Research Article

Taylor's Expansion for Composite Functions

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We build a Taylor's expansion for composite functions. Some applications are introduced, where the proposed technique allows the authors to obtain an asymptotic expansion of high order in many small parameters of solutions.

1. Introduction

Let $f \in C^{N+1}([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})$ and $u_\alpha \in W = \{v \in L^\infty(0, T; H^2(0, 1)) : v_t \in L^\infty(0, T; H^1(0, 1))\}, \alpha \in \mathbb{Z}_+^n, |\alpha| \leq N$.

Let two functions $h(x, t, \varepsilon)$ and $F(x, t, \varepsilon)$, which depend on (x, t) and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, be defined as follows:

$$h(x, t, \varepsilon) = \sum_{|\alpha| \leq N} u_\alpha(x, t) \varepsilon^\alpha, \quad (1)$$

where

$$\begin{aligned} \varepsilon^\alpha &= \varepsilon_1^{\alpha_1} \cdots \varepsilon_n^{\alpha_n}, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n, \\ \alpha &= (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n, \\ |\alpha| &= \alpha_1 + \cdots + \alpha_n, \end{aligned} \quad (2)$$

$$F(x, t, \varepsilon) = f\left(x, t, h, h_x, h_t, \|h\|^2, \|h_x\|^2, \|h_t\|^2\right),$$

where notation $\|\cdot\|$ stands for the norm in $L^2(0, 1)$.

The function $F(x, t, \varepsilon)$ has the form of a composite function as follows:

$$\begin{aligned} F(x, t, \varepsilon) &= f(x, t, g_1(\varepsilon), g_2(\varepsilon), \\ &\quad g_3(\varepsilon), g_4(\varepsilon), g_5(\varepsilon), g_6(\varepsilon)), \end{aligned} \quad (3)$$

where

$$\begin{aligned} g_1(\varepsilon) &= h(x, t, \varepsilon) = \sum_{|\alpha| \leq N} u_\alpha(x, t) \varepsilon^\alpha, \\ g_2(\varepsilon) &= h_x(x, t, \varepsilon) = \sum_{|\alpha| \leq N} u_{\alpha x}(x, t) \varepsilon^\alpha, \\ g_3(\varepsilon) &= h_t(x, t, \varepsilon) = \sum_{|\alpha| \leq N} u_{\alpha t}(x, t) \varepsilon^\alpha, \\ g_4(\varepsilon) &= \|h(t, \varepsilon)\|^2 = \left\| \sum_{|\alpha| \leq N} u_\alpha(t) \varepsilon^\alpha \right\|^2 \\ &= \sum_{|\delta| \leq 2N} \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_\alpha(t), u_\beta(t) \rangle \varepsilon^\delta \\ &= \sum_{|\alpha| \leq 2N} g_\alpha^{[4]}(t) \varepsilon^\alpha, \\ g_5(\varepsilon) &= \|h_x(t, \varepsilon)\|^2 = \left\| \sum_{|\alpha| \leq N} u_{\alpha x}(t) \varepsilon^\alpha \right\|^2 \\ &= \sum_{|\delta| \leq 2N} \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_{\alpha x}(t), u_{\beta x}(t) \rangle \varepsilon^\delta \\ &= \sum_{|\alpha| \leq 2N} g_\alpha^{[5]}(t) \varepsilon^\alpha, \end{aligned}$$

$$\begin{aligned}
g_6(\varepsilon) &= \|h_t(t, \varepsilon)\|^2 = \left\| \sum_{|\alpha| \leq N} u_{\alpha t}(t) \varepsilon^\alpha \right\|^2 \\
&= \sum_{|\delta| \leq 2N} \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_{\alpha t}(t), u_{\beta t}(t) \rangle \varepsilon^\delta \\
&= \sum_{|\alpha| \leq 2N} g_\alpha^{[6]}(t) \varepsilon^\alpha,
\end{aligned} \tag{4}$$

where notation $\langle \cdot, \cdot \rangle$ stands for the scalar product in $L^2(0, 1)$.

Under the above assumptions, in order to obtain an asymptotic expansion of high order in many small parameters of solutions in recent papers [1–4], we need to solve the following problem.

Problem 1. Establish the functions $F_\alpha = F_\alpha(x, t, |\alpha|) \leq N$ (independent of ε) such that for $\|\varepsilon\| = \sqrt{\varepsilon_1^2 + \dots + \varepsilon_n^2}$ being small enough,

$$F(x, t, \varepsilon) = \sum_{|\alpha| \leq N} F_\alpha(x, t) \varepsilon^\alpha + O(\|\varepsilon\|^{N+1}); \tag{5}$$

it means that

$$\left\| F - \sum_{|\alpha| \leq N} F_\alpha \varepsilon^\alpha \right\|_{L^\infty(0, T; L^2(0, 1))} \leq C_N \|\varepsilon\|^{N+1}, \tag{6}$$

where the constant C_N is independent from ε .

By Taylor's expansion for $F(x, t, \varepsilon)$, it implies that

$$F_\alpha(x, t) = \frac{1}{\alpha!} D^\alpha F(x, t, 0), \quad |\alpha| \leq N. \tag{7}$$

However, for $F(x, t, \varepsilon)$ which is given in (3), it is very difficult to calculate $D^\alpha F(x, t, 0)$, and so, the Problem 1 cannot proceed. Which method can be used for solving the Problem 1? To answer, let us note that the function $F(x, t, \varepsilon)$ has the form of a composite function; this view suggests that we need to construct the Taylor-Maclaurin expansion of the composite function. So, first in Section 2 we solve the following problem.

Problem 2. Let Ω be an open subset of $\mathbb{R}^n$ and $0 \in \Omega$. Let $g = (g_1, \dots, g_p) \in C^{N+1}(\Omega; \mathbb{R}^p)$ and $f \in C^{N+1}(\mathbb{R}^p; \mathbb{R})$. Seek the representation formula for $f \circ g$, such that for $\|x\|$ being small enough,

$$\begin{aligned}
(f \circ g)(x) &= f(g_1(x), \dots, g_p(x)) \\
&= \sum_{|\alpha| \leq N} d_\alpha x^\alpha + O(\|x\|^{N+1}),
\end{aligned} \tag{8}$$

where $d_\alpha, |\alpha| \leq N$ are calculated from the values of the given functions $f, g_1, \dots, g_p$ and of their derivatives at a suitable point.

Next in Section 3, as an application of the method used, we study the Problem 1. This technique is also a great help for the authors to obtain an asymptotic expansion as they want in recent papers [1–4].

2. Solving the Problem 2

We use the following notations. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we put

$$\begin{aligned}
|\alpha| &= \alpha_1 + \dots + \alpha_n, & \alpha! &= \alpha_1! \dots \alpha_n!, \\
\|x\| &= \sqrt{x_1^2 + \dots + x_n^2}, & x^\alpha &= x_1^{\alpha_1} \dots x_n^{\alpha_n}, \\
\alpha, \beta &\in \mathbb{Z}_+^n, & \alpha \leq \beta &\iff \alpha_i \leq \beta_i \quad \forall i = 1, \dots, n.
\end{aligned} \tag{9}$$

The following lemma is useful to solve the Problems 1 and 2.

Lemma 3. Let $m, N \in \mathbb{N}$ and $a_\alpha \in \mathbb{R}, \alpha \in \mathbb{Z}_+^n, 1 \leq |\alpha| \leq N$. Then

$$\left(\sum_{1 \leq |\alpha| \leq N} a_\alpha x^\alpha \right)^m = \sum_{m \leq |\alpha| \leq mN} T_N^{(m)}[a]_\alpha x^\alpha, \tag{10}$$

where the coefficients $T_N^{(m)}[a]_\alpha, m \leq |\alpha| \leq mN$ depending on $a = (a_\alpha), \alpha \in \mathbb{Z}_+^n, 1 \leq |\alpha| \leq N$ are defined by the recurrent formulas

$$\begin{aligned}
T_N^{(m)}[a]_\alpha &= \begin{cases} u_\alpha, & 1 \leq |\alpha| \leq N, \quad m = 1, \\ \sum_{\beta \in A_\alpha^{(m)}(N)} a_{\alpha-\beta} T_N^{(m-1)}[u]_\beta, & m \leq |\alpha| \leq mN, \quad m \geq 2, \end{cases} \\
A_\alpha^{(m)}(N) &= \{\beta \in \mathbb{Z}_+^n : \beta \leq \alpha, 1 \leq |\alpha - \beta| \leq N, \\
&\quad m - 1 \leq |\beta| \leq (m - 1)N\}.
\end{aligned} \tag{11}$$

The proof of Lemma 3 can be found in [2].

Now, using Taylor's expansion of the function f around the point $g = (g_1, \dots, g_p) \in \mathbb{R}^p$, we obtain that

$$\begin{aligned}
f(g_1 + H_1, \dots, g_p + H_p) &= f(g_1, \dots, g_p) \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g_1, \dots, g_p) H^\alpha \\
&\quad + R_N[f, H],
\end{aligned} \tag{12}$$

$H = (H_1, \dots, H_p) \in \mathbb{R}^p, \|H\|$ is small enough,

$R_N[f, H] = (N + 1)$

$$\begin{aligned}
&\times \sum_{|\alpha| = N+1} \frac{1}{\alpha!} H^\alpha \int_0^1 (1 - \theta)^N D^\alpha \\
&\quad \times f(g + \theta H) d\theta.
\end{aligned}$$

Similarly, we use Maclaurin's expansion of g_i as follows:

$$\begin{aligned} H_i &= g_i(x) - g_i(0) \\ &= \sum_{1 \leq |\beta| \leq N} \frac{1}{\beta!} D^\beta g_i(0) x^\beta + R_N[g_i, x] \\ &= P_i + R_i, \quad x \in \mathbb{R}^n, \\ &\quad \|x\| \text{ is small enough,} \\ P_i &= \sum_{1 \leq |\beta| \leq N} \frac{1}{\beta!} D^\beta g_i(0) x^\beta, \end{aligned} \quad (13)$$

$$\begin{aligned} R_i &= R_N[g_i, x] \\ &= (N+1) \sum_{|\beta|=N+1} \frac{1}{\beta!} x^\beta \int_0^1 (1-\theta)^N D^\beta g_i(\theta x) d\theta. \end{aligned}$$

Substituting (13) into (12), we get

$$\begin{aligned} f(g_1(x), \dots, g_p(x)) &= f(g_1(0), \dots, g_p(0)) \\ &\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g_1(0), \dots, g_p(0)) H^\alpha \\ &\quad + R_N[f, H] \\ &= f(g(0)) + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) P^\alpha \\ &\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) (H^\alpha - P^\alpha) \\ &\quad + R_N[f, H] \\ &= f(g(0)) + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) P^\alpha \\ &\quad + \|x\|^{N+1} R_N^{(1)}[f, g, H, R], \end{aligned} \quad (14)$$

where

$$\begin{aligned} \|x\|^{N+1} R_N^{(1)}[f, g, H, R] &= \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) (H^\alpha - P^\alpha) + R_N[f, H], \\ R_N[f, H] &= (N+1) \sum_{|\alpha|=N+1} \frac{1}{\alpha!} H^\alpha \int_0^1 (1-\theta)^N D^\alpha \\ &\quad \times f(g(0) + \theta H) d\theta. \end{aligned} \quad (15)$$

Applying Lemma 3, with $a = \sigma^{(i)} = (\sigma_\beta^{(i)}), \sigma_\beta^{(i)} = (1/\beta!) D^\beta g_i(0), 1 \leq |\beta| \leq N$, it implies that

$$\begin{aligned} P_i^{\alpha_i} &= \left(\sum_{1 \leq |\beta| \leq N} \frac{1}{\beta!} D^\beta g_i(0) x^\beta \right)^{\alpha_i} \equiv \left(\sum_{1 \leq |\beta| \leq N} \sigma_\beta^{(i)} x^\beta \right)^{\alpha_i} \\ &= \sum_{\alpha_i \leq |\gamma_i| \leq \alpha_i N} T_N^{(\alpha_i)}[\sigma^{(i)}]_{\gamma_i} x^{\gamma_i}. \end{aligned} \quad (16)$$

Hence,

$$\begin{aligned} P^\alpha &= P_1^{\alpha_1} \dots P_p^{\alpha_p} \\ &= \left(\sum_{1 \leq |\beta| \leq N} \frac{1}{\beta!} D^\beta g_1(0) x^\beta \right)^{\alpha_1} \\ &\quad \dots \left(\sum_{1 \leq |\beta| \leq N} \frac{1}{\beta!} D^\beta g_p(0) x^\beta \right)^{\alpha_p} \\ &= \sum_{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N} T_N^{(\alpha_1)}[\sigma^{(1)}]_{\gamma_1} x^{\gamma_1} \\ &\quad \dots \sum_{\alpha_p \leq |\gamma_p| \leq \alpha_p N} T_N^{(\alpha_p)}[\sigma^{(p)}]_{\gamma_p} x^{\gamma_p} \\ &= \sum_{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N,} T_N^{(\alpha_1)}[\sigma^{(1)}]_{\gamma_1} \dots T_N^{(\alpha_p)}[\sigma^{(p)}]_{\gamma_p} x^{\gamma_1 + \dots + \gamma_p} \\ &\quad \vdots \\ &\quad \alpha_p \leq |\gamma_p| \leq \alpha_p N \\ &= \sum_{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N,} T_N^{(\alpha_1)}[\sigma^{(1)}]_{\gamma_1} \dots T_N^{(\alpha_p)}[\sigma^{(p)}]_{\gamma_p} x^\delta \\ &\quad \vdots \\ &\quad \alpha_p \leq |\gamma_p| \leq \alpha_p N \\ &= \sum_{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N, \gamma_1 + \dots + \gamma_p = \delta} T_N^{(\alpha_1)}[\sigma^{(1)}]_{\gamma_1} \\ &\quad \vdots \\ &\quad \alpha_p \leq |\gamma_p| \leq \alpha_p N \\ &\quad \dots T_N^{(\alpha_p)}[\sigma^{(p)}]_{\gamma_p} x^\delta \\ &= \sum_{|\alpha| \leq |\delta| \leq |\alpha| N} \sum_{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N, \gamma_1 + \dots + \gamma_p = \delta} T_N^{(\alpha_1)}[\sigma^{(1)}]_{\gamma_1} \\ &\quad \vdots \\ &\quad \alpha_p \leq |\gamma_p| \leq \alpha_p N \\ &\quad \dots T_N^{(\alpha_p)}[\sigma^{(p)}]_{\gamma_p} x^\delta \\ &\equiv \sum_{|\alpha| \leq |\delta| \leq |\alpha| N} \Phi_\delta[\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \end{aligned}$$

$$\begin{aligned}
&= \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \\
&\quad + \sum_{N+1 \leq |\delta| \leq |\alpha|N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta, \\
&\Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] \\
&= \sum_{\substack{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N, \gamma_1 + \dots + \gamma_p = \delta \\ \vdots \\ \alpha_p \leq |\gamma_p| \leq \alpha_p N}} \sum T_N^{(\alpha_1)} [\sigma^{(1)}]_{\gamma_1} \\
&\quad \dots T_N^{(\alpha_p)} [\sigma^{(p)}]_{\gamma_p}.
\end{aligned}$$

(17)

Consequently,

$$\begin{aligned}
f(g(x)) &= f(g(0)) \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) P^\alpha \\
&\quad + \|x\|^{N+1} R_N^{(1)} [f, g, H, R] \\
&= f(g(0)) + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) \\
&\quad \times \sum_{|\alpha| \leq |\delta| \leq |\alpha|N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \\
&\quad + \|x\|^{N+1} R_N^{(1)} [f, g, H, R] \\
&= f(g(0)) \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) \\
&\quad \times \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) \\
&\quad \times \sum_{N+1 \leq |\delta| \leq |\alpha|N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \\
&\quad + \|x\|^{N+1} R_N^{(1)} [f, g, H, R] \\
&= f(g(0)) + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(g(0)) \\
&\quad \times \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \\
&\quad + \|x\|^{N+1} R_N^{(2)} [f, g, H, R, x] \\
&= f(g(0)) \\
&\quad + \sum_{1 \leq |\delta| \leq N} \sum_{1 \leq |\alpha| \leq |\delta|} \frac{1}{\alpha!} D^\alpha f(g(0)) \\
&\quad \times \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}] x^\delta \\
&\quad + \|x\|^{N+1} R_N^{(2)} [f, g, H, R, x]
\end{aligned}$$

$$\begin{aligned}
&= f(g(0)) + \sum_{1 \leq |\delta| \leq N} d_\delta x^\delta \\
&\quad + \|x\|^{N+1} R_N^{(2)} [f, g, H, R, x].
\end{aligned}$$

(18)

Clearly, the Problem 2 is solved with

$$d_\delta = \sum_{1 \leq |\alpha| \leq |\delta|} \frac{1}{\alpha!} D^\alpha f(g(0)) \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(p)}]. \quad (19)$$

3. Solving the Problem 1

As an application of the method used in Section 2 for $p = 6$, $\Omega = (0, 1)$, $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$, and $f = f(x, t, g_1, \dots, g_6) \in C^{N+1}(\mathbb{R}^3 \times \mathbb{R}_+^3; \mathbb{R})$, for each fixed $(x, t) \in [0, 1] \times \mathbb{R}_+$; $g_i \in C^{N+1}(\Omega; \mathbb{R})$, $i = 1, 2, 3$, for each fixed $(x, t) \in [0, 1] \times \mathbb{R}_+$; $g_i \in C^{N+1}(\Omega; \mathbb{R}_+)$, $i = 4, 5, 6$, for each fixed $t \in \mathbb{R}_+$, we now investigate the Problem 1.

For each fixed $(x, t) \in [0, 1] \times \mathbb{R}_+$, using Taylor's expansion of the function f around the point $g = (g_1, \dots, g_6) \in \mathbb{R}^3 \times \mathbb{R}_+^3$ up to order N , we obtain that

$$\begin{aligned}
f(x, t, g + H) &= f(x, t, g_1 + H_1, \dots, g_6 + H_6) \\
&= f(x, t, g_1, \dots, g_6) \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g_1, \dots, g_6) H^\alpha \\
&\quad + \|H\|^{N+1} R_N^{(1)} [f, H],
\end{aligned}$$

(20)

$$\begin{aligned}
&\|H\|^{N+1} R_N^{(1)} [f, H] \\
&= (N+1) \sum_{|\alpha|=N+1} \frac{1}{\alpha!} H^\alpha \int_0^1 (1-\theta)^N D^\alpha \\
&\quad \times f(g + \theta H) d\theta,
\end{aligned}$$

(21)

where $H = (H_1, \dots, H_6) \in \mathbb{R}^6$, $\|H\|$ is small enough, $D_i^{\alpha_i} f = \partial f / \partial g_i$, $i = 1, 2, \dots, 6$.

Next, the precise structure of the representation formulas for $g_1, \dots, g_6$ will be needed below to continue.

For each fixed $(x, t) \in [0, 1] \times \mathbb{R}_+$: we have the following.

(i) The representation formula for $g_i(\varepsilon) = g_i(x, t, \varepsilon)$, $i = 1, 2, 3$.

We rewrite $g_1(\varepsilon)$ as follows

$$\begin{aligned}
g_1(\varepsilon) &= \sum_{|\alpha| \leq N} u_\alpha(x, t) \varepsilon^\alpha \\
&\equiv g_1(0) + \sum_{1 \leq |\beta| \leq N} g_\beta^{[1]}(x, t) \varepsilon^\beta \\
&\equiv g_1(0) + H_1,
\end{aligned}$$

(22)

in which

$$\begin{aligned} g_1(0) &= g_1(x, t) = u_0(x, t), \quad |\beta| = 0, \\ g_\beta^{[1]}(x, t) &= u_\beta(x, t), \quad 1 \leq |\beta| \leq N. \end{aligned} \quad (23)$$

It is similar to $g_2(\varepsilon), g_3(\varepsilon)$; we write

$$\begin{aligned} g_2(\varepsilon) &= \sum_{|\alpha| \leq N} u_{\alpha x}(x, t) \varepsilon^\alpha \\ &\equiv g_2(0) + \sum_{1 \leq |\beta| \leq N} g_\beta^{[2]}(x, t) \varepsilon^\beta \equiv g_2(0) + H_2, \\ g_3(\varepsilon) &= \sum_{|\alpha| \leq N} u_{\alpha t}(x, t) \varepsilon^\alpha \\ &\equiv g_3(0) + \sum_{1 \leq |\beta| \leq N} g_\beta^{[3]}(x, t) \varepsilon^\beta \equiv g_3(0) + H_3, \end{aligned} \quad (24)$$

where

$$\begin{aligned} g_2(0) &= g_2(x, t) = u_{0x}(x, t), \\ g_3(0) &= g_3(x, t) = u_{0t}(x, t), \quad |\beta| = 0, \\ g_\beta^{[2]}(x, t) &= u_{\beta x}(x, t), \\ g_\beta^{[3]}(x, t) &= u_{\beta t}(x, t), \quad 1 \leq |\beta| \leq N. \end{aligned} \quad (25)$$

For each fixed $t \in \mathbb{R}_+$: we have the following.

(ii) The representation formula for $g_i(\varepsilon) = g_i(t, \varepsilon), i = 4, 5, 6$.

Write

$$\begin{aligned} g_4(\varepsilon) &= \left\| \sum_{|\alpha| \leq N} u_\alpha(t) \varepsilon^\alpha \right\|^2 \\ &= \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \langle u_\alpha(t), u_\beta(t) \rangle \varepsilon^{\alpha+\beta} \\ &= \|u_0(t)\|^2 \\ &\quad + \sum_{1 \leq |\delta| \leq N} G_\delta[u(t)] \varepsilon^\delta + \|\varepsilon\|^{N+1} R_N^{(1)}[u, \varepsilon] \\ &\equiv g_4(0) + \sum_{1 \leq |\beta| \leq N} g_\beta^{[4]}(t) \varepsilon^\beta + R_4 \end{aligned} \quad (26)$$

$$\equiv g_4(0) + H_4 \equiv g_4(0) + P_4 + R_4,$$

$$g_4(0) = \|u_0(t)\|^2,$$

$$\begin{aligned} g_\delta^{[4]}(t) &= G_\delta[u(t)] \\ &= \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_\alpha(t), u_\beta(t) \rangle, \\ &\quad 1 \leq |\delta| \leq N, \end{aligned} \quad (27)$$

$$G_\delta[u(t)] = \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_\alpha(t), u_\beta(t) \rangle,$$

$$1 \leq |\delta| \leq 2N,$$

$$R_4 = \|\varepsilon\|^{N+1} R_N^{(1)}[u, \varepsilon] = \sum_{N+1 \leq |\delta| \leq 2N} G_\delta[u(t)] \varepsilon^\delta.$$

Similarly,

$$\begin{aligned} g_5(\varepsilon) &= \left\| \sum_{|\alpha| \leq N} u_{\alpha x}(t) \varepsilon^\alpha \right\|^2 \\ &= \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \langle u_{\alpha x}(t), u_{\beta x}(t) \rangle \varepsilon^{\alpha+\beta} \\ &= \|u_{0x}(t)\|^2 + \sum_{1 \leq |\delta| \leq N} G_\delta[u_x(t)] \varepsilon^\delta \\ &\quad + \|\varepsilon\|^{N+1} R_N^{(1)}[u_x, \varepsilon] \\ &\equiv g_5(0) + \sum_{1 \leq |\beta| \leq N} g_\beta^{[5]}(t) \varepsilon^\beta + R_5 \\ &\equiv g_5(0) + H_5 \equiv g_5(0) + P_5 + R_5, \end{aligned} \quad (28)$$

in which

$$\begin{aligned} g_5(0) &= \|u_{0x}(t)\|^2, \\ g_\delta^{[5]}(t) &= G_\delta[u_x(t)] \\ &= \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_{\alpha x}(t), u_{\beta x}(t) \rangle, \\ &\quad 1 \leq |\delta| \leq N, \\ R_5 &= \|\varepsilon\|^{N+1} R_N^{(1)}[u_x, \varepsilon] = \sum_{N+1 \leq |\delta| \leq 2N} G_\delta[u_x(t)] \varepsilon^\delta, \end{aligned} \quad (29)$$

$$\begin{aligned} g_6(\varepsilon) &= \left\| \sum_{|\alpha| \leq N} u_{\alpha t}(t) \varepsilon^\alpha \right\|^2 \\ &= \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \langle u_{\alpha t}(t), u_{\beta t}(t) \rangle \varepsilon^{\alpha+\beta} \\ &= \|u_{0t}(t)\|^2 + \sum_{1 \leq |\delta| \leq N} G_\delta[u_t(t)] \varepsilon^\delta \\ &\quad + \|\varepsilon\|^{N+1} R_N^{(1)}[u_t, \varepsilon] \\ &\equiv g_6(0) + \sum_{1 \leq |\beta| \leq N} g_\beta^{[6]}(t) \varepsilon^\beta + R_6 \\ &\equiv g_6(0) + H_6 \equiv g_6(0) + P_6 + R_6, \end{aligned} \quad (30)$$

where

$$\begin{aligned} g_6(0) &= \|u_{0t}(t)\|^2, \\ g_\delta^{[6]}(t) &= G_\delta[u_t(t)] \\ &= \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{\alpha+\beta=\delta} \langle u_{\alpha t}(t), u_{\beta t}(t) \rangle, \\ &\quad 1 \leq |\delta| \leq N, \\ R_6 &= \|\varepsilon\|^{N+1} R_N^{(1)}[u_t, \varepsilon] = \sum_{N+1 \leq |\delta| \leq 2N} G_\delta[u_t(t)] \varepsilon^\delta. \end{aligned} \quad (31)$$

Then (22), (24), (26), (28), and (30) imply that

$$g_i(\varepsilon) - g_i(0) = H_i = P_i + R_i = \sum_{1 \leq |\beta| \leq N} g_\beta^{[i]} \varepsilon^\beta + R_i,$$

$$R_i = 0, \quad i = 1, 2, 3, \quad R_j = O(\|\varepsilon\|^{N+1}), \quad j = 4, 5, 6. \quad (32)$$

We also need the following lemma.

Lemma 4. For all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| \geq 1$, then

$$H^\alpha = H_1^{\alpha_1} H_2^{\alpha_2} H_3^{\alpha_3} H_4^{\alpha_4} H_5^{\alpha_5} H_6^{\alpha_6} \\ = \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta + O(\|\varepsilon\|^{N+1}), \quad (33)$$

where

$$\Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \\ = \sum_{\substack{\alpha_1 \leq |\gamma_1| \leq \alpha_1 N, \\ \gamma_1 + \dots + \gamma_6 = \delta}} T_N^{(\alpha_1)} [\sigma^{(1)}]_{\gamma_1} \dots T_N^{(\alpha_6)} [\sigma^{(6)}]_{\gamma_6}, \\ \vdots \\ \alpha_6 \leq |\gamma_6| \leq \alpha_6 N \\ \sigma^{(i)} = (\sigma_\beta^{(i)}), \quad \sigma_\beta^{(i)} = g_\beta^{[i]}, \quad 1 \leq |\beta| \leq N, \quad i = 1, 2, \dots, 6. \quad (34)$$

Proof of Lemma 4. We have

$$H^\alpha = H_1^{\alpha_1} H_2^{\alpha_2} H_3^{\alpha_3} H_4^{\alpha_4} H_5^{\alpha_5} H_6^{\alpha_6} \\ = P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} (P_4 + R_4)^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ = P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} P_4^{\alpha_4} P_5^{\alpha_5} P_6^{\alpha_6} + P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} \\ \times [(P_4 + R_4)^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} - P_4^{\alpha_4} P_5^{\alpha_5} P_6^{\alpha_6}]. \quad (35)$$

Note that $R_j = O(\|\varepsilon\|^{N+1})$, $j = 4, 5, 6$, so

$$(P_4 + R_4)^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} - P_4^{\alpha_4} P_5^{\alpha_5} P_6^{\alpha_6} \\ = (P_4 + R_4)^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ - P_4^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ + P_4^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} - P_4^{\alpha_4} P_5^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ + P_4^{\alpha_4} P_5^{\alpha_5} (P_6 + R_6)^{\alpha_6} - P_4^{\alpha_4} P_5^{\alpha_5} P_6^{\alpha_6} \\ = [(P_4 + R_4)^{\alpha_4} - P_4^{\alpha_4}] (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ + P_4^{\alpha_4} [(P_5 + R_5)^{\alpha_5} - P_5^{\alpha_5}] (P_6 + R_6)^{\alpha_6} \\ + P_4^{\alpha_4} P_5^{\alpha_5} [(P_6 + R_6)^{\alpha_6} - P_6^{\alpha_6}] \\ = R_4 \left[\sum_{j=0}^{\alpha_4-1} (P_4 + R_4)^j P_4^{\alpha_4-1-j} \right] \\ \times (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ + R_5 \left[\sum_{j=0}^{\alpha_5-1} (P_5 + R_5)^j P_5^{\alpha_5-1-j} \right] P_4^{\alpha_4} (P_6 + R_6)^{\alpha_6} \\ + R_6 \left[\sum_{j=0}^{\alpha_6-1} (P_6 + R_6)^j P_6^{\alpha_6-1-j} \right] P_4^{\alpha_4} P_5^{\alpha_5} \\ = \|\varepsilon\|^{N+1} R_N^{(1)} [H, \alpha, \varepsilon] = O(\|\varepsilon\|^{N+1}). \quad (36)$$

Combining (35) and (36) yields

$$H^\alpha = H_1^{\alpha_1} H_2^{\alpha_2} H_3^{\alpha_3} H_4^{\alpha_4} H_5^{\alpha_5} H_6^{\alpha_6} \\ = P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} (P_4 + R_4)^{\alpha_4} (P_5 + R_5)^{\alpha_5} (P_6 + R_6)^{\alpha_6} \\ = P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} P_4^{\alpha_4} P_5^{\alpha_5} P_6^{\alpha_6} \\ + \|\varepsilon\|^{N+1} R_N^{(1)} [H, \alpha, \varepsilon] \\ = \left(\sum_{1 \leq |\beta| \leq N} g_\beta^{(1)} \varepsilon^\beta \right)^{\alpha_1} \dots \left(\sum_{1 \leq |\beta| \leq N} g_\beta^{(6)} \varepsilon^\beta \right)^{\alpha_6} \\ + \|\varepsilon\|^{N+1} R_N^{(1)} [H, \alpha, \varepsilon] \quad (37) \\ = \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \\ + \sum_{N+1 \leq |\delta| \leq |\alpha| N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \\ + \|\varepsilon\|^{N+1} R_N^{(1)} [H, \alpha, \varepsilon] \\ \equiv \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \\ + \|\varepsilon\|^{N+1} R_N^{(2)} [H, \alpha, \varepsilon],$$

where

$$\|\varepsilon\|^{N+1} R_N^{(2)} [H, \alpha, \varepsilon] \\ = \sum_{N+1 \leq |\delta| \leq |\alpha| N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \quad (38) \\ + \|\varepsilon\|^{N+1} R_N^{(1)} [H, \alpha, \varepsilon].$$

Lemma 4 is proved. $\square$

Substituting (37) into (20), we obtain

$$F(x, t, \varepsilon^\alpha) \\ = f(x, t, h, h_x, h_t, \|h\|^2, \|h_x\|^2, \|h_t\|^2) \\ = f(x, t, g + H) = f(x, t, g(0) + H) \\ = f(x, t, g_1(0) + H_1, \dots, g_6(0) + H_6) \\ = f(x, t, g(0)) \\ + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) H^\alpha + \|H\|^{N+1} R_N^{(1)} [f, H] \\ = f(x, t, g(0)) \\ + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) \\ \times \left[\sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \right. \\ \left. + \|\varepsilon\|^{N+1} R_N^{(2)} [H, \alpha, \varepsilon] \right] \\ + \|H\|^{N+1} R_N^{(1)} [f, H]$$

$$\begin{aligned}
&= f(x, t, g(0)) \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) \\
&\quad \quad \times \left[\sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \right] \\
&\quad + \|\varepsilon\|^{N+1} \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) R_N^{(2)}[H, \alpha, \varepsilon] \\
&\quad \quad + \|H\|^{N+1} R_N^{(1)}[f, H] \\
&= f(x, t, g(0)) \\
&\quad + \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) \\
&\quad \quad \times \sum_{|\alpha| \leq |\delta| \leq N} \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \\
&\quad + \|\varepsilon\|^{N+1} R_N^{(3)}[f, g, H, \varepsilon] \\
&= f(x, t, g(0)) \\
&\quad + \sum_{1 \leq |\delta| \leq N} \sum_{|\alpha| \leq |\delta|} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) \Phi_\delta \\
&\quad \quad \times [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}] \varepsilon^\delta \\
&\quad + \|\varepsilon\|^{N+1} R_N^{(3)}[f, g, H, \varepsilon] \\
&= f(x, t, g(0)) + \sum_{1 \leq |\delta| \leq N} F_\delta(x, t) \varepsilon^\delta \\
&\quad + \|\varepsilon\|^{N+1} R_N^{(3)}[f, g, H, \varepsilon],
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
&\|\varepsilon\|^{N+1} R_N^{(3)}[f, g, H, \varepsilon] \\
&= \|\varepsilon\|^{N+1} \sum_{1 \leq |\alpha| \leq N} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) R_N^{(2)}[H, \alpha, \varepsilon] \\
&\quad + \|H\|^{N+1} R_N^{(1)}[f, H], \\
F_\delta(x, t) &= \sum_{|\alpha| \leq |\delta|} \frac{1}{\alpha!} D^\alpha f(x, t, g(0)) \Phi_\delta [\alpha, N, \sigma^{(1)}, \dots, \sigma^{(6)}], \\
&\quad \delta \in \mathbb{Z}_+^6, 1 \leq |\delta| \leq N.
\end{aligned} \tag{40}$$

Since $u_\alpha \in W = \{v \in L^\infty(0, T, H^2(0, 1)) : v_t \in L^\infty(0, T, H^1(0, 1)), \alpha \in \mathbb{Z}_+^n, |\alpha| \leq N, \text{ hence } u_\alpha, u_{\alpha x}, u_{\alpha t} \in L^\infty(0, T, H^1(0, 1)) \subset L^\infty(Q_T)\}$. Then from (39) and (40)₁, we get

$$\begin{aligned}
&\left\| F(\varepsilon) - \sum_{|\delta| \leq N} F_\delta \varepsilon^\delta \right\|_{L^\infty(0, T, L^2(0, 1))} \\
&= \|\varepsilon\|^{N+1} \left\| R_N^{(3)}[f, g, H, \varepsilon] \right\|_{L^\infty(0, T, L^2(0, 1))} \leq C_N \|\varepsilon\|^{N+1},
\end{aligned} \tag{41}$$

for $\|\varepsilon\|$ is small enough; consequently the Problem 1 is solved.

Remark 5. (i) This result improves the one in [1], for $n = 1$.

(ii) In case of $F(x, t, \varepsilon) = \mu(x, t, h, \|h_x\|^2)$, where $\mu \in C^{N+1}([0, 1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+; \mathbb{R})$, $n = p$, this result is also obtained in [3].

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