Research Article

A Note on Elongations of Summable QTAG-Modules

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A right module M over an associative ring with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. In this paper we find a suitable condition under which a special ω -elongation of a summable QTAG-module by a $(\omega + k)$ -projective QTAG-module is also a summable QTAG-module.

1. Introduction and Preliminaries

The study of QTAG-modules was initiated by Singh [1]. Mehdi et al. studied different notions and structures on QTAG-modules, developed the theory of these modules by introducing several notions and some interesting properties of these modules and characterized different submodules of QTAG-modules. Yet there is much to explore.

Throughout this paper, all rings will be associative with unity and modules M are unital QTAG-modules. An element $x \in M$ is uniform, if xR is a nonzero uniform (hence uniserial) module and for any R -module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\{d(yR/xR) \mid y \in M, x \in yR \text{ and } y \text{ uniform}\}$ are the exponent and height of x in M , respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k . A QTAG-module M is called a Σ -module, if all of its high submodules are direct sums of uniserial modules. M is h -divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$.

A module M is summable if $\text{Soc}(M) = \bigoplus_{\alpha < \tau} S_\alpha$, where S_α is the set of all elements of $H_\alpha(M)$ which are not in $H_{\alpha+1}(M)$, where τ is the length of M .

A submodule $N \subset M$ is nice [2, Definition 2.3] in M , if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ ; that is, every coset of M modulo N may be represented by an element of the same height.

Recall that a QTAG-module M is $(\omega+1)$ -projective if there exists a submodule $N \subset H^1(M)$ such that M/N is a direct sum of uniserial modules and a QTAG-module M is $(\omega+k)$ -projective if there exists a submodule $N \subset H^k(M)$ such that M/N is a direct sum of uniserial modules [3].

A QTAG-module is an ω -elongation of a totally projective QTAG-module by a $(\omega+k)$ -projective QTAG-module if and only if $H_\omega(M)$ is totally projective and $M/H_\omega(M)$ is $(\omega+k)$ -projective.

2. Main Results

In [4], we studied strong ω -elongations of summable modules by $(\omega+k)$ -projective QTAG-modules. It was found that the module M is a special ω -elongation if M^1 is summable and there exists a submodule $N \subseteq H^k(M)$ such that $M/(N + M^1)$ is a direct sum of uniserial modules. We showed there that under certain additional circumstances on N these elongations are of necessity summable modules; in fact $N \cap H_k(M) \subseteq M^1$ was taken; that is, N is a height finite submodule of M .

In [5], the class of n -layered module was introduced which is a proper subclass of Σ -modules as follows: M is an n -layered module if $H^k(M) = \bigcup_{i < \omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^k(M)$, and for all $i \geq 1$, $M_i \cap H_i(M) \subseteq M^1$. It was proved there that every n -layered module which is a strong ω -elongation of a totally projective module by a $(\omega+k)$ -projective module is totally projective and vice versa; in particular each n -layered

module is $(\omega + k)$ -projective uniquely when it is a direct sum of modules of length at most $(\omega + n)$.

Motivated by these elongations we investigate the relationship between the classes of n -layered modules and strong ω -elongations of summable modules by $(\omega + k)$ -projective modules, that is, how n -layered modules are situated inside these special ω -elongations of summable modules by $(\omega + k)$ -projective modules and whether there is an analogue with the strong ω -elongations of a totally projective modules by $(\omega + k)$ -projective modules.

Before doing that, we need the following preliminaries.

A module M is said to be pillared [6], if M/M^1 is a direct sum of uniserial modules. Clearly, such a module is necessarily n -layered module and hence a Σ -module [5], whereas the converse is not true in general. We investigate the conditions under which the converse holds good.

A module M is said to be a strong ω -elongation (of a summable module) by a $(\omega + k)$ -projective module if there exists a submodule $N \subseteq H^k(M)$ with $M/(N + M^1)$ a direct sum of uniserial modules. The first Ulm-factor of this module is $(\omega + k)$ -projective. There are modules with $(\omega + k)$ -projective first Ulm-factor which are not strong ω -elongations by a $(\omega + k)$ -projective module. These modules are with strongly $(\omega + k)$ -projective first Ulm-factor.

Now we are in the state to prove the following main result.

Theorem 1. *An n -layered module is a strong ω -elongation by a $(\omega + k)$ -projective module if and only if it is a pillared module.*

Proof. Suppose that M is an n -layered module, so $H^k(M) = \bigcup_{i < \omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^k(M)$, and for all $i \geq 1$, $M_i \cap H_i(M) \subseteq M^1$ and suppose that there exists a submodule N of M such that $N \subseteq H^k(M)$ with $M/(N + M^1)$ is a direct sum of uniserial modules. Clearly,

$$\frac{M/M^1}{(N + M^1)/M^1} \cong \frac{M}{N + M^1}. \quad (1)$$

Since $N \subseteq \bigcup_{i < \omega} M_i$, we infer that $N = \bigcup_{i < \omega} (M_i \cap N)$ and thus $(N + M^1)/M^1 = \cup((N + M^1)/M^1)$ by putting $N_i = M_i \cap N$.

Using modular law, we compute that

$$\begin{aligned} \frac{N_i + M^1}{M^1} \cap H_i\left(\frac{M}{M^1}\right) &= \frac{(N_i \cap H_i(M)) + M^1}{M^1} \\ &= \{0\}. \end{aligned} \quad (2)$$

Hence M/M^1 is a direct sum of uniserial modules and therefore M is pillared. The proof is complete as the converse is trivial. $\square$

Proposition 2. *An n -layered module is a strong ω -elongation of a summable module by a $(\omega + k)$ -projective module if and only if it is a summable pillared module.*

Proof. Suppose n -layered module is a strong ω -elongation of a summable module by a $(\omega + k)$ -projective module. Since M is a Σ -module and M^1 is summable, M has to be summable as well and Theorem 1 ensures that M^1 must be pillared.

Suppose that M is summable pillared module, so it ensures that M^1 is summable and pillared modules are both n -layered and strong ω -elongations by $(\omega + k)$ -projective module by taking $N = 0$, which completes the proof. $\square$

As an immediate consequence of the above, we have the following corollary.

Corollary 3. *Suppose M is a Σ -module which is a strong ω -elongation of a summable module by a $(\omega + k)$ -projective module and the $(\omega + n)$ th Ulm-Kaplansky invariants of M are zero for each n such that $0 \leq n \leq k - 1$, if $k > 1$. Then M is a summable pillared module.*

Proof. Since $(\omega + n)$ th Ulm-Kaplansky invariants of M are zero, $H^k(M) = H^k(N) \oplus H^k(M^1)$ where N is a high submodule of M . Since it is a direct sum of uniserial modules, $H^k(N) = \bigcup_{i < \omega} N_i$, $N_i \subseteq N_{i+1} \subseteq H^k(N)$ where $N_i \cap H_i(N) = 0$. Putting $M_i = N_i \oplus H^k(M^1)$, we obtain $H^k(M) = \bigcup_{i < \omega} M_i$. Since N is h -pure in M , we have

$$\begin{aligned} M_i \cap H_i(M) &\subseteq M^1 + N_i \cap H_i(M) \\ &= M^1 + N_i \cap H_i(M) \\ &= M^1. \end{aligned} \quad (3)$$

By Proposition 2, M is an n -layered module. $\square$

Proposition 4. *A module of length not exceeding $(\omega + n - 1)$ is an n -layered module if and only if it is a direct sum of countably generated modules.*

Proof. Suppose M is an n -layered module of length not exceeding $(\omega + n - 1)$; then $M^1 \subseteq H^{n-1}(M)$ and hence

$$\text{Soc}\left(\frac{M}{M^1}\right) = \bigcap_{i < \omega} \frac{H_i(M) + \text{Soc}(M)}{M^1} \subseteq \frac{H^n(M)}{M^1} \quad (4)$$

since $H(\bigcap_{i < \omega} (H_i(M) + \text{Soc}(M))) \subseteq M^1$. Moreover, we write $H^n(M) = \bigcup_{i < \omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^n(M)$ and $M_i \cap H_i(M) \subseteq M^1$. Consequently, $\text{Soc}(M/M^1) = \bigcup_{i < \omega} T_i$, where $T_i = (M_i + M^1)/M^1 \cap \text{Soc}(M/M^1)$. We compute

$$\begin{aligned} T_i \cap H_i\left(\frac{M}{M^1}\right) &= T_i \cap \left(\frac{H_i(M)}{M^1}\right) \\ &= \frac{(M_i + M^1) \cap H_i(M)}{M^1} \\ &= \frac{(M_i \cap H_i(M)) + M^1}{M^1} \\ &= 0 \end{aligned} \quad (5)$$

hence M is pillared and as M^1 is bounded, M is a direct sum of countable modules.

The sufficiency is obvious and follows from [5]. $\square$

As an immediate consequence of the above, we have the following corollary.

Corollary 5. *A module M is an n -layered module if and only if each of its $H_{(\omega+n-1)}(M)$ -high submodules is a direct sum of countable modules.*

Proof. Let M be an n -layered module and N be its $H_{(\omega+n-1)}(M)$ -high submodule. In [5], we show that M is an n -layered module precisely when N is an n -layered module. Hence using the Proposition 4, result follows immediately. $\square$

With the help of the last statement we can verify once again the validity of Corollary 3 as it is checked that a submodule N of M is h -high in M if and only if N is $(\omega+n-1)$ -high in M whenever the $(\omega+m)$ th Ulm-Kaplansky invariants of M are zero for $0 \leq m \leq n-1$; that is,

$$\text{Soc}(H_{\omega}(M)) = \cdots = \text{Soc}(H_{\omega+n-1}(M)). \quad (6)$$

A module M is said to be strong $(\omega+n-1)$ -elongation of a summable module by a totally projective module if $H_{\omega+n-1}(M)$ is summable and there is a nice submodule K of M such that $K \cap H_{\omega+n-1}(M) = 0$ and $M/(K \oplus H_{\omega+n-1}(M))$ is totally projective [3].

Now we are in the state to prove our final result.

Theorem 6. *An n -layered module is a strong $(\omega+n-1)$ -elongation of a summable module by a totally projective module if and only if it is a summable pillared module.*

Proof. Clearly

$$\frac{M}{K \oplus H_{\omega+n-1}(M)} \cong \frac{M/H_{\omega+n-1}(M)}{(K \oplus H_{\omega+n-1}(M))/H_{\omega+n-1}(M)} \quad (7)$$

is totally projective. Since $K \cap H_{\omega+n-1}(M) = 0$, K is contained in some $(\omega+n-1)$ -high submodule of M , say N . By Corollary 5, N is totally projective of length at most $(\omega+n-1)$, so we may write $N = \bigcup_{i < \omega} N_i$, where $N_i \subseteq N_{i+1} \subseteq N$ and all N_i 's are height finite in N and hence in M as N is isotype in M . Therefore

$$\begin{aligned} & \frac{K \oplus H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)} \\ &= \bigcup_{i < \omega} \left[\left(\frac{N_i + H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)} \right) \cap \left(\frac{K \oplus H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)} \right) \right]. \end{aligned} \quad (8)$$

In the same way, we can show that $(N_i + H_{\omega+n-1}(M))/H_{\omega+n-1}(M)$ are height finite in $M/H_{\omega+n-1}(M)$ and by [3], $(K \oplus H_{\omega+n-1}(M))/H_{\omega+n-1}(M)$ is nice in $M/H_{\omega+n-1}(M)$ and hence $M/H_{\omega+n-1}(M)$ is totally projective. Then

$$\frac{M}{H_{\omega}(M)} \cong \frac{M/H_{\omega+n-1}(M)}{H_{\omega}(M)/H_{\omega+n-1}(M)} = \frac{M/H_{\omega+n-1}(M)}{H_{\omega}(M/H_{\omega+n-1}(M))} \quad (9)$$

should be a direct sum of uniserial modules and hence M is pillared.

On the other hand, $H_{\omega+n-1}(M)$ being summable implies that $H_{\omega}(M)$ is summable and hence M has to be summable; thus it is summable pillared, which completes the proof. $\square$

As an immediate consequence for $n = 1$, we have the following.

Corollary 7. *A Σ -module is a strong ω -elongation of a summable module by a totally projective module if and only if it is a summable pillared module.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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