter estimation techniques such as those based on maximum likelihood [5] can only be used to estimate/optimize the modal parameters if a good guess of the poles, mode shapes, and modal participation factors are known a priori as discussed, for instance, in [6].

Unlike in classic modal analysis, it is not possible to estimate the modal mass in operational modal analysis. On the other hand, it is perfectly possible to estimate the modal participation factors from output-only vibration measurements. Both in classic and in operational modal analysis, the estimation of the modal participation factors is carried in a linear least squares sense when the poles and mode shape vectors are estimated from the vibration data, as described, for instance, in [7, 8].

In this paper, another approach is proposed to estimate the modal participation factors from the response correlation matrix. The main advantage of such approach with regard to the traditional least squares based ones is that the former estimates the modal participation factors by means of a closed expression. In order to illustrate the performance of the proposed approach from a practical point of view, it is applied to two application examples in the final part of the paper.

2. Theory

Before proceeding to the new idea that we like to propose in this paper, we will shortly introduce the classic ways of obtaining the modal participation from the operating responses. Considering the response vector $\mathbf{y}(t) \in \mathbb{R}^{N_o}$ containing the N_o responses of a linear system at time t, the correlation function (CF) matrix, $\mathbf{R}(\tau) \in \mathbb{R}^{N_o \times N_o}$, can be defined in the classic way [9] as

$$\mathbf{R}(\tau) = \mathbf{E}\left[\mathbf{y}(t)\mathbf{y}^T(t+\tau)\right], \tag{1}$$

with $\tau \in \mathbb{R}$ denoting a *time lag*. Assuming M modes with mode shape vectors $\mathbf{b}_n \in \mathbb{C}^{N_o}$, poles $\lambda_n \in \mathbb{C}$, and a constant input load spectral density (SD) matrix $\mathbf{G}_x \in \mathbb{C}^{N_o \times N_o}$ of the load vector $\mathbf{x}(t) \in \mathbb{R}^{N_o}$ containing N_o inputs, it can be shown (refer to, for instance, [4, 10]) that the analytical solution for the CF matrix is given by

$$\mathbf{R}(\tau) = \begin{cases} 2\pi \sum_{n=1}^{M} \left(\mathbf{A}_n \mathbf{G}_x \mathbf{B}_n e^{-\lambda_n \tau} + \mathbf{A}_n^* \mathbf{G}_x \mathbf{B}_n^* e^{-\lambda_n^* \tau}\right), & \tau \le 0, \\ 2\pi \sum_{n=1}^{M} \left(\mathbf{B}_n \mathbf{G}_x \mathbf{A}_n e^{\lambda_n \tau} + \mathbf{B}_n^* \mathbf{G}_x \mathbf{A}_n^* e^{\lambda_n^* \tau}\right), & \tau \ge 0, \end{cases} \tag{2}$$

where $(\cdot)^*$ denotes the conjugate of a complex quantity and $\mathbf{A}_n \in \mathbb{C}^{N_o \times N_o}$ are the rank one residues defined by the mode shapes and the generalized modal mass a_n:

$$\mathbf{A}_n = \frac{\mathbf{b}_n \mathbf{b}_n^T}{a_n}. \tag{3}$$

The matrices $\mathbf{B}_n \in \mathbb{C}^{N_o \times N_o}$ are a weighted sum of the residual matrices $\mathbf{A}_s$ and $\mathbf{A}_s^*$ given by

$$\mathbf{B}_n = \sum_{s=1}^{M} \left(\frac{\mathbf{A}_s}{-\lambda_n - \lambda_s} + \frac{\mathbf{A}_s^*}{-\lambda_n - \lambda_s^*} \right). \tag{4}$$

In [4, 10], the modal participation vector, $\boldsymbol{\gamma}_n \in \mathbb{C}^{N_o}$, is defined as

$$\boldsymbol{\gamma}_n = \frac{\mathbf{B}_n \mathbf{G}_x \mathbf{b}_n}{a_n}. \tag{5}$$

This eventually leads to the following simplified expressions for the theoretical CF matrix solution:

$$\mathbf{R}(\tau) = \begin{cases} 2\pi \sum_{n=1}^{M} \left(\mathbf{b}_n \boldsymbol{\gamma}_n^T e^{-\lambda_n \tau} + \mathbf{b}_n^* \boldsymbol{\gamma}_n^H e^{-\lambda_n^* \tau}\right), & \tau \le 0, \\ 2\pi \sum_{n=1}^{M} \left(\boldsymbol{\gamma}_n \mathbf{b}_n^T e^{\lambda_n \tau} + \boldsymbol{\gamma}_n^* \mathbf{b}_n^H e^{\lambda_n^* \tau}\right), & \tau \ge 0, \end{cases} \tag{6}$$

where $(\cdot)^H$ denotes the conjugate transpose (Hermitian) of a complex matrix. In [4], it is proposed to estimate the modal participation vectors, $\boldsymbol{\gamma}_n$, either in the time domain fitting the empirical CF function matrices to the theoretical solutions (2) and (6) or in the frequency domain by fitting the empirical SD matrices to the similar theoretical solution for the SD matrix of the response, $\mathbf{G}_y(\omega) \in \mathbb{C}^{N_o \times N_o}$, where $\mathbf{G}_y(\omega)$ is the Fourier transform of $\mathbf{R}(\tau)$. This normally would include either the positive or the negative part of the CF matrix and thus leads to fitting on the half spectral density function matrix.

When the modal parameters including the modal participation vectors are known, the corresponding response SD matrix can easily be found from equation (6) by taking the Fourier transform of both the negative and the positive time part of the CF matrix solution [4]:

$$\mathbf{G}_y(\omega) = \sum_{n=1}^{M} \left(\frac{\mathbf{b}_n \boldsymbol{\gamma}_n^T}{-i\omega - \lambda_n} + \frac{\mathbf{b}_n^* \boldsymbol{\gamma}_n^H}{-i\omega - \lambda_n^*} + \frac{\boldsymbol{\gamma}_n \mathbf{b}_n^T}{i\omega - \lambda_n} + \frac{\boldsymbol{\gamma}_n^* \mathbf{b}_n^H}{i\omega - \lambda_n^*} \right). \tag{7}$$

However, since this solution contains both terms of the type $\mathbf{b}_n \boldsymbol{\gamma}_n^T$ and their respective transpose of the type $\boldsymbol{\gamma}_n \mathbf{b}_n^T$, there is no simple known way to fit the classic SD function matrix to find the modal participation vectors. Therefore, in modal identification with modern parametric frequency-domain identification techniques such as the polyreference least squares frequency-domain (pLSCF) [11, 12], the fitting is carried out by making use of the half spectrum (i.e, the SD computed solely from the CF function matrix with positive *time lags* τ) which consists only of the last two terms in equation (7).

However, in this paper a more simple approach is proposed based on the response correlation matrix. Since the response correlation matrix is the value of the CF matrix $\mathbf{R}(\tau)$ at time lag zero, but also equal to the integral of the SD matrix $\mathbf{G}_y(\omega)$ as expressed by the Parseval equation

$$\mathbf{C} = \mathbf{R}(0) = \int_{-\infty}^{+\infty} \mathbf{G}_y(\omega) \mathrm{d}\omega, \tag{8}$$

the approach can be considered as a common approach for both time domain and frequency domain. In the following section, we will show that the solution given by equations (2) and (6) leads to an expression for the response correlation matrix that is real and symmetric. Furthermore, we will formulate a simple way to estimate the modal participation vectors from the correlation matrix, and finally we will illustrate how this information can be used to synthesize the SD matrix in case of random response data.

2.1. General Form of the Response Correlation Matrix. It is obvious that since the correlation matrix $\mathbf{C} = \mathbf{E}[\mathbf{y}(t)\mathbf{y}^T(t)]$ can now be calculated in two different ways using equations (2) and (6), $\mathbf{C}_+ = \mathbf{R}(0_+)$ and $\mathbf{C}_- = \mathbf{R}(0_-)$, we will start showing that the two different ways of calculation leads to the same result.

From the bottom part of equation (2), we have

$$\mathbf{C}_+ = 2\pi \sum_{n=1}^{M} \left(\mathbf{B}_n \mathbf{G}_x \mathbf{A}_n + \mathbf{B}_n^* \mathbf{G}_x \mathbf{A}_n^*\right). \tag{9}$$

Substituting equation (4) in equation (9) and carrying out the multiplications, we get

$$\begin{aligned}\mathbf{C}_+ &= 2\pi \sum_{n=1}^{M} \left(\sum_{s=1}^{M} \left(\frac{\mathbf{A}_s}{-\lambda_n - \lambda_s} + \frac{\mathbf{A}_s^*}{-\lambda_n - \lambda_s^*} \right) \mathbf{G}_x \mathbf{A}_n \right. \\ &\quad \left. + \sum_{s=1}^{M} \left(\frac{\mathbf{A}_s^*}{-\lambda_n^* - \lambda_s^*} + \frac{\mathbf{A}_s}{-\lambda_n^* - \lambda_s} \right) \mathbf{G}_x \mathbf{A}_n^* \right) \\ &= 2\pi \sum_{n=1}^{M} \sum_{s=1}^{M} \frac{\mathbf{A}_s}{-\lambda_n - \lambda_s} \mathbf{G}_x \mathbf{A}_n + \frac{\mathbf{A}_s}{-\lambda_n^* - \lambda_s} \mathbf{G}_x \mathbf{A}_n^* \\ &\quad + \frac{\mathbf{A}_s^*}{-\lambda_n^* - \lambda_s^*} \mathbf{G}_x \mathbf{A}_n^* + \frac{\mathbf{A}_s^*}{-\lambda_n - \lambda_s^*} \mathbf{G}_x \mathbf{A}_n.\end{aligned} \tag{10}$$

Swapping the summation indexes in equation (10) and contracting the equation back to short form yields

$$\begin{aligned}\mathbf{C}_+ &= 2\pi \sum_{s=1}^{M} \sum_{n=1}^{M} \mathbf{A}_s \mathbf{G}_x \frac{\mathbf{A}_n}{-\lambda_s - \lambda_n} + \mathbf{A}_s \mathbf{G}_x \frac{\mathbf{A}_n^*}{-\lambda_s - \lambda_n^*} \\ &\quad + \mathbf{A}_s^* \mathbf{G}_x \frac{\mathbf{A}_n^*}{-\lambda_s^* - \lambda_n^*} + \mathbf{A}_s^* \mathbf{G}_x \frac{\mathbf{A}_n}{-\lambda_s^* - \lambda_n} \\ &= 2\pi \sum_{s=1}^{M} \mathbf{A}_s \mathbf{G}_x \sum_{n=1}^{M} \left(\frac{\mathbf{A}_n}{-\lambda_s - \lambda_n} + \frac{\mathbf{A}_n^*}{-\lambda_s - \lambda_n^*} \right) \\ &\quad + \mathbf{A}_s^* \mathbf{G}_x \sum_{n=1}^{M} \left(\frac{\mathbf{A}_n^*}{-\lambda_s^* - \lambda_n^*} + \frac{\mathbf{A}_s}{-\lambda_s^* - \lambda_n} \right) \\ &= 2\pi \sum_{s=1}^{M} \left(\mathbf{A}_s \mathbf{G}_x \mathbf{B}_s + \mathbf{A}_s^* \mathbf{G}_x \mathbf{B}_s^* \right),\end{aligned} \tag{11}$$

which actually is equal to what we get from the top part of equation (2) for $\mathbf{R}(0_-)$, meaning that the two formulations $\mathbf{C}_+ = \mathbf{R}(0_+)$ and $\mathbf{C}_- = \mathbf{R}(0_-)$ are identical.

In order to get a little closer to the structure of this theoretical solution for the response correlation matrix, we continue manipulating equations (9) and (10). By substituting equation (3) in these equations and renaming the summation indexes ($s \longrightarrow r$ and $n \longrightarrow c$), we obtain

$$\begin{aligned}\mathbf{C}_+ &= 2\pi \sum_{r,c=1}^{M} \mathbf{b}_r \frac{\mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c}{a_r(-\lambda_r - \lambda_c)a_c} \mathbf{b}_c^T + \mathbf{b}_r^* \frac{\mathbf{b}_r^H \mathbf{G}_x \mathbf{b}_c}{a_r^*(-\lambda_r^* - \lambda_c)a_c} \mathbf{b}_c^T \\ &\quad + \mathbf{b}_r \frac{\mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c^*}{a_r(-\lambda_r - \lambda_c^*)a_c^*} \mathbf{b}_c^H + \mathbf{b}_r^* \frac{\mathbf{b}_r^H \mathbf{G}_x \mathbf{b}_c^*}{a_r^*(-\lambda_r^* - \lambda_c^*)a_c^*} \mathbf{b}_c^H,\end{aligned} \tag{12}$$

or in a more compact form

$$\mathbf{C}_+ = 2\pi \sum_{r,c=1}^{M} \mathbf{b}_r t_{1_{rc}} \mathbf{b}_c^T + \mathbf{b}_r^* t_{2_{rc}} \mathbf{b}_c^T + \mathbf{b}_r t_{3_{rc}} \mathbf{b}_c^H + \mathbf{b}_r^* t_{4_{rc}} \mathbf{b}_c^H, \tag{13}$$

with $t_{i_{rc}} = t_i(r,c) \in \mathbb{C}\,(\forall i \in \{1,2,3,4\})$ given by

$$\begin{aligned} t_{1_{rc}} &= \frac{\mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c}{a_r(-\lambda_r - \lambda_c)a_c}, \\ t_{2_{rc}} &= \frac{\mathbf{b}_r^H \mathbf{G}_x \mathbf{b}_c}{a_r^*(-\lambda_r^* - \lambda_c)a_c}, \\ t_{3_{rc}} &= \frac{\mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c^*}{a_r(-\lambda_r - \lambda_c^*)a_c^*}, \\ t_{4_{rc}} &= \frac{\mathbf{b}_r^H \mathbf{G}_x \mathbf{b}_c^*}{a_r^*(-\lambda_r^* - \lambda_c^*)a_c^*}. \end{aligned} \tag{14}$$

As a side remark, it is worth noticing that we without any loss of generality can remove the generalized modal masses a_n from the equation using the mass scaled mode shapes $\mathbf{b}_n/\sqrt{a_n}$. However, since we later will use mode shapes with arbitrary scaling, we will keep the explicit modal masses in the equations.

Considering the first term in the summation of equation (12), we see that the scalar $t_{1_{rc}} = \mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c / (a_r(-\lambda_r - \lambda_c)a_c)$ can be seen as a matrix element with row index r and column index c and that we can write the summation as the outer product $\mathbf{B}\mathbf{T}_1\mathbf{B}^T$ where $\mathbf{B}$ is the mode shape matrix $\mathbf{B} = [\,\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \cdots \;\; \mathbf{b}_M\,] \in \mathbb{C}^{N_o \times M}$ and

$$\begin{aligned}\mathbf{T}_1 = \left[t_{1_{rc}}\right] = \left[t_1(r,c)\right] &= \begin{bmatrix} t_1(1,1) & \cdots & t_1(1,M) \\ \vdots & \ddots & \vdots \\ t_1(M,1) & \cdots & t_1(M,M) \end{bmatrix} \\ &= \left[\frac{\mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c}{a_r(-\lambda_r - \lambda_c)a_c}\right] \in \mathbb{C}^{M \times M}\end{aligned} \tag{15}$$

is the matrix containing the elements $t_{1_{rc}}\,(\forall r,c \in \{1,2,\ldots,M\})$. Using this notation on all four terms of equation (12), we get

$$\mathbf{C}_+ = 2\pi\left(\mathbf{B}\mathbf{T}_1\mathbf{B}^T + \mathbf{B}^*\mathbf{T}_2\mathbf{B}^T + \mathbf{B}\mathbf{T}_3\mathbf{B}^H + \mathbf{B}^*\mathbf{T}_4\mathbf{B}^H\right), \tag{16}$$

where

$$\begin{aligned}\mathbf{T}_2 &= \left[\frac{\mathbf{b}_r^H \mathbf{G}_x \mathbf{b}_c}{a_r^*(-\lambda_r^* - \lambda_c)a_c}\right] \in \mathbb{C}^{M \times M}, \\ \mathbf{T}_3 &= \left[\frac{\mathbf{b}_r^T \mathbf{G}_x \mathbf{b}_c^*}{a_r(-\lambda_r - \lambda_c^*)a_c^*}\right] \in \mathbb{C}^{M \times M}, \\ \mathbf{T}_4 &= \left[\frac{\mathbf{b}_r^H \mathbf{G}_x \mathbf{b}_c^*}{a_r^*(-\lambda_r^* - \lambda_c^*)a_c^*}\right] \in \mathbb{C}^{M \times M}.\end{aligned} \tag{17}$$

Evaluating the bottom part of equation (6) for $\tau = 0$, we now obtain a similar relation expressed in the modal participation vectors:

$$\begin{aligned}\mathbf{C}_+ &= 2\pi \sum_{n=1}^{M} \left(\boldsymbol{\gamma}_n \mathbf{b}_n^T + \boldsymbol{\gamma}_n^* \mathbf{b}_n^H\right) \\ &= 2\pi\left(\boldsymbol{\Gamma}\mathbf{B}^T + \boldsymbol{\Gamma}^*\mathbf{B}^H\right),\end{aligned} \tag{18}$$

where $\boldsymbol{\Gamma} = [\,\boldsymbol{\gamma}_1 \;\; \boldsymbol{\gamma}_2 \;\; \cdots \;\; \boldsymbol{\gamma}_M\,] \in \mathbb{C}^{N_o \times M}$ is the modal participation vector matrix. Since the two first terms of equation (16) come from the first term of equation (9) and thus equal $\boldsymbol{\Gamma}\mathbf{B}^T$, we can conclude that the modal participation matrix can be expressed by

$$\boldsymbol{\Gamma} = \mathbf{B}\mathbf{T}_1 + \mathbf{B}^*\mathbf{T}_2, \tag{19}$$

and we see that the last term in equation (18) $\boldsymbol{\Gamma}^*\mathbf{B}^H$ equals the last two terms in equation (16).

Taking origin in the top part of equation (2) instead of equation (6) and using the definition of the modal participation matrix given by equation (19) leads to satisfaction of the equation $\mathbf{C}_- = 2\pi(\mathbf{B}\boldsymbol{\Gamma}^T + \mathbf{B}^*\boldsymbol{\Gamma}^H)$, which is the version of equation (18) corresponding to the top part of equation (6).

Finally, we can conclude that equation (19) is a general result for the modal participation matrix and that our final and general result for the response correlation matrix is

$$\boldsymbol{\Gamma} = 2\pi\left[\left(\mathbf{B}\mathbf{T}_1 + \mathbf{B}^*\mathbf{T}_2\right)\mathbf{B}^T + \left(\mathbf{B}^*\mathbf{T}_1^* + \mathbf{B}\mathbf{T}_2^*\right)\mathbf{B}^H\right]. \tag{20}$$

2.2. Correlation Matrix for Proportional Damping. In case of real modes, equation (20) becomes

$$\mathbf{C} = 4\pi\mathbf{B}\mathrm{Re}(\mathbf{T})\mathbf{B}^T, \tag{21}$$

where $\mathrm{Re}(\cdot)$ denotes the real value of a complex quantity and

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2. \tag{22}$$

For proportional damping, the generalized modal mass becomes pure imaginary and is given by

$$a_n = 2i\omega_{dn}m_n, \tag{23}$$

where ω_{dn} is the damped angular natural frequency and m_n is the classic modal mass. Furthermore, it is known (as seen, for instance, in [7]) that from the orthogonality equations, we have in the case of general damping

$$\begin{cases}(\lambda_r + \lambda_c)m_{rc} + c_{rc} = 0, & r \neq c, \\ 2\lambda_n m_n + c_n, & r = c.\end{cases} \tag{24}$$

In the case of proportional damping, the full modal mass and damping matrices $[m_{rc}]$ and $[c_{rc}]$ in equation (24) reduce to the corresponding diagonal forms $[m_n]$ and $[c_n]$. Combining $\mathbf{T}_1$ and $\mathbf{T}_2$ from equations (15) and (17) with equation (23), we get

$$\begin{aligned}\mathbf{T}_1 &= \left[\frac{\mathbf{b}_r^T\mathbf{G}_x\mathbf{b}_c}{4\omega_{dr}m_r\omega_{dc}m_c}\,\frac{1}{\lambda_r + \lambda_c}\right], \\ \mathbf{T}_2 &= -\left[\frac{\mathbf{b}_r^T\mathbf{G}_x\mathbf{b}_c}{4\omega_{dr}m_r\omega_{dc}m_c}\,\frac{1}{\lambda_r^* + \lambda_c}\right],\end{aligned} \tag{25}$$

which leads to the following expression for the matrix $\mathbf{T}$:

$$\mathbf{T} = \left[\frac{\mathbf{b}_r^T\mathbf{G}_x\mathbf{b}_c}{4\omega_{dr}m_r\omega_{dc}m_c}\left(\frac{1}{\lambda_r + \lambda_c} - \frac{1}{\lambda_r^* + \lambda_c}\right)\right]. \tag{26}$$

If we want to obtain the real version in equation (21),

$$\mathrm{Re}(\mathbf{T}) = \left[\frac{\mathbf{b}_r^T\mathbf{G}_x\mathbf{b}_c}{4\omega_{dr}m_r\omega_{dc}m_c}\mathrm{Re}\left(\frac{1}{\lambda_r + \lambda_c} - \frac{1}{\lambda_r^* + \lambda_c}\right)\right]. \tag{27}$$

Equation (27) can also be obtained from equation (16). By combining equation (16) with equation (23) and assuming real mode shapes, it follows that

$$\begin{aligned}\mathbf{C} = 2\pi\Bigg\{\mathbf{B}\Bigg[&\frac{\mathbf{b}_r^T\mathbf{G}_x\mathbf{b}_c}{4\omega_{dr}m_r\omega_{dc}m_c}\Bigg(\frac{1}{\lambda_r + \lambda_c} - \frac{1}{\lambda_r^* + \lambda_c} - \frac{1}{\lambda_r + \lambda_c^*} \\ &- \frac{1}{\lambda_r^* + \lambda_c^*}\Bigg)\Bigg]\mathbf{B}^T\Bigg\}.\end{aligned} \tag{28}$$

Now, combining equation (28) with equation (21) leads directly to equation (27). It is worth noticing that having an estimate of the matrix $\mathbf{T}$ might be helpful in estimating the modal masses if the input SD matrix $\mathbf{G}_x$ is known or it might be helpful estimating the input matrix if the modal masses are known.

It is also worth checking the result against the classic solution for a 1-degree-of-freedom (1-DOF) system with the pole λ. In this case, the mode shape reduces to 1, the input SD matrix $\mathbf{G}_x$ becomes a scalar, and the correlation matrix $\mathbf{C}$ reduces to the variance σ_y^2 of the response of the 1-DOF system. Rewriting equations (21) and (27) for 1-DOF system and combining the resulting equations, we get

$$\sigma_y^2 = \pi\left[\frac{\mathbf{G}_x}{\omega_d^2 m^2}\mathrm{Re}\left(\frac{1}{2\lambda} + \frac{1}{2\mathrm{Re}(\lambda)}\right)\right]. \tag{29}$$

After some reduction using that the classic definition of the pole $\lambda = -\zeta\omega_0 + i\omega_d$, we obtain

$$\sigma_y^2 = \frac{\pi\mathbf{G}_x}{2\zeta m^2\omega_0^3} = \frac{\pi\mathbf{G}_x}{ck}, \tag{30}$$

where c is the damping coefficient and k is the stiffness coefficient in the classic 1-DOF differential equation. This is the solution known from the literature (see example 1 in chapter 8 of [13]).

At this point, it is worth highlighting that, similar to equation (20), equation (21) is only a general definition of the modal participation $\boldsymbol{\Gamma}$. In reality, since the load spectral density, $\mathbf{G}_x$, is not known in an output-only vibration test, the modal participation cannot be computed with this expression. Instead, a new expression needs to be formulated

to estimate the modal participation by making use of the covariance matrix, **C**. This is elaborated in the following section.

3. Estimating the Modal Participation from the Response CF Matrix (Proposed Approach)

In this section, we will consider several issues of practical interest using the expressions of the previous section. First let us simplify equation (21) by writing

$$\mathbf{C} = 4\pi \mathbf{B}\mathbf{T}_r\mathbf{B}^T, \tag{31}$$

where $\mathbf{T}_r = \mathrm{Re}(\mathbf{T})$. Similarly, we can now write equation (18) for real modes as

$$\mathbf{C} = 2\pi\left(\mathbf{\Gamma}\mathbf{B}^T + \mathbf{\Gamma}^*\mathbf{B}^H\right) = 4\pi\mathbf{\Gamma}_r\mathbf{B}^T, \tag{32}$$

where $\mathbf{\Gamma}_r = \mathrm{Re}(\mathbf{\Gamma})$, so that

$$\mathbf{\Gamma}_r = \mathbf{B}\mathbf{T}_r. \tag{33}$$

In case we have estimated all modal parameters from the operating responses of a structure and now want to estimate the modal participation from the response correlation matrix, the approach is simple. For a full set of modes, where we know the square mode shape matrix **B** and the response correlation matrix **C**, the real version of the modal participation matrix is simply found by solving equation (32) for $\mathbf{\Gamma}_r$, which gives

$$\mathbf{\Gamma}_r = \frac{1}{4\pi}\mathbf{C}\mathbf{B}^{-T}, \tag{34}$$

where $\mathbf{B}^{-T}$ is the inverse of the transpose mode shape matrix. However, when using experimental data, both the mode shape matrix and the response correlation matrix are estimated quantities and thus influenced by noise. This means that equations (31)–(33) can only be approximately satisfied, and we have to solve the equations in a least square sense.

If the number of modes is smaller than the number of degrees of freedom (DOFs), which is often the case in experimental dynamics, then we have the preferred situation of an over determinate problem, and we can then use a least square solution. If we start with equation (31) and solve it for $\mathbf{T}_r$, we obtain

$$\widehat{\mathbf{T}}_r = \frac{1}{4\pi}\mathbf{B}^+\mathbf{C}\mathbf{B}^{+T}, \tag{35}$$

where $\widehat{\mathbf{T}}_r$ is the least square estimate $\mathbf{T}_r$ and $\mathbf{B}^+$ and $\mathbf{B}^{+T}$ are the pseudo inverse of the mode shape matrix and its transpose, respectively. Substituting this solution into equation (31) defines the smoothed version of the experimentally obtained response correlation matrix:

$$\widehat{\mathbf{C}} = 4\pi\mathbf{B}\widehat{\mathbf{T}}_r\mathbf{B}^T. \tag{36}$$

Furthermore, using equation (33) defines the estimated modal participation matrix as

$$\widehat{\mathbf{\Gamma}}_r = \mathbf{B}\widehat{\mathbf{T}}_r. \tag{37}$$

We could also start with equation (32), and we now get the following estimate:

$$\widehat{\mathbf{\Gamma}}_r = \frac{1}{4\pi}\mathbf{C}\mathbf{B}^{+T}, \tag{38}$$

which is the least square equivalent to the solution given by equation (34). The two estimates given by equations (37) and (38) are only the same if we use the smoothed version of the response correlation matrix in equation (38). In this case, substituting equation (36) into equation (38), we obtain

$$\widehat{\mathbf{\Gamma}}_r = \mathbf{B}\widehat{\mathbf{T}}_r\mathbf{B}^T\mathbf{B}^{+T} = \mathbf{B}\widehat{\mathbf{T}}_r. \tag{39}$$

If there is some noise of significance on the response correlation matrix estimate, it should be considered to use the smoothed estimate.

In the cases where the number of modes (M) is equal or greater than measured DOFs (N_o), we might need to look at ways of reducing the number of modes to secure an overdetermined problem so that we can use the abovementioned least square solutions. We can do that using a (frequency) band-by-band approach.

Another reason for using such approach might be that we like to estimate the input SD matrix $\mathbf{G}_x$ or its modal load counterpart:

$$\mathbf{B}^T\mathbf{G}_x\mathbf{B} = \left[\mathbf{b}_r^T\mathbf{G}_x\mathbf{b}_c\right]. \tag{40}$$

Also, we like to be able to obtain different estimates of $\mathbf{G}_x$ or $\mathbf{B}^T\mathbf{G}_x\mathbf{B}$ in different frequency bands. We solve this problem by generalizing equation (5) to the case where the modal participation vectors are frequency dependent $\boldsymbol{\gamma}_n = \boldsymbol{\gamma}_n(\omega)$:

$$\boldsymbol{\gamma}_n(\omega) = \frac{\mathbf{B}_n\mathbf{G}_x\mathbf{b}_n}{a_n}. \tag{41}$$

We can now think about modelling a general response SD matrix similar to that given in equation (7) using the frequency dependent modal participation vectors instead in the constant ones. The band-by-band approach now materializes by using the Parseval equation given by equation (8) but now formulated for a certain frequency band

$$\Delta B = [\omega; \omega + \Delta\omega]. \tag{42}$$

Also, we have

$$\Delta\mathbf{C} = 2\mathrm{Re}\left(\int_{\Delta B}\mathbf{G}_y(\omega)\mathrm{d}\omega\right). \tag{43}$$

Here, we have used that we need to integrate over the frequency band and its negative frequency counterpart, which provides the complex conjugate of the positive frequency integration, so by adding the two terms, we get the real result as given by equation (43).

Thus, if we have a frequency band where a number of modes are dominating and we integrate over this frequency band as given by equation (43), then $\Delta\mathbf{C}$ is approximately the contribution to the response correlation matrix from these modes. $\Delta\mathbf{C}$ is exactly the contribution to the response correlation matrix from the frequency band ΔB but only approximately the contribution from the considered modes because the response from these modes covers the whole

frequency band, but we will assume that the response from the modes is negligible outside of ΔB.

Finally, let us discuss how noise can be reduced using the Parseval equation integrating the response SD matrix as given by equations (8) and (43). It is well known (as seen, for instance, in [14]) that the physics in the SD matrix can be decoupled from noise using a singular value decomposition like

$$\mathbf{G}_y(\omega) = \mathbf{U}(\omega)\mathbf{S}(\omega)\mathbf{U}^H(\omega), \tag{44}$$

where the singular values in the diagonal matrix $\mathbf{S}(\omega)$ contain information about the modal coordinates in the frequency domain, and the singular vectors of the matrix $\mathbf{U}(\omega)$ contain information about the mode shapes. Furthermore, it is known that the large singular values (and the corresponding singular vectors) represent the physics and lower singular values represent the noise.

So, if we are considering N_m number of modes (in the abovementioned frequency band), then we can reduce the noise by keeping only the first n singular values of $\mathbf{S}(\omega)$ and setting the rest of the singular values to zero. In this case, $n \geq N_m$ because if $n < N_m$, then we will exclude information from one of the mode shapes when estimating $\Delta\mathbf{C}$ using equation (43).

In practice, we will be using a sampled version of the SD matrix $\mathbf{G}_y(n) = \mathbf{G}_y(n\Delta\omega)$. If we are using a sampled version of the theoretical SD matrix, then we obtain

$$\Delta\mathbf{C} = 2\mathrm{Re}\left(\sum_{n\Delta\omega\in\Delta B}\mathbf{G}_y(n)\Delta\omega\right) = 2\Delta\omega\left(\sum_{n\Delta\omega\in\Delta B}\mathrm{Re}\left(\mathbf{G}_y(n)\right)\right), \tag{45}$$

where $\Delta\omega$ is the frequency resolution measured in rad/s, given by $\Delta\omega = 2\pi\Delta f$, where Δf is the traditional frequency resolution measured in Hertz.

4. Simulation Case

We are considering a case with 5 modes and modal parameters as shown in Table 1; all modal masses are assumed equal to unity. We are assuming a time step $\Delta t = 0.103$ s corresponding to the frequency resolution $\Delta f = 0.0095$ Hz or $\Delta\omega = 0.0596$ rad/s for the assumed time domain segment size equal to $N_s = 1024$ data points. This corresponds to the Nyquist frequency $f_\nu = 4.8544$ Hz. In the following, the approach introduced in Section 3 is applied to estimate the modal participation both from the exact and empirical SD matrices simulated with the modal parameters shown in Table 1.

4.1. Estimation Using the Exact SD Matrix. The exact complex modal participation vectors are then calculated according to equation (5), and the exact real modal participation vectors are calculated according to equations (27) and (33). A full band estimate of $\mathbf{C}$ is found according to equation (45), i.e, by using numerical integration of the response SD matrix over the Nyquist band and subsequently by estimating the modal participation for all five modes

Table 1: Modal participation vector estimates for the first mode.

Parameter	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
Natural frequency (Hz)	1.0	1.2	3.0	3.2	4.0
Damping ratio (%)	2.0	1.0	1.5	1.0	0.5
Mode shape (mass scaled)	−0.3073	−0.4975	−0.4028	0.2276	−17.240
	−11.535	0.6350	−0.2009	−0.9302	−10.723
	0.5554	−0.0830	−0.580	−0.6404	0.2468
	0.2995	−0.4454	−0.766	−0.1306	0.2896
	−0.0115	0.8423	17.276	0.5856	0.3342
	−0.0200	−0.1401	11.322	−0.9856	−0.0964

using equations (35) and (37). Similarly, a band-by-band estimate is found by using partial integration according to equation (45) over the three bands

(i) ΔB_1, containing the 2 first modes: $f \in [0 : 2.08]$ Hz

(ii) ΔB_2, containing modes 3 and 4: $f \in [2.08 : 3.59]$ Hz

(iii) ΔB_3, containing mode 5: $f \in [3.59 : 4.85]$ Hz

to obtain the contributions to the correlation matrix from each of the bands and then by using equations (35) and (37) for each of the correlation matrix contributions. When the modal participation vectors have been estimated, the resulting synthesized SD matrix is obtained from equation (7).

The resulting SD plots are shown in Figure 1, where the top and middle plots are based on the full band estimate of $\mathbf{C}$ and the bottom plot is based on the band-by-band estimates. In the top plot, the singular values of the SD matrix synthesized with the estimated modal participation is compared to those of the exact SD matrix, whereas in the two bottom plots, this comparison is illustrated in terms of the cross spectral density $g_{12}(f) = \mathbf{G}_y(1,2)(f)$.

In general, we can see that the real modal participation vector estimates of the spectral densities are quite good. We see from the single density plots (2 bottom plots of the Figure 1) that the estimates mainly influence the zeros of the SD function, but since it is known that the zeros do not have much physical meaning for SD functions, this should not be of any concern. This influence is mainly due to the fact that the zero estimates are more sensitive to the loss of mode shape phase and, in turn, to the assumption of real mode shapes. Finally, we see that the difference between the full band and band-by-band estimates is quite small—only the zeros are moving a bit.

Table 2 shows the values of the modal participation for the first mode. We see that the real modal participation vector is just the real value of the vector, and the full band and band-by-band estimates show only small deviation from the exact real vector.

4.2. Estimation Using the SD Matrix Estimated from the Vibration Responses. In order to illustrate the application of the approach introduced in Section 3 from an experimental perspective, the modal participation was also computed using the empirical SD matrix estimated from the simulated responses.

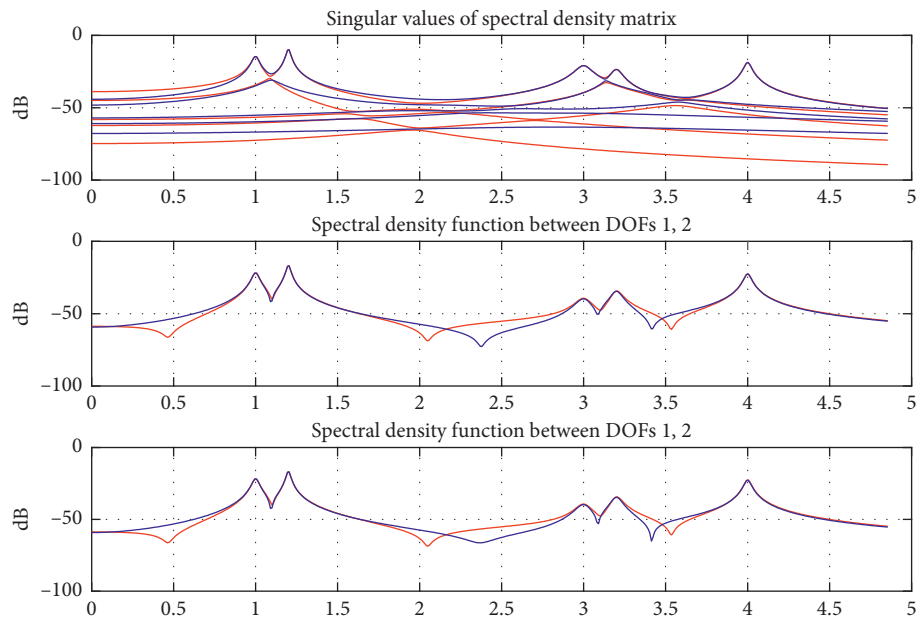

Figure 1: Exact density functions of the simulation case (red) and estimated density functions (blue). Top plot shows the singular values of the spectral density matrix, $\mathbf{S}(\omega)$, whereas the two bottom plots show the cross spectral density function, $g_{12}(f) = \mathbf{G}_y(1,2)(f)$, between the first and second DOFs. The middle plot shows the full band estimate, and the bottom plot shows the band-by-band estimate.

Table 2: Modal participation vector estimates for the first mode.

Exact complex	Exact real	Full band estimate	First band estimate
$-0.3670 - 0.0557i$	-0.3670	-0.3668	-0.3728
$-1.4270 + 0.1262i$	-14.270	-14.266	-14.283
$0.6827 - 0.0264i$	0.6827	0.6826	0.6805
$0.3762 - 0.0754i$	0.3762	0.3760	0.3736
$-0.0309 + 0.1238i$	-0.0309	-0.0308	-0.0234
$-0.0221 + 0.0005i$	-0.0221	-0.0219	-0.0232

These responses were obtained using Gaussian white noise uncorrelated univariate inputs in all DOFs with 32 k data points. The empirical SD matrix is found using Welch averaging with 50% overlap with 1024 data in each segment. In this case, the correlation matrix is estimated as a full band estimate from the SD matrix estimate just like in the previous application example. The results are shown in Figure 2. In this case, where the average of the empirical (i.e., estimated) spectral density is equal to the correlation matrix, the theoretical SD matrix obtained from equation (7) has to be divided by the total band width measured in rad/s $N_s \Delta\omega = 2\pi f_s$ in order to provide a fit to the empirical spectral densities.

The obtained result is shown in Figure 2. Also in this figure, the singular values of the SD matrix estimated from the simulated responses are compared to those from SD matrix synthesized with the estimated modal participation in the top plot, whereas the bottom plot shows this comparison in terms of the cross spectral density $g_{12}(f) = \mathbf{G}_y(1,2)(f)$. Again, we see that the fit is quite good and only the zeros of the single SD function show some deviation.

Many different simulation cases have been carried out, and the results shown here are typical. Cases with more closely spaced modes do not show results that are different in nature.

5. Application on the Heritage Court Tower Data

In order to illustrate the approach proposed in Section 3 to estimate the modal participation factors, the vibration responses of the Heritage Court Tower (HCT) depicted in Figure 3 are used as primary data. The HCT vibration responses are a well-known set of data in the literature of operational modal analysis.

The original publication of the ambient vibration test of the HCT is found in [15], and a full operational modal analysis was presented at the same year of this publication by Brincker and Andersen [16]. The case is also one of the examples in the text book by Brincker and Ventura [4], and the data are a part of the associated Matlab toolbox [17].

The HCT data consist of four datasets, and the first datasets to be considered here have six measurement channels (remaining datasets have eight measurements channels). We will use the modal parameters estimated by example 9.1 in [4], where identification is carried out using the time-domain polyreference technique [18, 19].

In this section, we will just synthesize the first three modes of dataset 1 using equation (45), where we have estimated the partial response correlation matrix by numerical integration from 0.5 Hz up to 3.0 Hz. The result is shown in Figure 4. The sampling time step was 0.025 s corresponding to a Nyquist frequency of 50 Hz. Spectral density was estimated by Welch averaging using 2048 data points in the data segments. This gives only a few averages and thus a relatively high estimation error.

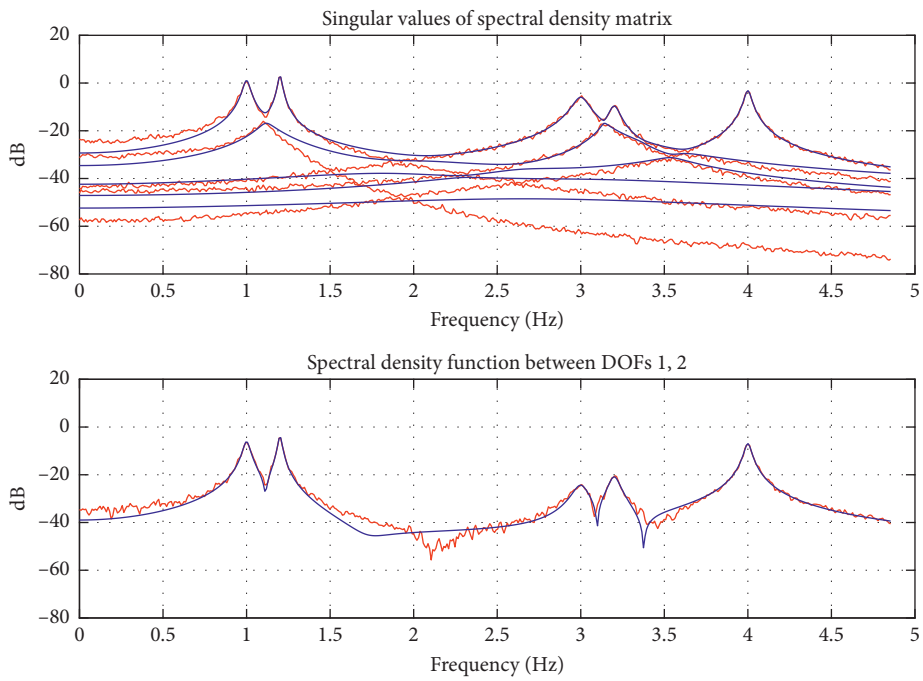

Figure 2: Estimated empirical density functions of the simulation case (red) and synthesized density functions (blue). Top plot shows the singular values, $\mathbf{S}(\omega)$, of the spectral density matrix, whereas the bottom plot shows the cross spectral density function, $g_{12}(f) = \mathbf{G}_y(1,2)(f)$, between the first and second DOFs.

(a) (b)

Figure 3: North (a) and east (b) building faces of the Heritage Court Tower in Vancouver, Canada.

Again, we see that the fit is quite good, both for the singular value plot and for single SD function plot. Even though the experimental data might look quite noisy, the obvious noise content is not due to measurement noise, but nearly solely due to the abovementioned estimation error. Therefore, in this case, noise cannot be reduced by cancelling of lowest singular values as described in equation (44), and thus, the raw SD matrix has been used as the basis of the this analysis.

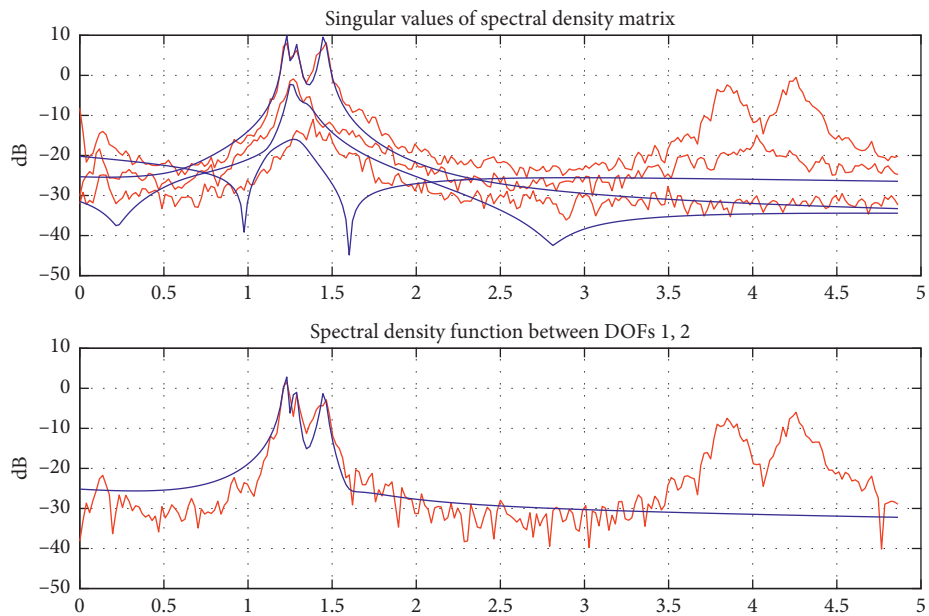

Figure 4: Estimated empirical density functions of the Heritage Court Tower case (red) and synthesized density functions (blue). Top plot shows the singular values of the spectral density matrix, $\mathbf{S}(\omega)$, and the bottom plot shows the cross spectral density function, $g_{12}(f) = \mathbf{G}_y(1,2)(f)$, between the first and second DOFs.

6. Conclusions

We have shown that two matrices can be defined, a symmetric matrix $\mathbf{T}_1$ and a Hermitian matrix $\mathbf{T}_2$, so that in the case of general damping, the modal participation matrix $\mathbf{\Gamma}$ can be obtained as $\mathbf{\Gamma} = \mathbf{B}\mathbf{T}_1 + \mathbf{B}^*\mathbf{T}_2$, where $\mathbf{B}$ is the mode shape matrix of the considered system.

A general expression for the response correlation matrix $\mathbf{C}$ has been formulated based on the above-mentioned matrices $\mathbf{T}_1$ and $\mathbf{T}_2$. In case of real mode shapes, this expressions turn into the simple relation $\mathbf{C} = 4\pi\mathbf{B}\mathbf{T}_r\mathbf{B}^T$, where $\mathbf{T}_r$ is the real part of the sum of the two matrices $\mathbf{T}_1$ and $\mathbf{T}_2$. This also naturally leads to a corresponding real version of the participation matrix given by $\mathbf{\Gamma}_r = \mathbf{B}\mathbf{T}_r$, which has the right physical interpretation since it relates the response correlation matrix with the mode shape matrix in a simple way as $\mathbf{C} = 4\pi\mathbf{\Gamma}_r\mathbf{B}^T$.

Finally, it has been shown how the real version of modal participation matrix can be estimated from the response correlation matrix and how the so-estimated modal participation vectors can be used to synthesize empirical spectral density functions. This has been illustrated both on simulated as well as real data.

Data Availability

The vibration data of the HCT used to support this study are part of the OMA toolbox used in [4] and can be downloaded at http://www.brinckerdynamics.com/oma-toolbox/.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] B. Schwartz, S. Richardson, and M. Richardson, "Curve fitting analytical mode shapes to experimental data: topics in modal analysis & testing," in *Conference Proceedings of the Society for Experimental Mechanics Series*, vol. 10, Orlando, FL, USA, October 2016.

[2] W. Fladung, "Implementation of an SDOF parameter estimation—method with generalized residuals: topics in modal analysis I," in *Conference Proceedings of the Society for Experimental Mechanics Series*, vol. 5, Jacksonville, FL, USA, March 2012.

[3] J. M. Liu, S. Shen, M. Ying, and S. W. Dong, "The optimization and autonomous identification of modal parameters: modal analysis topics," in *Conference Proceedings of the Society for Experimental Mechanics Series*, vol. 3, Jacksonville, FL, USA, February 2011.

[4] R. Brincker and C. E. Ventura, *Introduction to Operational Modal Analysis*, Wiley, Hoboken, NJ, USA, 1st edition, 2015.

[5] M. El-Kafafy, T. De-Troyer, B. Peeters, and P. Guillaume, "Fast maximum-likelihood identification of modal parameters with uncertainty intervals: a modal model-based formulation," *Mechanical Systems and Signal Processing*, vol. 37, no. 1-2, pp. 422–439, 2013.

[6] S. Diord, F. Magalhães, A. Cunha, and E. Caetano, "High spatial resolution modal identification of a stadium suspension roof: assessment of the estimates uncertainty and of modal contributions," *Engineering Structures*, vol. 135, pp. 117–135, 2017.

[7] W. Heylen, S. Lammens, and P. Sass, *Modal Analysis Theory and Testing*, Kathoelieke Universiteit Leuven, Department of Mechanical Engineering, Leuven, Belgium, 1997.

[8] N. M. M. Maia and J. M. M. Silva, *Theoretical and Experimental Modal Analysis*, Research Studies Press Ltd, Hertfordshire, UK, 2nd edition, 1998.

[9] J. S. Bendat and A. G. Piersol, *Random Data: Analysis and Measurement Procedures*, John Wiley & Sons Inc, Hoboken, NJ, USA, 2010.

[10] R. Brincker, "On the application of correlation function matrices in OMA," *Mechanical Systems and Signal Processing*, vol. 87, pp. 17–22, 2017.

[11] B. Peeters, H. Van-Der-Auweraera, F. Vanhollebeck, and P. Guillaume, "Operational modal analysis for estimating the dynamic properties of a stadium structure during a football game," *Journal of Shock and Vibration*, vol. 14, no. 4, pp. 283–303, 2007.

[12] P. Guillaume, P. Verboven, S. Vanlanduit, H. Van der Auweraer, and B. Peeters, "A poly-reference implementation of the least-squares complex frequency-domain estimator," in *Proceedings of the 21st International Modal Analysis Conference (IMACXXI)*, Kissimmee, FL, USA, February 2003.

[13] D. E. Newland, *An Introduction to Random Vibrations, Spectral & Wavelet Analysis*, Longman, Harlow, UK, 3rd edition, 1997.

[14] R. Brincker, L. Zhang, and P. Andersen, "Modal identification of output-only systems using frequency domain decomposition," *Smart Materials and Structures*, vol. 10, no. 3, pp. 441–445, 2001.

[15] C. E. Ventura and T. Horyna, "Measured and calculated modal characteristics of the heritage court tower in Vancouver, B.C.," in *Proceedings of the XVIII International Modal Analysis Conference (IMAC)*, pp. 1970–1974, San Antonio, TX, USA, February 2000.

[16] R. Brincker and P. Andersen, "Ambient response analysis of the heritage court tower building structure," in *Proceedings of the XVIII International Modal Analysis Conference (IMAC)*, pp. 1081–1094, San Antonio, TX, USA, February 2000.

[17] R. Brincker, "OMA matlab toolbox," 2015, http://www.wiley.com/go/brincker.

[18] H. Vold, J. Kundrat, G. T. Rocklin, and R. Russell, *A Multi-Input Modal Estimation Algorithm for Mini-Computers*, SAE Paper, Pittsburgh, PA, USA, 1982.

[19] H. Vold and G. T. Rocklin, "The numerical implementation of a multi-input modal estimation method for mini-computers," in *Proceedings of the International Modal Analysis Conference (IMAC)*, pp. 542–548, Orlando, FL, USA, November 1982.