Research Article

ϵ -Henig Saddle Points and Duality of Set-Valued Optimization Problems in Real Linear Spaces

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We study ϵ -Henig saddle points and duality of set-valued optimization problems in the setting of real linear spaces. Firstly, an equivalent characterization of ϵ -Henig saddle point of the Lagrangian set-valued map is obtained. Secondly, under the assumption of the generalized cone subconvexlikeness of set-valued maps, the relationship between the ϵ -Henig saddle point of the Lagrangian set-valued map and the ϵ -Henig properly efficient element of the set-valued optimization problem is presented. Finally, some duality theorems are given.

1. Introduction

Saddle points and duality of set-valued optimization, which have close relationship to the generalized convexity of set-valued maps and the efficiency of solutions of set-valued optimization, are two important topics in optimization theory. Recently, many new researches involving saddle points and duality of set-valued optimization have appeared in the literature. Li [1] introduced Benson proper saddle point of the Lagrangian set-valued maps for a set-valued optimization problem in locally convex spaces and established saddle point theorems and duality theorems under the assumption of the cone subconvexlikeness of set-valued maps. Zhao and Rong [2] and Li et al. [3] investigated ϵ -strict saddle points and duality of set-valued optimization problems under the assumption of cone convexlikeness and ic-cone convexlikeness of set-valued maps, respectively. Xia and Qiu [4] obtained saddle point theorems and duality theorems of set-valued optimization problems in the sense of super efficiency under the nearly subconvexlikeness of set-valued maps.

However, in the above mentioned references, saddle points and duality of set-valued optimization were studied in locally convex spaces. How to generalize saddle point theorems and duality theorems of set-valued optimization from locally convex spaces to linear spaces is interesting. Adán and Novo [5] studied saddle points and duality for convexlike vector optimization problems in real linear spaces.

In the ϵ -global proper efficiency, Zhou et al. [6] introduced the concept of the ϵ -global proper saddle point and obtained the relationships between the ϵ -global proper saddle points of Lagrangian set-valued maps and the ϵ -global properly efficient element of set-valued optimization problems.

This paper is a continuation of the research work [7]. The aim of this paper is to investigate ϵ -Henig proper saddle points and duality of set-valued optimization problems in linear spaces. This paper is organized as follows. In Section 2, some preliminaries, including notations and lemmas, are given. In Section 3, we introduce a new notion of ϵ -Henig proper saddle point of the Lagrangian set-valued map in linear spaces and obtain several saddle point theorems. In Section 4, we present three duality theorems including ϵ -weak duality, ϵ -converse duality, and ϵ -strong duality.

2. Preliminaries

Let X be a real linear space, and let Y and Z be two real ordered linear spaces. 0 denotes the zero element of every space. Let K be a nonempty subset in Y . The generated cone of K is defined as $\text{cone}(K) := \{\lambda k \mid k \in K, \lambda \geq 0\}$. K is called a cone if and only if $\lambda K \subseteq K$ for any $\lambda \geq 0$. A cone K is said to be pointed if and only if $K \cap (-K) = \{0\}$. K is said to be nontrivial if and only if $K \neq \{0\}$ and $K \neq Y$.

The algebraic dual of Y and Z is denoted by Y^* and Z^* , respectively. Let C and D be two nontrivial pointed convex cones in Y and Z , respectively. The algebraic dual cone C^+ and strictly algebraic dual cone C^{+i} of C are, respectively, defined as

$$\begin{aligned} C^+ &:= \{y^* \in Y^* \mid \langle y, y^* \rangle \geq 0, \forall y \in C\}, \\ C^{+i} &:= \{y^* \in Y^* \mid \langle y, y^* \rangle > 0, \forall y \in C \setminus \{0\}\}, \end{aligned} \quad (1)$$

where $\langle y, y^* \rangle$ denotes the value of the linear functional y^* at the point y . The meaning of D^+ is similar to that of C^+ .

Definition 1 (see [8]). Let K be a nonempty subset in Y . The algebraic interior of K is the set

$$\begin{aligned} \text{cor}(K) &:= \{k \in K \mid \forall h \in Y, \exists \lambda' > 0, \\ &\forall \lambda \in [0, \lambda'], k + \lambda h \in K\}. \end{aligned} \quad (2)$$

Definition 2 (see [9]). Let K be a nonempty subset in Y . K is balanced if and only if, for all $x \in K$, for all $\lambda \in [-1, 1]$, $\lambda k \in K$. K is called absorbent if and only if $0 \in \text{cor}(K)$.

Definition 3 (see [10]). Let K be a nonempty subset in Y . The vector closure of K is the set

$$\begin{aligned} \text{vcl}(K) &:= \{k \in Y \mid \exists h \in Y, \forall \lambda' > 0, \exists \lambda \in]0, \lambda'], \\ &k + \lambda h \in K\}. \end{aligned} \quad (3)$$

Following Adán and Novo [10], K is vectorially closed (v-closed) if $\text{vcl}(K) = K$.

Definition 4 (see [7]). Let B be a nonempty convex subset in Y . B is a base of C if and only if $C = \text{cone}(B)$ and there exists a balanced, absorbent, and convex set V such that $0 \notin B + V$ in Y . Write $C_V(B) := \text{cone}(V + B)$.

From now on, we suppose that $\text{cor}(C) \times \text{cor}(D) \neq \emptyset$ and B is a base of C . We recall a notion of ϵ -Henig properly efficient point introduced by Zhou et al. [7] in linear spaces.

Definition 5 (see [7]). Let $K \subseteq Y$ and $\epsilon \in C$. $\bar{y} \in K$ is called an ϵ -Henig properly minimal efficient point with respect to B (denoted by $\bar{y} \in \epsilon\text{-}H_{\min}(K, B)$) if and only if there exists a balanced, absorbent, and convex set V with $0 \notin B + V$ such that $\text{cone}(K - \bar{y} + \epsilon) \cap (-C_V(B)) = \{0\}$. $\bar{y} \in K$ is called an ϵ -Henig properly maximal efficient point with respect to B (denoted by $\bar{y} \in \epsilon\text{-}H_{\max}(K, B)$) if and only if there exists a balanced, absorbent, and convex set V with $0 \notin B + V$ such that $\text{cone}(K - \bar{y} - \epsilon) \cap C_V(B) = \{0\}$.

Let A be a nonempty set in X and $F : A \rightrightarrows Y$ and $G : A \rightrightarrows Z$ be two set-valued maps on A . Write $F(A) := \bigcup_{x \in A} F(x)$, $\langle F(x), y^* \rangle := \{\langle y, y^* \rangle \mid y \in F(x)\}$, and $\langle G(x), z^* \rangle := \bigcup_{x \in A} \langle F(x), y^* \rangle$. The meanings of $G(A)$, $\langle G(x), z^* \rangle$, and $\langle G(A), z^* \rangle$ are similar to those of $F(A)$, $\langle F(x), y^* \rangle$, and $\langle F(A), y^* \rangle$, respectively.

Definition 6 (see [11]). A set-valued map $F : A \rightrightarrows Y$ is called generalized C -subconvexlike on A if and only if $\text{cone}(F(A)) + \text{cor}(C)$ is a convex set in Y .

Lemma 7 (see [6]). Let Z be a linear space, and let $M, N \subseteq Z$ be two nonempty sets such that $M - N$ is a convex set in Z . If $\text{cor}(M - N) \neq \emptyset$ and $0 \notin \text{vcl}(M - N)$, then there exists $z^* \in Z^* \setminus \{0\}$ such that $\sup_{z_2 \in N} \langle z_2, z^* \rangle < \inf_{z_1 \in M} \langle z_1, z^* \rangle$.

3. ϵ -Henig Saddle Points

In this section, we will establish approximate saddle point theorems of set-valued optimization problems in the sense of ϵ -Henig proper efficiency.

Let $F : A \rightrightarrows Y$ and $G : A \rightrightarrows Z$ be two set-valued maps on A . We consider the following vector optimization problem with set-valued maps:

$$(\text{VP}) \min F(x) \text{ subject to } G(x) \cap (-D) \neq \emptyset. \quad (4)$$

The feasible set of (VP) is defined by $S := \{x \in A \mid G(x) \cap (-D) \neq \emptyset\}$.

Definition 8 (see [7]). Let $\epsilon \in C$. $\bar{x} \in S$ is called an ϵ -Henig properly efficient solution of (VP) if and only if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in \epsilon\text{-}H_{\min}(F(S), B)$. The pair $(\bar{x}, \bar{y})$ is called an ϵ -Henig properly efficient element of (VP).

We denote by $L(Z, Y)$ the set of all linear operators from Z to Y . A subset $L^+(Z, Y)$ of $L(Z, Y)$ is defined as $L^+(Z, Y) := \{T \in L(Z, Y) \mid T(D) \subseteq C\}$. The Lagrangian set-valued map of (VP) is defined by

$$L(x, T) := F(x) + T(G(x)), \quad \forall (x, T) \in A \times L^+(Z, Y). \quad (5)$$

Consider the following unconstrained vector optimization problem with set-valued maps:

$$(\text{UVP})_T \min L(x, T) \text{ subject to } (x, T) \in A \times L^+(Z, Y). \quad (6)$$

Lemma 9 (see [7]). Let $\epsilon \in C$, $\bar{x} \in S$, and $\bar{y} \in F(\bar{x})$. If there exists $\bar{T} \in L^+(Z, Y)$ such that $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of $(\text{UVP})_{\bar{T}}$, then $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (VP).

Now, we will introduce a new notion called ϵ -Henig proper saddle point of the Lagrangian set-valued map $L(x, T)$ in linear spaces.

Definition 10. $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is called an ϵ -Henig proper saddle point of the Lagrangian set-valued map $L(x, T)$ if and only if

$$\begin{aligned} &L(\bar{x}, \bar{T}) \cap \epsilon\text{-}H_{\min}\left(\bigcup_{x \in A} L(x, \bar{T}), B\right) \\ &\cap \epsilon\text{-}H_{\max}\left(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T), B\right) \neq \emptyset. \end{aligned} \quad (7)$$

The following proposition is an important equivalent characterization for an ϵ -Henig proper saddle point of the Lagrangian set-valued map $L(x, T)$.

Proposition 11. Let D be v -closed and $\epsilon \in C$. Then, $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is an ϵ -Henig proper saddle point of the Lagrangian set-valued map $L(x, \bar{T})$ if and only if there exist $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, and a balanced, absorbent, and convex set V with $0 \notin B + V$ such that

- (i) $\bar{y} + \bar{T}(\bar{z}) \in \epsilon\text{-}H_{\min}(\bigcup_{x \in A} L(x, \bar{T}), B)$;
- (ii) $G(\bar{x}) \subseteq -D$;
- (iii) $-\bar{T}(\bar{z}) \in C \setminus (\epsilon + C \setminus \{0\})$;
- (iv) $\text{cone}(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z}) - \epsilon) \cap C_V(B) = \{0\}$.

Proof. Necessity. Let $(\bar{x}, \bar{T})$ be an ϵ -Henig proper saddle point of $L(x, T)$. Then, there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that

$$\bar{y} + \bar{T}(\bar{z}) \in \epsilon\text{-}H_{\min}\left(\bigcup_{x \in A} L(x, \bar{T}), B\right), \quad (8)$$

$$\bar{y} + \bar{T}(\bar{z}) \in \epsilon\text{-}H_{\max}\left(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T), B\right). \quad (9)$$

Equation (8) shows that (i) holds. By (9), there exists a balanced, absorbent, and convex set V with $0 \notin B + V$ such that

$$\text{cone}\left(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T) - \bar{y} - \bar{T}(\bar{z}) - \epsilon\right) \cap C_V(B) = \{0\}. \quad (10)$$

Taking $T = 0$ in (10), we obtain

$$\text{cone}(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z}) - \epsilon) \cap C_V(B) = \{0\}. \quad (11)$$

Therefore, (iv) holds. Since $\bar{y} \in F(\bar{x})$ and V is absorbent in Y , it follows from (11) that

$$\text{cone}(\bar{T}(\bar{z}) + \epsilon) \cap (-B) = \emptyset. \quad (12)$$

Because $\text{cone}(B) = C$, it follows from (12) that $\text{cone}(\bar{T}(\bar{z}) + \epsilon) \cap (-C \setminus \{0\}) = \emptyset$. Clearly, $(\bar{T}(\bar{z}) + \epsilon) \cap (-C \setminus \{0\}) = \emptyset$. Therefore,

$$-\bar{T}(\bar{z}) \notin \epsilon + C \setminus \{0\}. \quad (13)$$

We assert that $-\bar{z} \in D$. Otherwise, by Lemma 7, it is easy to prove (see the proof of Proposition 4.1 in [6]) that there exists $z_1^* \in D^+ \setminus \{0\}$ such that $\langle \bar{z}, z_1^* \rangle > 0$. Taking $b_1 \in B$, we define a vector-valued map $T_1 : Z \rightarrow Y$ as follows:

$$T_1(z) = \frac{\langle z, z_1^* \rangle}{\langle \bar{z}, z_1^* \rangle} (b_1 + \epsilon) + \bar{T}(z), \quad \forall z \in Z. \quad (14)$$

Clearly, $T_1 \in L^+(Z, Y)$ and $T_1(\bar{z}) - \bar{T}(\bar{z}) - \epsilon = b_1 \in C_V(B)$. On the other hand, $T_1(\bar{z}) - \bar{T}(\bar{z}) - \epsilon \in \text{cone}(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T) - \bar{y} - \bar{T}(\bar{z}) - \epsilon)$. Since $0 \notin B + V$, $b_1 \neq 0$. Therefore,

$$\text{cone}\left(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T) - \bar{y} - \bar{T}(\bar{z}) - \epsilon\right) \cap C_V(B) \neq \{0\}, \quad (15)$$

which contradicts (10). Hence, $-\bar{z} \in D$. Since $\bar{T} \in L^+(Z, Y)$, we have

$$-\bar{T}(\bar{z}) \in C. \quad (16)$$

It follows from (13) and (16) that (iii) holds. We assert that $G(\bar{x}) \subseteq -D$. Otherwise, there exists $z_1 \in G(\bar{x})$ such that $z_1 \notin -D$. Similar to the above proof, there exists $z_2^* \in D^+ \setminus \{0\}$ such that $\langle z_1, z_2^* \rangle > 0$. Taking $b_2 \in B$, we define a vector-valued map $T_2 : Z \rightarrow Y$ as follows:

$$T_2(z) = \frac{\langle z, z_2^* \rangle}{\langle z_1, z_2^* \rangle} (b_2 + \epsilon), \quad \forall z \in Z. \quad (17)$$

Clearly,

$$T_2 \in L^+(Z, Y), \quad (18)$$

$$T_2(z_1) - \bar{T}(\bar{z}) - \epsilon = b_2 - \bar{T}(\bar{z}) \in B + C \subseteq C \setminus \{0\}. \quad (19)$$

Since $\bar{y} \in F(\bar{x})$ and $z_1 \in G(\bar{x})$, it follows from (10) and (18) that

$$\text{cone}(T_2(z_1) - \bar{T}(\bar{z}) - \epsilon) \cap C_V(B) = \{0\}. \quad (20)$$

By (20), it is easy to check that $T_2(z_1) - \bar{T}(\bar{z}) - \epsilon \notin C \setminus \{0\}$, which contradicts (19). Therefore, (ii) holds.

Sufficiency. Since $\bar{y} + \bar{T}(\bar{z}) \in L(\bar{x}, \bar{T})$, by condition (i), we only prove that

$$\bar{y} + \bar{T}(\bar{z}) \in \epsilon\text{-}H_{\max}\left(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T), B\right). \quad (21)$$

We assert

$$\text{cone}\left(\bigcup_{T \in L^+(Z, Y)} L(\bar{x}, T) - \bar{y} - \bar{T}(\bar{z}) - \epsilon\right) \cap C_V(B) = \{0\}. \quad (22)$$

Otherwise, there exists $r_1 > 0$, $r_2 > 0$, $T_3 \in L^+(Z, Y)$, $y' \in F(\bar{x})$, $z' \in G(\bar{x})$, $v \in V$, and $b_3 \in B$ such that $r_1(y' + T_3(z')) - \bar{y} - \bar{T}(\bar{z}) - \epsilon = r_2(v + b_3)$. Clearly,

$$r_1(y' - \bar{y} - \bar{T}(\bar{z}) - \epsilon) = r_2(v + b_3) - r_1 T_3(z'). \quad (23)$$

By condition (ii), $z' \in -D$. Since $T_3 \in L^+(Z, Y)$, $-T_3(z') \in C$. Therefore, there exist $r_3 \geq 0$ and $b_4 \in B$ such that

$$-T_3(z') = r_3 b_4. \quad (24)$$

It follows from (23) and (24) that

$$\begin{aligned} & r_1(y' - \bar{y} - \bar{T}(\bar{z}) - \epsilon) \\ &= r_2(v + b_3) + r_1 r_3 b_4 \\ &= r_2 v + (r_2 b_3 + r_1 r_3 b_4) = (r_2 + r_1 r_3) \\ & \quad \times \left[\frac{r_2}{r_2 + r_1 r_3} v + \left(\frac{r_2}{r_2 + r_1 r_3} b_3 + \frac{r_1 r_3}{r_2 + r_1 r_3} b_4 \right) \right]. \end{aligned} \quad (25)$$

Since $0 < r_2/(r_2 + r_1 r_3) \leq 1$, it follows from the balance of V that

$$\frac{r_2}{r_2 + r_1 r_3} v \in V. \quad (26)$$

It follows from the convexity of B that

$$\frac{r_2}{r_2 + r_1 r_3} b_3 + \frac{r_1 r_3}{r_2 + r_1 r_3} b_4 \in B. \quad (27)$$

Since $0 \notin V + B$ and $r_2 + r_1 r_3 > 0$, it follows from (25)–(27) that

$$\begin{aligned} 0 &\neq r_1 (y' - \bar{y} - \bar{T}(\bar{z}) - \epsilon) \\ &\in \text{cone}(F(\bar{x}) - \bar{y} - \bar{T}(\bar{z}) - \epsilon) \cap C_V(B), \end{aligned} \quad (28)$$

which contradicts condition (iv). Therefore, (22) holds. Thus, (21) holds. $\square$

Remark 12. According to Theorem 1 in [12], the notion of ϵ -strictly efficient point is equivalent to the notion of ϵ -Henig properly efficient point in locally convex spaces. Moreover, the generalized subconvexlikeness of the set-valued map F is equivalent to ϵ -cone convexlikeness of the set-valued map F introduced by Sach [13] when the topological interior $\text{int}(C) \neq \emptyset$. Therefore, Proposition 11 generalizes Proposition 5.1 in [3] from locally convex spaces to real linear spaces.

Theorem 13. Let D be v -closed and $\epsilon \in C$. If $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is an ϵ -Henig proper saddle point of the Lagrangian set-valued map $L(x, T)$, then there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that $(\bar{x}, \bar{y})$ is an $\bar{\epsilon}$ -Henig properly efficient element of (VP), where $\bar{\epsilon} = \epsilon - \bar{T}(\bar{z})$.

Proof. Since $(\bar{x}, \bar{T}) \in A \times L^+(Z, Y)$ is an ϵ -Henig proper saddle point of the Lagrangian set-valued map $L(x, T)$, it follows from Proposition 11 that there exist $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, and a balanced, absorbent, and convex set V with $0 \notin B + V$ such that conditions (i)–(iv) hold. By condition (ii), $\bar{x} \in S$. By condition (iii), $\bar{\epsilon} = \epsilon - \bar{T}(\bar{z}) \in C + C \setminus (\epsilon + C \setminus \{0\}) \subseteq C$. We assert that $(\bar{x}, \bar{y})$ is an $\bar{\epsilon}$ -Henig properly efficient element of (VP). Otherwise, for any balanced, absorbent, and convex set U with $0 \notin B + U$, $\text{cone}(F(S) - \bar{y} + \bar{\epsilon}) \cap (-C_U(B)) \neq \{0\}$. Therefore, there exist $r_1 > 0$, $r_2 > 0$, $x_1 \in S$, $y_1 \in F(S)$, $b_1 \in B$, and $u_1 \in U$ such that

$$r_1 (y_1 - \bar{y} + \bar{\epsilon}) = -r_2 (b_1 + u_1). \quad (29)$$

It follows from (29) that

$$\begin{aligned} r_1 (y_1 + \bar{T}(z) + \epsilon - (\bar{y} + \bar{T}(\bar{z}))) &= -r_2 (b_1 + u_1) + r_1 \bar{T}(z), \\ \forall z &\in G(\bar{x}) \cap (-D). \end{aligned} \quad (30)$$

Since $\bar{T} \in L^+(Z, Y)$, for any $z \in G(\bar{x}) \cap (-D)$, it follows from (30) that there exists $c \in C$ such that

$$r_1 (y_1 + \bar{T}(z) + \epsilon - (\bar{y} + \bar{T}(\bar{z}))) = -r_2 (b_1 + u_1) - r_1 c. \quad (31)$$

Because B is a base of C , there exist $r_3 \geq 0$ and $b_2 \in B$ such that $c = r_3 b_2$. By (31), we obtain

$$\begin{aligned} r_1 (y_1 + \bar{T}(z) + \epsilon - (\bar{y} + \bar{T}(\bar{z}))) &= -r_2 (b_1 + u_1) - r_1 r_3 b_2 \\ &= -[r_2 u_1 + (r_2 b_1 + r_1 r_3 b_2)] \\ &= -(r_2 + r_1 r_3) \\ &\quad \times \left[\frac{r_2}{r_2 + r_1 r_3} u_1 \right. \\ &\quad \left. + \left(\frac{r_2}{r_2 + r_1 r_3} b_1 + \frac{r_1 r_3}{r_2 + r_1 r_3} b_2 \right) \right] \in -C_U(B). \end{aligned} \quad (32)$$

Clearly, $r_1 (y_1 + \bar{T}(z) + \epsilon - (\bar{y} + \bar{T}(\bar{z}))) \neq 0$ for all $z \in G(\bar{x}) \cap (-D)$. Therefore, we obtain

$$\text{cone} \left(\bigcup_{x \in A} L(x, \bar{T}) + \epsilon - (\bar{y} + \bar{T}(\bar{z})) \right) \cap (-C_U(B)) \neq \{0\}, \quad (33)$$

which contradicts $\bar{y} + \bar{T}(\bar{z}) \in \epsilon - H_{\min}(\bigcup_{x \in A} L(x, \bar{T}), B)$. Hence, $(\bar{x}, \bar{y})$ is an $\bar{\epsilon}$ -Henig properly efficient element of (VP). $\square$

Remark 14. Comparing Theorem 13 with Theorem 4.1 in [6], the notion of ϵ -global proper efficiency has been replaced by the notion of ϵ -Henig proper efficiency and the condition $0 \in G(\bar{x})$ has been dropped.

In order to obtain sufficient conditions of ϵ -Henig proper saddle point under the assumption of the generalized cone subconvexlikeness, we need the following lemma.

Lemma 15 (see [7]). Let $\epsilon \in C$, $\bar{x} \in S$, and $0 \in G(\bar{x})$. Suppose that the following conditions hold:

- (i) $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (VP);
- (ii) $\bar{I}(x)$ is generalized $C \times D$ -subconvexlike on A , where $\bar{I}(x) = (F(x) - \bar{y} + \epsilon) \times G(x)$;
- (iii) $\text{vcl}(\text{cone}(G(A) + D)) = Z$.

Then, there exists $\bar{T} \in L^+(Z, Y)$ such that $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (UVP) $_{\bar{T}}$.

By Lemma 15, we easily obtain the following theorem involving the generalized cone subconvexlikeness of set-valued maps.

Theorem 16. Let D be v -closed, $\epsilon \in C$, $\bar{x} \in S$, and $0 \in G(\bar{x})$. Suppose that the following conditions hold:

- (i) $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (VP);
- (ii) $\bar{I}(x)$ is generalized $C \times D$ -subconvexlike on A , where $\bar{I}(x) = (F(x) - \bar{y} + \epsilon) \times G(x)$;

$$(iii) \text{vcl}(\text{cone}(G(A) + D)) = Z;$$

$$(iv) \bar{y} \in \epsilon\text{-}H_{\max}(\bigcup_{T \in L^+(Z,Y)} L(\bar{x}, T), C).$$

Then, there exists $\bar{T} \in L^+(Z, Y)$ such that $(\bar{x}, \bar{T})$ is an ϵ -Henig proper saddle point of L .

4. ϵ -Duality

In this section, we will give several duality theorems characterized by ϵ -Henig proper efficiency of set-valued optimization problems in linear spaces.

Definition 17. Let $\epsilon \in C$ and let B be a base of C . The set-valued map $\Phi : L^+(Z, Y) \rightrightarrows Y$, defined by $\Phi(T) = \epsilon\text{-}H_{\min}(\bigcup_{x \in A} L(x, T), B)$, is called an ϵ -Henig properly dual map of (VP).

Now, we construct the following duality problem of the primal problem (VP):

$$(VD) \max \bigcup_{T \in L^+(Z,Y)} \Phi(T). \quad (34)$$

Definition 18. Let $\epsilon \in C$. $\bar{y} \in \bigcup_{T \in L^+(Z,Y)} \Phi(T)$ is called an ϵ -efficient point of (VD) if and only if

$$\left(\bigcup_{T \in L^+(Z,Y)} \Phi(T) - \bar{y} - \epsilon \right) \cap (C \setminus \{0\}) = \emptyset. \quad (35)$$

Theorem 19 (ϵ -weak duality). Let $\epsilon \in C$, $\bar{x} \in S$, and $\bar{y} \in \bigcup_{T \in L^+(Z,Y)} \Phi(T)$. Then, $(\bar{y} - F(\bar{x}) - \epsilon) \cap (C \setminus \{0\}) = \emptyset$.

Proof. Since $\bar{y} \in \bigcup_{T \in L^+(Z,Y)} \Phi(T)$, there exists $\bar{T} \in L^+(Z, Y)$ such that $\bar{y} \in \Phi(\bar{T})$. Clearly, $\bar{y} \in \epsilon\text{-}H_{\min}(\bigcup_{x \in A} L(x, \bar{T}), B)$. Thus, there exists a balanced, absorbent, and convex set V with $0 \notin B + V$ such that $\text{cone}(\bigcup_{x \in A} L(x, \bar{T}) - \bar{y} + \epsilon) \cap (-C_V(B)) = \{0\}$. It is easy to check $C \setminus \{0\} \subseteq C_V(B) \setminus \{0\}$. Therefore, $(\bar{y} - \bigcup_{x \in A} L(x, \bar{T}) - \epsilon) \cap (C \setminus \{0\}) = \emptyset$. Since $\bar{x} \in S \subseteq A$, $(\bar{y} - F(\bar{x}) - \bar{T}(G(\bar{x})) - \epsilon) \cap (C \setminus \{0\}) = \emptyset$. Because $\bar{x} \in S$, there exists $\bar{z} \in G(\bar{x}) \cap (-D)$ such that $(\bar{y} - F(\bar{x}) - \bar{T}(\bar{z}) - \epsilon) \cap (C \setminus \{0\}) = \emptyset$. Since $\bar{T}(\bar{z}) \in C$, it is easy to check that $(\bar{y} - F(\bar{x}) - \epsilon) \cap (C \setminus \{0\}) = \emptyset$. $\square$

Theorem 20 (ϵ -converse duality). Let $\epsilon \in C$ and $\bar{x} \in S$. If $\bar{y} \in F(\bar{x}) \cap \bigcup_{T \in L^+(Z,Y)} \Phi(T)$ and $0 \in G(\bar{x})$, then $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (VP) and $\bar{y}$ is ϵ -efficient point of (VD).

Proof. Since $\bar{y} \in F(\bar{x}) \cap \bigcup_{T \in L^+(Z,Y)} \Phi(T)$, there exists $\bar{T} \in L^+(Z, Y)$ such that $\bar{y} \in \Phi(\bar{T})$. It follows from $0 \in G(\bar{x})$ and the definition of Φ that $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (UVP) $_{\bar{T}}$. According to Lemma 9, $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (VP). Because $\bar{x} \in S$ and $\bar{y} \in \bigcup_{T \in L^+(Z,Y)} \Phi(T)$, using Theorem 19, we have $(\bigcup_{T \in L^+(Z,Y)} \Phi(T) - F(\bar{x}) - \epsilon) \cap (C \setminus \{0\}) = \emptyset$. Clearly, $(\bigcup_{T \in L^+(Z,Y)} \Phi(T) - \bar{y} - \epsilon) \cap (C \setminus \{0\}) = \emptyset$. Therefore, $\bar{y}$ is ϵ -efficient point of (VD). $\square$

Theorem 21 (ϵ -strong duality). Let $\epsilon \in C$, $\bar{x} \in S$, and $0 \in G(\bar{x})$. Suppose that the following conditions hold:

- (i) $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (VP);
- (ii) $\bar{I}(x)$ is generalized $C \times D$ -subconvexlike on A , where $\bar{I}(x) = (F(x) - \bar{y} + \epsilon) \times G(x)$;
- (iii) $\text{vcl}(\text{cone}(G(A) + D)) = Z$.

Then, $\bar{y}$ is ϵ -efficient point of (VD).

Proof. According to Lemma 15, there exists $\bar{T} \in L^+(Z, Y)$ such that $(\bar{x}, \bar{y})$ is an ϵ -Henig properly efficient element of (UVP) $_{\bar{T}}$. Since $0 \in G(\bar{x})$, we have

$$\bar{y} \in \epsilon\text{-}H_{\min}\left(\bigcup_{x \in A} L(x, \bar{T}), B\right) = \Phi(\bar{T}) \subseteq \bigcup_{T \in L^+(Z,Y)} \Phi(T). \quad (36)$$

Since $\bar{y} \in F(\bar{x})$, it follows from Theorem 19 that $\bar{y}$ is ϵ -efficient point of (VD). $\square$

5. Conclusions

Based on [7], we introduce the concept of ϵ -Henig saddle point of the set-valued map in linear spaces. The relationships between the ϵ -Henig saddle point of the set-valued map and the ϵ -Henig properly efficient element of the set-valued optimization problem are established. Some duality theorems are obtained in the sense of ϵ -Henig proper efficiency. When ϵ -Henig proper efficiency is replaced by ϵ -super efficiency in linear spaces, whether the conclusions of this paper hold is an interesting topic.

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