Research Article

On a Family of Multivariate Modified Humbert Polynomials

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This paper attempts to present a multivariable extension of generalized Humbert polynomials. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties, and also some special cases for these multivariable polynomials.

1. Introduction

The generalized Humbert polynomials are generated by [1]

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C) t^n, \quad (1)$$

where m is a positive integer and other parameters are unrestricted in general (see also [2, pages 77, 86] and [3, 4]). This definition includes many well-known special polynomials such as Humbert, Louville, Gegenbauer, Legendre, Tchebycheff, Pincherle, and Kinney polynomials.

In this paper, we consider the following multivariable extension of the generalized Humbert polynomials which are completely different from the polynomials introduced in [5]. This class of polynomials is generated by

$$\begin{aligned} & \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha} \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \cdots t_r^{n_r}, \quad (2) \\ & \left(\left| \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right| < 1 \right), \end{aligned}$$

where $\mathbf{x} = (x_1, \dots, x_r)$, $\mathbf{y} = (y_1, \dots, y_r)$, $\mathbf{m} = (m_1, \dots, m_r)$ and m_i ($i = 1, 2, \dots, r$) is a positive integer. It follows from (2) that

$$\begin{aligned} & \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &= \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(\alpha)_{n_1+\dots+n_r-(m_1-1)k_1-\dots-(m_r-1)k_r}}{k_1! \cdots k_r! (n_1 - m_1 k_1)! \cdots (n_r - m_r k_r)!} \\ & \quad \times (-1)^{k_1+\dots+k_r} (m_1 x_1)^{n_1-m_1 k_1} \cdots (m_r x_r)^{n_r-m_r k_r} y_1^{k_1} \cdots y_r^{k_r}, \quad (3) \end{aligned}$$

where $(\lambda)_k := \lambda(\lambda+1) \cdots (\lambda+k-1)$, $(\lambda)_0 := 1$ is the Pochhammer symbol.

The aim of this paper is to derive various families of multilinear and multilateral generating functions and to give several recurrence relations and expansions in the series of orthogonal polynomials for the family of multivariable polynomials given explicitly by (3). We present some special cases of our results and also obtain some other properties for these special cases.

2. Bilinear and Bilateral Generating Functions

In this section, with the help of the similar method as considered in [5–9], we derive several families of bilinear and

bilateral generating functions for the family of multivariable polynomials generated by (2) and given explicitly by (3).

We begin by stating the following theorem.

Theorem 1. Corresponding to an identically nonvanishing function $\Omega_{\mu}(\mathbf{z})$ of s complex variables $z_1, \dots, z_s$ ($s \in \mathbb{N}$) and of complex order μ , let

$$\Lambda_{\mu, \nu}(\mathbf{z}; w) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(\mathbf{z}) w^k, \quad (4)$$

where ($a_k \neq 0$, $\mu, \nu \in \mathbb{C}$); $\mathbf{z} = (z_1, \dots, z_s)$. Then, for $p \in \mathbb{N}$; $\mathbf{x} = (x_1, \dots, x_r)$; $\mathbf{y} = (y_1, \dots, y_r)$; $\mathbf{m} = (m_1, \dots, m_r)$, $m_i \in \mathbb{N}$ ($i = 1, 2, \dots, r$), one has

$$\begin{aligned} & \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} a_k \Theta_{n_1-pk, n_2, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times t_2^{n_2} \dots t_r^{n_r} \Omega_{\mu+\nu k}(\mathbf{z}) \eta^k t_1^{n_1-pk} \\ & = \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha} \Lambda_{\mu, \nu}(\mathbf{z}; \eta) \end{aligned} \quad (5)$$

provided that each member of (5) exists.

Proof. For convenience, let S denote the first member of the assertion (5) of Theorem 1. Replacing n_1 by $n_1 + pk$, we may write that

$$\begin{aligned} S &= \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k=0}^{\infty} a_k \Theta_{n_1, n_2, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \Omega_{\mu+\nu k}(\mathbf{z}) \eta^k t_1^{n_1} \dots t_r^{n_r} \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, n_2, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(\mathbf{z}) \eta^k \\ &= \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha} \Lambda_{\mu, \nu}(\mathbf{z}; \eta), \end{aligned} \quad (6)$$

which completes the proof. $\square$

By using a similar idea, we also get the next result immediately.

Theorem 2. For a nonvanishing function $\varphi_{\lambda_1, \dots, \lambda_r}(\mathbf{z})$ of s complex variables $z_1, \dots, z_s$ ($s \in \mathbb{N}$) and for $p_i \in \mathbb{N}$, $\lambda_i, \eta_i \in \mathbb{C}$, $\mathbf{z} = (z_1, \dots, z_s)$, $\mathbf{w} = (w_1, \dots, w_r)$, $\lambda := (\lambda_1, \dots, \lambda_r)$, $\eta := (\eta_1, \dots, \eta_r)$ let

$$\begin{aligned} & \Omega_{\lambda, \eta, \alpha, \beta, \mathbf{m}}^{\mathbf{n}, \mathbf{p}}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \mathbf{w}) \\ & := \sum_{k_1=0}^{[n_1/p_1]} \dots \sum_{k_r=0}^{[n_r/p_r]} a_{k_1, \dots, k_r} \Theta_{n_1-p_1 k_1, \dots, n_r-p_r k_r}^{(\alpha+\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times \varphi_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}(\mathbf{z}) w_1^{k_1} \dots w_r^{k_r}, \end{aligned} \quad (7)$$

where $a_{k_1, \dots, k_r} \neq 0$; $n_i, (i = 1, 2, \dots, r) \in \mathbb{N}_0$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then, one has

$$\begin{aligned} & \sum_{l_1=0}^{n_1} \dots \sum_{l_r=0}^{n_r} \sum_{k_1=0}^{[l_1/p_1]} \dots \sum_{k_r=0}^{[l_r/p_r]} a_{k_1, \dots, k_r} \Theta_{n_1-l_1, \dots, n_r-l_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times \Theta_{l_1-p_1 k_1, \dots, l_r-p_r k_r}^{(\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times \varphi_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}(\mathbf{z}) \\ & \quad \times w_1^{k_1} \dots w_r^{k_r} \\ & = \Omega_{\lambda, \eta, \alpha, \beta, \mathbf{m}}^{\mathbf{n}, \mathbf{p}}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \mathbf{w}) \end{aligned} \quad (8)$$

provided that each member of (8) exists.

3. Special Cases

As an application of the above theorems, when the multivariable function $\Omega_{\mu+\nu k}(\mathbf{z})$, $\mathbf{z} = (z_1, \dots, z_s)$, $k \in \mathbb{N}_0$, $s \in \mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$s = r, \quad \Omega_{\mu+\nu k}(\mathbf{z}) = g_{\mu+\nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) \quad (9)$$

in Theorem 1, where the Chan-Chyan-Srivastava polynomials $g_{\mu+\nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z})$ [10] are generated by

$$\begin{aligned} & \prod_{i=1}^r (1 - x_i t)^{-\alpha_i} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n \\ & (\alpha_i \in \mathbb{C} (i = 1, 2, \dots, r); |t| < \min \{|x_1|^{-1}, \dots, |x_r|^{-1}\}). \end{aligned} \quad (10)$$

We are thus led to the following result which provides a class of bilateral generating functions for the Chan-Chyan-Srivastava polynomials and the family of multivariable polynomials given explicitly by (3).

Corollary 3. If $\Lambda_{\mu, \nu}(\mathbf{z}; w) := \sum_{k=0}^{\infty} a_k g_{\mu+\nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) w^k$, $a_k \neq 0$, $\mu, \nu \in \mathbb{C}$, $\mathbf{z} = (z_1, \dots, z_r)$ then, one has

$$\begin{aligned} & \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} a_k \Theta_{n_1-pk, n_2, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times t_2^{n_2} \dots t_r^{n_r} g_{\mu+\nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) \eta^k t_1^{n_1-pk} \\ & = \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha} \Lambda_{\mu, \nu}(\mathbf{z}; \eta) \end{aligned} \quad (11)$$

provided that each member of (11) exists.

Remark 4. Using the generating relation (10) for the Chan-Chyan-Srivastava polynomials and getting $a_k = 1$, $\mu = 0$, $\nu = 1$, we find that

$$\begin{aligned} & \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} \Theta_{n_1-pk, n_2, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_2^{n_2} \dots t_r^{n_r} g_k^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) \\ & \quad \times \eta_1^k t_1^{n_1-pk} \\ & = \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i})\right)^{-\alpha} \prod_{i=1}^r (1 - z_i \eta_i)^{-\beta_i}, \\ & |\eta| < \min \{|z_1|^{-1}, \dots, |z_r|^{-1}\}, \left| \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right| < 1. \end{aligned} \quad (12)$$

On the other hand, choosing $s = 2r$ and

$$\varphi_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}(\mathbf{z}) = \Theta_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}^{(\gamma)}(\mathbf{u}, \mathbf{v}, \mathbf{m}), \quad (13)$$

λ_i, η_i ($i = 1, 2, \dots, r$) $\in \mathbb{N}_0$, $\mathbf{u} = (u_1, \dots, u_r)$, $\mathbf{v} = (v_1, \dots, v_r)$ in Theorem 2, we obtain the following class of bilinear generating functions for the family of multivariable polynomials given explicitly by (3).

Corollary 5. If

$$\begin{aligned} & \Lambda_{\lambda, \eta, \alpha, \beta, \gamma, \mathbf{m}}^{\mathbf{n}, \mathbf{p}}(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v}; \mathbf{w}) \\ & := \sum_{k_1=0}^{[n_1/p_1]} \dots \sum_{k_r=0}^{[n_r/p_r]} a_{k_1, \dots, k_r} \Theta_{n_1-p_1 k_1, \dots, n_r-p_r k_r}^{(\alpha+\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times \Theta_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}^{(\gamma)}(\mathbf{u}, \mathbf{v}, \mathbf{m}) w_1^{k_1} \dots w_r^{k_r}, \end{aligned} \quad (14)$$

where $a_{k_1, \dots, k_r} \neq 0$; $p_i \in \mathbb{N}$; $n_i, \lambda_i, \eta_i \in \mathbb{N}_0$, $i = 1, 2, \dots, r$, then, we get

$$\begin{aligned} & \sum_{l_1=0}^{n_1} \dots \sum_{l_r=0}^{n_r} \sum_{k_1=0}^{[l_1/p_1]} \dots \sum_{k_r=0}^{[l_r/p_r]} a_{k_1, \dots, k_r} \Theta_{n_1-l_1, \dots, n_r-l_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times \Theta_{l_1-p_1 k_1, \dots, l_r-p_r k_r}^{(\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad \times \Theta_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}^{(\gamma)}(\mathbf{u}, \mathbf{v}, \mathbf{m}) \\ & \quad \times w_1^{k_1} \dots w_r^{k_r} \\ & = \Lambda_{\lambda, \eta, \alpha, \beta, \gamma, \mathbf{m}}^{\mathbf{n}, \mathbf{p}}(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v}; \mathbf{w}) \end{aligned} \quad (15)$$

provided that each member of (15) exists.

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable functions $\Omega_{\mu+\nu k}(\mathbf{y})$ and $\varphi_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}(\mathbf{y})$, $\mathbf{y} = (y_1, \dots, y_s)$, ($s \in \mathbb{N}$), are expressed as an appropriate product of several simpler functions, the assertions of Theorems 1 and 2 can be applied in order to derive various families of multilinear and multilateral

generating functions for the family of multivariable polynomials given explicitly by (3).

4. Some Miscellaneous Properties

In this section, we now discuss some further properties of the family of multivariable polynomials given by (3). We start with the following theorems.

Theorem 6. Let $\Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ be a family of functions generated by

$$\begin{aligned} & G(m_1 x_1 t_1 - y_1 t_1^{m_1}, \dots, m_r x_r t_r - y_r t_r^{m_r}) \\ & = \sum_{n_1, \dots, n_r=0}^{\infty} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r}. \end{aligned} \quad (16)$$

The following relations hold:

$$\begin{aligned} n_i \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) & = x_i \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & \quad - y_i \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \end{aligned} \quad (17)$$

for $n_i \geq m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$ and

$$n_i \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = x_i \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \quad (18)$$

for $n_i = 0, 1, \dots, m_i - 2$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$. Also, one finds that

$$\begin{aligned} & m_i x_i \frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) - m_i y_i \frac{\partial}{\partial y_i} \\ & \quad \times \Psi_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ & = -(n_i - m_i + 1) \Psi_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \end{aligned} \quad (19)$$

for $n_i \geq m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$ and

$$\frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = 0 \quad (20)$$

for $n_i = 0, 1, \dots, m_i - 2$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$.

Proof. Fix $i = 1, 2, \dots, r$. Then, by differentiating (16) with respect to x_i and t_i , after making necessary calculations we obtain that

$$\begin{aligned} & \sum_{n_1, \dots, n_r=0}^{\infty} n_i \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\ &= (x_i - y_i t_i^{m_i-1}) \sum_{n_1, \dots, n_r=0}^{\infty} \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\ &= x_i \sum_{n_1, \dots, n_r=0}^{\infty} \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\ &\quad - y_i \sum_{\substack{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_r=0 \\ (n_i \geq m_i-1)}}^{\infty} \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &\quad \times t_1^{n_1} \dots t_r^{n_r}. \end{aligned} \quad (21)$$

Comparing the coefficients of $t_1^{n_1} \dots t_r^{n_r}$, we obtain (17) and (18) for the fixed i . Similarly, if we differentiate (16) with respect to y_i and t_i , we can find the relation (19). $\square$

Theorem 7. If $\Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ is a family of functions generated by (16), then it satisfies the relations

$$\begin{aligned} & -m_i \frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &= \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_{i-1}, n_i-m_i, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \end{aligned} \quad (22)$$

for $n_i \geq m_i$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$ and

$$\frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = 0 \quad (23)$$

for $n_i = 1, 2, \dots, m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$.

Proof. By comparing the derivatives of (16) with respect to x_i and y_i , we have

$$\begin{aligned} & -m_i t_i \sum_{n_1, \dots, n_r=0}^{\infty} \frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\ &= t_i^{m_i} \sum_{n_1, \dots, n_r=0}^{\infty} \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r}, \end{aligned} \quad (24)$$

which implies that

$$\begin{aligned} & -m_i \frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &= \frac{\partial}{\partial x_i} \Psi_{n_1, \dots, n_{i-1}, n_i-m_i, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \end{aligned} \quad (25)$$

for $n_i \geq m_i$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$ and

$$\frac{\partial}{\partial y_i} \Psi_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_r}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = 0 \quad (26)$$

for $n_i = 1, 2, \dots, m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$. Thus, the proof is completed. $\square$

As a result of these theorems, if we choose

$$\begin{aligned} & G(m_1 x_1 t_1 - y_1 t_1^{m_1}, \dots, m_r x_r t_r - y_r t_r^{m_r}) \\ &= \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha}, \end{aligned} \quad (27)$$

then we can give some recurrence relations for the family of multivariable polynomials given explicitly by (3). In the view of Theorem 6, we get

Corollary 8. For the family of multivariable polynomials generated by (2), the following relations hold

$$\begin{aligned} n_i \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) &= x_i \frac{\partial}{\partial x_i} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &\quad - y_i \frac{\partial}{\partial x_i} \Theta_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}), \end{aligned} \quad (28)$$

$$\begin{aligned} & m_i x_i \frac{\partial}{\partial y_i} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) - m_i y_i \frac{\partial}{\partial y_i} \\ &\quad \times \Theta_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &= -(n_i - m_i + 1) \Theta_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \end{aligned} \quad (29)$$

for $n_i \geq m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$. Also, for $n_i = 0, 1, \dots, m_i - 2$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$, we have

$$n_i \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = x_i \frac{\partial}{\partial x_i} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}), \quad (30)$$

$$\frac{\partial}{\partial y_i} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = 0. \quad (31)$$

Corollary 9. By summing the relations given by (28) and (29), respectively, for $i = 1, 2, \dots, r$, we get

$$\begin{aligned} & \sum_{i=1}^r n_i \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &= \sum_{i=1}^r x_i \frac{\partial}{\partial x_i} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\ &\quad - \sum_{i=1}^r y_i \frac{\partial}{\partial x_i} \Theta_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}), \end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^r m_i x_i \frac{\partial}{\partial y_i} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
& - \sum_{i=1}^r m_i y_i \frac{\partial}{\partial y_i} \Theta_{n_1, \dots, n_{i-1}, n_i - m_i + 1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
& = - \sum_{i=1}^r (n_i - m_i + 1) \\
& \quad \times \Theta_{n_1, \dots, n_{i-1}, n_i - m_i + 1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \quad (32)
\end{aligned}$$

for $n_i \geq m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$.

Similarly, as a consequence of Theorem 7, we can give the next result at once.

Corollary 10. Other recurrence relations for the family of multivariable polynomials $\Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ are

$$\begin{aligned}
& - m_i \frac{\partial}{\partial y_i} \Theta_{n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
& = \frac{\partial}{\partial x_i} \Theta_{n_1, \dots, n_{i-1}, n_i - m_i, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \quad (33)
\end{aligned}$$

for $n_i \geq m_i$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$ and

$$\frac{\partial}{\partial y_i} \Theta_{n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = 0 \quad (34)$$

for $n_i = 1, 2, \dots, m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$.

Theorem 11. The generating relation (2) yields the following addition formula for the family of multivariable polynomials given by (3)

$$\begin{aligned}
\Theta_{n_1, \dots, n_r}^{(\alpha+\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \Theta_{n_1-k_1, \dots, n_r-k_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
&\quad \times \Theta_{k_1, \dots, k_r}^{(\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}). \quad (35)
\end{aligned}$$

Proof. From (2), we have

$$\begin{aligned}
& \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, \dots, n_r}^{(\alpha+\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \cdots t_r^{n_r} \\
&= \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-(\alpha+\beta)} \\
&= \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha} \\
&\quad \times \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\beta}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \cdots t_r^{n_r} \\
&\quad \times \sum_{k_1, \dots, k_r=0}^{\infty} \Theta_{k_1, \dots, k_r}^{(\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{k_1} \cdots t_r^{k_r}. \quad (36)
\end{aligned}$$

Replacing n_i by $n_i - k_i$, $i = 1, 2, \dots, r$, the right-hand side of the last equality is

$$\begin{aligned}
& \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \Theta_{n_1-k_1, \dots, n_r-k_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
&\quad \times \Theta_{k_1, \dots, k_r}^{(\beta)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \cdots t_r^{n_r}. \quad (37)
\end{aligned}$$

Comparing the coefficients of $t_1^{n_1} \cdots t_r^{n_r}$ completes the proof. $\square$

We now give expansions of the family of multivariable polynomials $\Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ given explicitly by (3) in series of Legendre, Gegenbauer, Hermite, and Laguerre polynomials.

Theorem 12. Expansions of $\Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ in series of Legendre, Gegenbauer, Hermite, and Laguerre polynomials are as follows

$$\begin{aligned}
& \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
&= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{[(n_i-m_i k_i)/2]} \frac{(2n_i - 2m_i k_i - 4s_i + 1)}{k_i! s_i! (3/2)_{n_i - m_i k_i - s_i}} \right. \\
&\quad \times (-1)^{k_i} y_i^{k_i} P_{n_i - m_i k_i - 2s_i} \left(\frac{m_i x_i}{2} \right) \Bigg\} \\
&\quad \times (\alpha)_{n_1 + \cdots + n_r - (m_1 - 1)k_1 - \cdots - (m_r - 1)k_r},
\end{aligned}$$

$$\begin{aligned}
& \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
&= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{[(n_i-m_i k_i)/2]} \frac{(\nu_i + n_i - m_i k_i - 2s_i)}{k_i! s_i! (\nu_i)_{n_i - m_i k_i - s_i + 1}} \right. \\
&\quad \times (-y_i)^{k_i} C_{n_i - m_i k_i - 2s_i}^{\nu_i} \left(\frac{m_i x_i}{2} \right) \Bigg\} \\
&\quad \times (\alpha)_{n_1 + \cdots + n_r - (m_1 - 1)k_1 - \cdots - (m_r - 1)k_r},
\end{aligned}$$

$$\begin{aligned}
& \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
&= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{[(n_i-m_i k_i)/2]} \frac{(-y_i)^{k_i} H_{n_i - m_i k_i - 2s_i}(m_i x_i/2)}{k_i! s_i! (n_i - m_i k_i - 2s_i)!} \right\} \\
&\quad \times (\alpha)_{n_1 + \cdots + n_r - (m_1 - 1)k_1 - \cdots - (m_r - 1)k_r},
\end{aligned}$$

$$\begin{aligned}
& \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) \\
&= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{n_i-m_i k_i} \frac{(\beta_i+1)_{n_i-m_i k_i} 2^{n_i-m_i k_i} (-1)^{s_i}}{k_i! (n_i-m_i k_i-s_i)! (\beta_i+1)_{s_i}} \right. \\
&\quad \left. \times (-y_i)^{k_i} L_{s_i}^{(\beta_i)} \left(\frac{m_i x_i}{2} \right) \right\} \\
&\quad \times (\alpha)_{n_1+\dots+n_r-(m_1-1)k_1-\dots-(m_r-1)k_r}.
\end{aligned} \quad (38)$$

Proof. By (2), we get

$$\begin{aligned}
& \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\
&= \left(1 - \sum_{i=1}^r (m_i x_i t_i - y_i t_i^{m_i}) \right)^{-\alpha} \\
&= \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\alpha)_{n_1+\dots+n_r+k_1+\dots+k_r}}{k_1! \dots k_r! n_1! \dots n_r!} (-1)^{k_1+\dots+k_r} \\
&\quad \times (m_1 x_1)^{n_1} \dots (m_r x_r)^{n_r} y_1^{k_1} \dots y_r^{k_r} \\
&\quad \times t_1^{n_1+m_1 k_1} \dots t_r^{n_r+m_r k_r}.
\end{aligned} \quad (39)$$

If we use the result in [11, page 181]

$$\frac{(mx)^n}{n!} = \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(2n-4s+1) P_{n-2s}(mx/2)}{s! (3/2)_{n-s}}, \quad (40)$$

we can write that

$$\begin{aligned}
& \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\
&= \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{\infty} \sum_{s_i=0}^{\lfloor n_i/2 \rfloor} \frac{(2n_i-4s_i+1) P_{n_i-2s_i}(m_i x_i/2)}{s_i! k_i! (3/2)_{n_i-s_i}} P_{n_i-2s_i} \left(\frac{m_i x_i}{2} \right) \right. \\
&\quad \left. \times (-y_i)^{k_i} t_i^{n_i+m_i k_i} \right\} (\alpha)_{n_1+\dots+n_r+k_1+\dots+k_r}.
\end{aligned} \quad (41)$$

Getting $n_i - m_i k_i$ instead of n_i in the last equality, we have

$$\begin{aligned}
& \sum_{n_1, \dots, n_r=0}^{\infty} \Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m}) t_1^{n_1} \dots t_r^{n_r} \\
&= \sum_{n_1, \dots, n_r=0}^{\infty} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{(n_i-m_i k_i)/2} \frac{(2n_i-2m_i k_i-4s_i+1)}{s_i! k_i! (3/2)_{n_i-m_i k_i-s_i}} \right. \\
&\quad \left. \times P_{n_i-m_i k_i-2s_i} \left(\frac{m_i x_i}{2} \right) (-y_i)^{k_i} t_i^{n_i} \right\} \\
&\quad \times (\alpha)_{n_1+\dots+n_r-(m_1-1)k_1-\dots-(m_r-1)k_r} \\
&= \sum_{n_1, \dots, n_r=0}^{\infty} \left\{ \prod_{i=1}^r \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{(n_i-m_i k_i)/2} \frac{(2n_i-2m_i k_i-4s_i+1)}{s_i! k_i! (3/2)_{n_i-m_i k_i-s_i}} \right. \\
&\quad \left. \times P_{n_i-m_i k_i-2s_i} \left(\frac{m_i x_i}{2} \right) (-y_i)^{k_i} \right\} \\
&\quad \times (\alpha)_{n_1+\dots+n_r-(m_1-1)k_1-\dots-(m_r-1)k_r} t_1^{n_1} \dots t_r^{n_r}.
\end{aligned} \quad (42)$$

Comparing the coefficients of $t_1^{n_1} \dots t_r^{n_r}$ gives the desired relation.

In a similar manner, in (39), using the following results, respectively, [11, page 283 (36), page 194 (4), page 207 (2)]

$$\begin{aligned}
\frac{(mx)^n}{n!} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(\nu+n-2k) C_{n-2k}^{\nu}(mx/2)}{k! (\nu)_{n+1-k}}, \\
\frac{(mx)^n}{n!} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{H_{n-2k}(mx/2)}{k! (n-2k)!},
\end{aligned} \quad (43)$$

$$\frac{(mx)^n}{n!} = 2^n \sum_{k=0}^n \frac{(-1)^k (\alpha+1)_n L_k^{(\alpha)}(mx/2)}{(n-k)! (\alpha+1)_k},$$

one can easily obtain the other expansions of $\Theta_{n_1, \dots, n_r}^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ in series of Gegenbauer, Hermite, and Laguerre polynomials. $\square$

5. The Special Cases of $\Theta_n^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ and Some Properties

In this section, we discuss some special cases of the family of multivariable polynomials $\Theta_n^{(\alpha)}(\mathbf{x}, \mathbf{y}, \mathbf{m})$ and give their several properties.

5.1. The Case of $m_i = 2$, $y_i = 1$, $i = 1, 2, \dots, r$ in (2). This case gives a multivariable analogue of Gegenbauer polynomials

$$\begin{aligned}
& (1 - 2x_1 t_1 + t_1^2 - \dots - 2x_r t_r + t_r^2)^{-\alpha} \\
&= \sum_{n_1, \dots, n_r=0}^{\infty} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) t_1^{n_1} \dots t_r^{n_r},
\end{aligned} \quad (44)$$

which reduces to two variables Gegenbauer polynomials given by [12] for $r = 2$. Equation (44) yields the following formula:

$$\begin{aligned} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \sum_{k_1=0}^{[n_1/2]} \cdots \sum_{k_r=0}^{[n_r/2]} \left(\binom{(\alpha)_{n_1+\dots+n_r-k_1-\dots-k_r}}{(k_1! \cdots k_r! (n_1-2k_1)! \cdots (n_r-2k_r)!)} \right. \\ &\quad \times (-1)^{k_1+\dots+k_r} (2x_1)^{n_1-2k_1} \cdots (2x_r)^{n_r-2k_r}. \end{aligned} \quad (45)$$

It follows that $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ is a polynomial of degree n_i with respect to the fixed variable x_i ($i = 1, 2, \dots, r$). Thus, $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ is a polynomial of total degree $(n_1 + \dots + n_r)$ with respect to the variables $x_1, \dots, x_r$. Equation (45) also yields

$$\begin{aligned} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \frac{2^{n_1+\dots+n_r} (\alpha)_{n_1+\dots+n_r}}{n_1! \cdots n_r!} x_1^{n_1} \cdots x_r^{n_r} + \Pi(x_1, \dots, x_r), \end{aligned} \quad (46)$$

where $\Pi(x_1, \dots, x_r)$ is a polynomial of degree $(n_1 + \dots + n_r - 2)$ with respect to the variables $x_1, \dots, x_r$. In (44), by getting $x_1 \rightarrow -x_1$ and $t_1 \rightarrow -t_1$, we have

$$C_{n_1, \dots, n_r}^{(\alpha)}(-x_1, x_2, \dots, x_r) = (-1)^{n_1} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r). \quad (47)$$

Similarly, for $i = 1, 2, \dots, r$, we get

$$\begin{aligned} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_r) &= (-1)^{n_i} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r). \end{aligned} \quad (48)$$

Taking $x_i \rightarrow -x_i$ and $t_i \rightarrow -t_i$, $i = 1, 2, \dots, r$, in (44), we obtain

$$C_{n_1, \dots, n_r}^{(\alpha)}(-x_1, \dots, -x_r) = (-1)^{n_1+\dots+n_r} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r). \quad (49)$$

Theorem 13. For the polynomials $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$, one has

$$C_{2n_1, \dots, 2n_r}^{(\alpha)}(0, 0, \dots, 0) = \frac{(-1)^{n_1+\dots+n_r} (\alpha)_{n_1+\dots+n_r}}{n_1! \cdots n_r!} \quad (50)$$

and if at least one of n_i , $i = 1, 2, \dots, r$, is odd; then

$$C_{n_1, \dots, n_r}^{(\alpha)}(0, 0, \dots, 0) = 0. \quad (51)$$

Proof. If we set all $x_i = 0$, $i = 1, 2, \dots, r$ in (44), we have

$$(1 + t_1^2 + \dots + t_r^2)^{-\alpha} = \sum_{n_1, \dots, n_r=0}^{\infty} C_{n_1, \dots, n_r}^{(\alpha)}(0, 0, \dots, 0) t_1^{n_1} \cdots t_r^{n_r}. \quad (52)$$

On the other hand, we get

$$\begin{aligned} (1 + t_1^2 + \dots + t_r^2)^{-\alpha} &= \sum_{n_1, \dots, n_r=0}^{\infty} \frac{(-1)^{n_1+\dots+n_r}}{n_1! \cdots n_r!} (\alpha)_{n_1+\dots+n_r} t_1^{2n_1} \cdots t_r^{2n_r}. \end{aligned} \quad (53)$$

By comparing the coefficients of $t_1^{n_1} \cdots t_r^{n_r}$, we obtain the desired. $\square$

From the theorems and corollaries given in Section 4, we can give some other properties of $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$.

Remark 14. By Corollaries 8 and 9, for the family of multivariable polynomials generated by (44), the following relations:

$$\begin{aligned} n_i C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= x_i \frac{\partial}{\partial x_i} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\ &\quad - \frac{\partial}{\partial x_i} C_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r), \\ \sum_{i=1}^r n_i C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \sum_{i=1}^r x_i \frac{\partial}{\partial x_i} C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\ &\quad - \sum_{i=1}^r \frac{\partial}{\partial x_i} C_{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \end{aligned} \quad (54)$$

hold for $n_i \geq 1$, $i = 1, 2, \dots, r$.

Remark 15. From Theorem 11, the multivariable polynomials $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ satisfy the following addition formula:

$$\begin{aligned} C_{n_1, \dots, n_r}^{(\alpha+\beta)}(x_1, \dots, x_r) &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} C_{n_1-k_1, \dots, n_r-k_r}^{(\alpha)}(x_1, \dots, x_r) \\ &\quad \times C_{k_1, \dots, k_r}^{(\beta)}(x_1, \dots, x_r). \end{aligned} \quad (55)$$

Remark 16. As a result of Theorem 12, expansions of $C_{n_1, \dots, n_r}^{(\alpha)}$ ($x_1, \dots, x_r$) in series of Legendre, Gegenbauer, Hermite, and Laguerre polynomials are as follows:

$$\begin{aligned}
 C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/2]} \sum_{s_i=0}^{[(n_i-2k_i)/2]} \frac{(2n_i - 4k_i - 4s_i + 1)}{k_i! s_i! (3/2)_{n_i-2k_i-s_i}} \right. \\
 &\quad \left. \times (-1)^{k_i} P_{n_i-2k_i-2s_i}(x_i) \right\} \\
 &\quad \times (\alpha)_{n_1+\dots+n_r-k_1-\dots-k_r}, \\
 C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/2]} \sum_{s_i=0}^{[(n_i-2k_i)/2]} \frac{(\nu_i + n_i - 2k_i - 2s_i)}{k_i! s_i! (\nu_i)_{n_i-2k_i-s_i+1}} \right. \\
 &\quad \left. \times (-1)^{k_i} C_{n_i-2k_i-2s_i}^{\nu_i}(x_i) \right\} \\
 &\quad \times (\alpha)_{n_1+\dots+n_r-k_1-\dots-k_r}, \\
 C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/2]} \sum_{s_i=0}^{[(n_i-2k_i)/2]} \frac{(-1)^{k_i} H_{n_i-2k_i-2s_i}(x_i)}{k_i! s_i! (n_i - 2k_i - 2s_i)!} \right\} \\
 &\quad \times (\alpha)_{n_1+\dots+n_r-k_1-\dots-k_r}, \\
 C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/2]} \sum_{s_i=0}^{n_i-2k_i} \frac{(\beta_i + 1)_{n_i-2k_i} 2^{n_i-2k_i} (-1)^{s_i}}{k_i! (n_i - 2k_i - s_i)! (\beta_i + 1)_{s_i}} \right. \\
 &\quad \left. \times (-1)^{k_i} L_{s_i}^{(\beta_i)}(x_i) \right\} \\
 &\quad \times (\alpha)_{n_1+\dots+n_r-k_1-\dots-k_r}.
 \end{aligned} \tag{56}$$

We now give a hypergeometric representation for the multivariable polynomials $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ given by (44).

Theorem 17. The multivariable polynomials $C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ have the following hypergeometric representation:

$$\begin{aligned}
 C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \frac{(\alpha)_{n_1+\dots+n_r}}{n_1! \dots n_r!} (2x_1)^{n_1} \dots (2x_r)^{n_r} \\
 &= \frac{(\alpha)_{n_1+\dots+n_r}}{n_1! \dots n_r!} (2x_1)^{n_1} \dots (2x_r)^{n_r}
 \end{aligned}$$

$$\begin{aligned}
 &\times F_B^{(r)} \left[-\frac{n_1}{2}, \dots, -\frac{n_r}{2}, \frac{-n_1+1}{2}, \dots, \frac{-n_r+1}{2}; \right. \\
 &\quad \left. 1 - \alpha - n_1 - \dots - n_r; \frac{1}{x_1^2}, \dots, \frac{1}{x_r^2} \right], \\
 &\quad \left(\max \left\{ \frac{1}{x_1^2}, \dots, \frac{1}{x_r^2} \right\} < 1 \right),
 \end{aligned} \tag{57}$$

where $F_B^{(r)}$ is a Lauricella function of r -variable defined by [13] (see also [2])

$$\begin{aligned}
 F_B^{(r)}[a_1, \dots, a_r, b_1, \dots, b_r; c; x_1, \dots, x_r] &= \sum_{n_1, \dots, n_r=0}^{\infty} \frac{(a_1)_{n_1} \dots (a_r)_{n_r} (b_1)_{n_1} \dots (b_r)_{n_r} x_1^{n_1} \dots x_r^{n_r}}{(c)_{n_1+\dots+n_r} n_1! \dots n_r!}, \\
 &\quad (\max \{|x_1|, \dots, |x_r|\} < 1).
 \end{aligned} \tag{58}$$

Proof. We have the following results from [11]:

$$\begin{aligned}
 (\alpha)_{n-k} &= \frac{(-1)^k (\alpha)_n}{(1 - \alpha - n)_k}, \quad (-n)_{2k} = 2^{2k} \left(-\frac{n}{2} \right)_k \left(\frac{-n+1}{2} \right)_k, \\
 (n-2k)! &= \frac{n!}{(-n)_{2k}}.
 \end{aligned} \tag{59}$$

In view of these relations, the equality (45) can be written as follows:

$$\begin{aligned}
 C_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \sum_{k_1=0}^{[n_1/2]} \dots \sum_{k_r=0}^{[n_r/2]} \left(\left((\alpha)_{n_1+\dots+n_r} 2^{(k_1+\dots+k_r)} \left(-\frac{n_1}{2} \right)_{k_1} \right. \right. \\
 &\quad \times \left(\frac{-n_1+1}{2} \right)_{k_1} \dots \left(-\frac{n_r}{2} \right)_{k_r} \\
 &\quad \times \left(\frac{-n_r+1}{2} \right)_{k_r} \Big) \\
 &\quad \times \left(k_1! \dots k_r! n_1! \dots n_r! \right. \\
 &\quad \times (1 - \alpha - n_1 - \dots - n_r)_{k_1+\dots+k_r} \Big)^{-1} \\
 &\quad \times (2x_1)^{n_1-2k_1} \dots (2x_r)^{n_r-2k_r} \\
 &= \frac{(\alpha)_{n_1+\dots+n_r}}{n_1! \dots n_r!} (2x_1)^{n_1} \dots (2x_r)^{n_r}
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{k_1=0}^{[n_1/2]} \cdots \sum_{k_r=0}^{[n_r/2]} \left(\left(\left(-\frac{n_1}{2} \right)_{k_1} \left(\frac{-n_1+1}{2} \right)_{k_1} \right. \right. \\
& \quad \cdots \left(-\frac{n_r}{2} \right)_{k_r} \left(\frac{-n_r+1}{2} \right)_{k_r} \\
& \quad \times \left((1-\alpha-n_1-\cdots-n_r)_{k_1+\cdots+k_r} \right. \\
& \quad \left. \left. \times k_1! \cdots k_r! \right)^{-1} \right) \\
& \quad \times \left(\frac{1}{x_1^2} \right)^{k_1} \cdots \left(\frac{1}{x_r^2} \right)^{k_r} \\
& = F_B^{(r)} \left[-\frac{n_1}{2}, \dots, -\frac{n_r}{2}, \frac{-n_1+1}{2}, \dots, \frac{-n_r+1}{2}; \right. \\
& \quad \left. 1-\alpha-n_1-\cdots-n_r; \frac{1}{x_1^2}, \dots, \frac{1}{x_r^2} \right] \\
& \quad \times \frac{(\alpha)_{n_1+\cdots+n_r}}{n_1! \cdots n_r!} (2x_1)^{n_1} \cdots (2x_r)^{n_r},
\end{aligned} \tag{60}$$

where $\max \{1/x_1^2, \dots, 1/x_r^2\} < 1$. The proof is completed. $\square$

Remark 18. For $r = 2$, Theorem 17 reduces to the known result for two variable Gegenbauer polynomials given by [12].

5.2. The Case of $x_i = 0$, $y_i = -x_i$, $i = 1, 2, \dots, r$ in (2). This case yields

$$\begin{aligned}
& (1 - x_1 t_1^{m_1} - x_2 t_2^{m_2} - \cdots - x_r t_r^{m_r})^{-\alpha} \\
& = \sum_{n_1, \dots, n_r=0}^{\infty} U_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) t_1^{n_1} \cdots t_r^{n_r},
\end{aligned} \tag{61}$$

which is a different unification from that in [8].

Remark 19. From Corollaries 8 and 9, the multivariable polynomials $U_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ satisfy

$$\begin{aligned}
& m_i x_i \frac{\partial}{\partial x_i} U_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\
& = (n_i - m_i + 1) U_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r),
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^r m_i x_i \frac{\partial}{\partial x_i} \\
& \quad \times U_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\
& = \sum_{i=1}^r (n_i - m_i + 1) \\
& \quad \times U_{n_1, \dots, n_{i-1}, n_i-m_i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r).
\end{aligned} \tag{62}$$

for $n_i \geq m_i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$.

Remark 20. As result of Theorem 11, the multivariable polynomials $U_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ have the following addition formula:

$$\begin{aligned}
U_{n_1, \dots, n_r}^{(\alpha+\beta)}(x_1, \dots, x_r) & = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} U_{n_1-k_1, \dots, n_r-k_r}^{(\alpha)}(x_1, \dots, x_r) \\
& \quad \times U_{k_1, \dots, k_r}^{(\beta)}(x_1, \dots, x_r).
\end{aligned} \tag{63}$$

5.2.1. The Case of $m_i = 1$, $i = 1, 2, \dots, r$ in Section 5.2. In this case, we get a r -variable analogue of Lagrange polynomials (see [14]) which is different from that in [10]

$$\begin{aligned}
& (1 - x_1 t_1 - \cdots - x_r t_r)^{-\alpha} \\
& = \sum_{n_1, \dots, n_r=0}^{\infty} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) t_1^{n_1} \cdots t_r^{n_r}.
\end{aligned} \tag{64}$$

They are given explicitly by

$$G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) = \frac{(\alpha)_{n_1+\cdots+n_r}}{n_1! \cdots n_r!} x_1^{n_1} \cdots x_r^{n_r}. \tag{65}$$

Remark 21. From Corollaries 8 and 9, the multivariable polynomials $G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ satisfy

$$n_i G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) = x_i \frac{\partial}{\partial x_i} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \tag{66}$$

for $i = 1, 2, \dots, r$ and

$$\begin{aligned}
& \sum_{i=1}^r n_i G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\
& = \sum_{i=1}^r x_i \frac{\partial}{\partial x_i} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r).
\end{aligned} \tag{67}$$

Remark 22. By Theorem 11, the multivariable polynomials $G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ have the following addition formula:

$$\begin{aligned}
G_{n_1, \dots, n_r}^{(\alpha+\beta)}(x_1, \dots, x_r) & = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} G_{n_1-k_1, \dots, n_r-k_r}^{(\alpha)}(x_1, \dots, x_r) \\
& \quad \times G_{k_1, \dots, k_r}^{(\beta)}(x_1, \dots, x_r).
\end{aligned} \tag{68}$$

Remark 23. From Theorem 12, expansions of $G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ in series of Legendre, Gegenbauer, Hermite, and Laguerre polynomials are given by

$$\begin{aligned}
 G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{s_i=0}^{[n_i/2]} \frac{(2n_i - 4s_i + 1)}{s_i! (3/2)_{n_i - s_i}} P_{n_i - 2s_i} \left(\frac{x_i}{2} \right) \right\} \\
 &\quad \times (\alpha)_{n_1 + \dots + n_r}, \\
 G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{s_i=0}^{[n_i/2]} \frac{(\nu_i + n_i - 2s_i)}{s_i! (\nu_i)_{n_i - s_i + 1}} C_{n_i - 2s_i}^{\nu_i} \left(\frac{x_i}{2} \right) \right\} \\
 &\quad \times (\alpha)_{n_1 + \dots + n_r}, \\
 G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{s_i=0}^{[n_i/2]} \frac{H_{n_i - 2s_i}(x_i/2)}{s_i! (n_i - 2s_i)!} \right\} (\alpha)_{n_1 + \dots + n_r}, \\
 G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) &= \prod_{i=1}^r \left\{ \sum_{s_i=0}^{n_i} \frac{(\beta_i + 1)_{n_i} 2^{n_i} (-1)^{s_i}}{(n_i - s_i)! (\beta_i + 1)_{s_i}} L_{s_i}^{(\beta_i)} \left(\frac{x_i}{2} \right) \right\} \\
 &\quad \times (\alpha)_{n_1 + \dots + n_r}.
 \end{aligned} \tag{69}$$

We can discuss some generating functions for the multi-variable polynomials $G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$.

Theorem 24. For the polynomials $G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$, the following generating function holds true for $k \in \mathbb{N}_0$:

$$\begin{aligned}
 \sum_{n_1, \dots, n_r=0}^{\infty} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) [k + n_i]_m z_1^{n_1} \dots z_r^{n_r} \\
 = m! (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} \\
 \times g_m^{(\alpha, -k)} \left(\frac{x_i z_i}{1 - x_1 z_1 - \dots - x_r z_r}, -1 \right),
 \end{aligned} \tag{70}$$

for each $i = 1, 2, \dots, r$, where

$$[k]_m := k(k-1) \dots (k-m+1), \quad m \in \mathbb{N}, [k]_0 = 1 \tag{71}$$

and $g_n^{(\alpha, \beta)}(x, y)$ denotes the classical Lagrange polynomials defined by [14]

$$\begin{aligned}
 (1 - xt)^{-\alpha} (1 - yt)^{-\beta} &= \sum_{n=0}^{\infty} g_n^{(\alpha, \beta)}(x, y) t^n \\
 (\alpha, \beta \in \mathbb{C}; |t| < \min \{|x|^{-1}, |y|^{-1}\}).
 \end{aligned} \tag{72}$$

Proof. Fix $i = 1, 2, \dots, r$. Let $\mathcal{C}_{\xi_i}$ and $\mathcal{C}_{\xi_i}^*$ denote the circles of radius ε_i , centered $\xi_i = z_i$ and $\xi_i = 0$, respectively, where

$$0 < \varepsilon_i < |x_i|^{-1} \left| 1 - \sum_{k=1}^r x_k z_k \right|. \tag{73}$$

Here these circles are described in the positive direction. By Cauchy's Integral Formula, we have

$$\begin{aligned}
 D_{z_i}^m \{ (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} z_i^k \} \\
 = \frac{m!}{2\pi i} \oint_{\mathcal{C}_{\xi_i}} \frac{\xi_i^k (1 - x_1 z_1 - \dots - x_i \xi_i - \dots - x_r z_r)^{-\alpha}}{(\xi_i - z_i)^{m+1}} d\xi_i \\
 = \frac{m!}{2\pi i} \oint_{\mathcal{C}_{\xi_i}^*} \frac{(\xi_i + z_i)^k (1 - x_1 z_1 - \dots - x_i (\xi_i + z_i) - \dots - x_r z_r)^{-\alpha}}{\xi_i^{m+1}} d\xi_i \\
 = \frac{m!}{2\pi i} z_i^k (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} \\
 \times \oint_{\mathcal{C}_{\xi_i}^*} \frac{(1 + \xi_i/z_i)^k}{\xi_i^{m+1}} \left(1 - \frac{x_i \xi_i}{1 - x_1 z_1 - \dots - x_r z_r} \right)^{-\alpha} d\xi_i \\
 = m! z_i^k (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} \\
 \times g_m^{(\alpha, -k)} \left(\frac{x_i}{1 - x_1 z_1 - \dots - x_r z_r}, -\frac{1}{z_i} \right) \\
 = m! z_i^{k-m} (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} \\
 \times g_m^{(\alpha, -k)} \left(\frac{x_i z_i}{1 - x_1 z_1 - \dots - x_r z_r}, -1 \right).
 \end{aligned} \tag{74}$$

On the other hand, we can write that

$$\begin{aligned}
 D_{z_i}^m \{ (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} z_i^k \} \\
 = D_{z_i}^m \left\{ z_i^k \sum_{n_1, \dots, n_r=0}^{\infty} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) z_1^{n_1} \dots z_r^{n_r} \right\} \\
 = \sum_{n_1, \dots, n_r=0}^{\infty} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \frac{d^m}{dz_i^m} z_1^{n_1} \dots z_i^{n_i+k} \dots z_r^{n_r} \\
 = z_i^{k-m} \sum_{n_1, \dots, n_r=0}^{\infty} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) [k + n_i]_m z_1^{n_1} \dots z_r^{n_r}.
 \end{aligned} \tag{75}$$

From (74) and (75), we see that

$$\begin{aligned}
 \sum_{n_1, \dots, n_r=0}^{\infty} G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) [k + n_i]_m z_1^{n_1} \dots z_r^{n_r} \\
 = m! (1 - x_1 z_1 - \dots - x_r z_r)^{-\alpha} \\
 \times g_m^{(\alpha, -k)} \left(\frac{x_i z_i}{1 - x_1 z_1 - \dots - x_r z_r}, -1 \right). \quad \square
 \end{aligned} \tag{76}$$

Corollary 25. From (71), one observes that

$$\begin{aligned}
 [n_i + m]_m &= (n_i + m)(n_i + m - 1) \dots (n_i + 1) \\
 &= (n_i + 1)_m
 \end{aligned} \tag{77}$$

for each $i = 1, 2, \dots, r$. By setting $k = m$ in (70) and then using (77), one has

$$\begin{aligned} & \sum_{n_1, \dots, n_r=0}^{\infty} (n_i + 1)_m G_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) z_1^{n_1} \cdots z_r^{n_r} \\ &= m! (1 - x_1 z_1 - \cdots - x_r z_r)^{-\alpha} \\ & \times g_m^{(\alpha, -m)} \left(\frac{x_i z_i}{1 - x_1 z_1 - \cdots - x_r z_r}, -1 \right). \end{aligned} \quad (78)$$

5.2.2. The Case of $m_i = i$, $i = 1, 2, \dots, r$ in Section 5.2. This case reduces to a multivariable analogue of Lagrange-Hermite polynomials

$$\begin{aligned} & (1 - x_1 t_1 - x_2 t_2^2 - \cdots - x_r t_r^r)^{-\alpha} \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} H_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) t_1^{n_1} \cdots t_r^{n_r}, \end{aligned} \quad (79)$$

which is different from that in [6].

Remark 26. By Corollaries 8 and 9, the multivariable polynomials $H_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ satisfy

$$\begin{aligned} & i x_i \frac{\partial}{\partial x_i} H_{n_1, \dots, n_{i-1}, n_i-i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\ &= (n_i - i + 1) H_{n_1, \dots, n_{i-1}, n_i-i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r), \\ & \sum_{i=1}^r i x_i \frac{\partial}{\partial x_i} H_{n_1, \dots, n_{i-1}, n_i-i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r) \\ &= \sum_{i=1}^r (n_i - i + 1) \\ & \times H_{n_1, \dots, n_{i-1}, n_i-i+1, n_{i+1}, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r). \end{aligned} \quad (80)$$

for $n_i \geq i - 1$, $i = 1, 2, \dots, r$; $n_j \geq 0$, $j \neq i$.

Remark 27. From Theorem 11, the multivariable polynomials $H_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$ have the following addition formula:

$$\begin{aligned} & H_{n_1, \dots, n_r}^{(\alpha+\beta)}(x_1, \dots, x_r) \\ &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} H_{n_1-k_1, \dots, n_r-k_r}^{(\alpha)}(x_1, \dots, x_r) \\ & \times H_{k_1, \dots, k_r}^{(\beta)}(x_1, \dots, x_r). \end{aligned} \quad (81)$$

Furthermore, setting $m_i = i$, $x_i = 0$, $y_i = -x_i$, $i = 1, 2, \dots, r$ in Corollary 5, we obtain the following class of bilinear generating functions for the polynomials $H_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r)$.

Remark 28. If

$$\begin{aligned} & \Xi_{\lambda, \eta, \alpha, \beta, \gamma}^{\mathbf{n}, \mathbf{p}}(x_1, \dots, x_r; u_1, \dots, u_r; \mathbf{w}) \\ &:= \sum_{k_1=0}^{[n_1/p_1]} \cdots \sum_{k_r=0}^{[n_r/p_r]} a_{k_1, \dots, k_r} H_{n_1-p_1 k_1, \dots, n_r-p_r k_r}^{(\alpha+\beta)}(x_1, \dots, x_r) \\ & \times H_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}^{(\gamma)}(u_1, \dots, u_r) w_1^{k_1} \cdots w_r^{k_r}, \end{aligned} \quad (82)$$

where $a_{k_1, \dots, k_r} \neq 0$; $p_i \in \mathbb{N}$; $n_i, \lambda_i, \eta_i \in \mathbb{N}_0$, $i = 1, 2, \dots, r$, and $\lambda = (\lambda_1, \dots, \lambda_r)$, $\eta = (\eta_1, \dots, \eta_r)$, $\mathbf{w} = (w_1, \dots, w_r)$, then we have

$$\begin{aligned} & \sum_{l_1=0}^{n_1} \cdots \sum_{l_r=0}^{n_r} \sum_{k_1=0}^{[l_1/p_1]} \cdots \sum_{k_r=0}^{[l_r/p_r]} a_{k_1, \dots, k_r} H_{n_1-l_1, \dots, n_r-l_r}^{(\alpha)}(x_1, \dots, x_r) \\ & \times H_{l_1-p_1 k_1, \dots, l_r-p_r k_r}^{(\beta)}(x_1, \dots, x_r) \\ & \times H_{\lambda_1+\eta_1 k_1, \dots, \lambda_r+\eta_r k_r}^{(\gamma)}(u_1, \dots, u_r) \\ & \times w_1^{k_1} \cdots w_r^{k_r} \\ &= \Xi_{\lambda, \eta, \alpha, \beta, \gamma}^{\mathbf{n}, \mathbf{p}}(x_1, \dots, x_r; u_1, \dots, u_r; \mathbf{w}). \end{aligned} \quad (83)$$

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