Research Article

A Note on Decomposing a Square Matrix as Sum of Two Square Nilpotent Matrices over an Arbitrary Field

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Let K be an arbitrary field and X a square matrix over K . Then X is sum of two square nilpotent matrices over K if and only if, for every algebraic extension L of K and arbitrary nonzero $\alpha \in L$, there exist idempotent matrices P_1 and P_2 over L such that $X = \alpha P_1 - \alpha P_2$.

1. Introduction

Botha (see [1]) proved that a square matrix A over a field K is a sum of two nilpotent matrices over K if and only if A is similar to a particular form. In an early paper, Pazzis (see [2]) gave necessary and sufficient conditions in which a matrix can be decomposed as a linear combination of two idempotents with given nonzero coefficients. The goal of this paper is to build a bridge that connects the result obtained in [1] with the result obtained in [2]. However, the relation between these two facts has not been formally discussed yet (more details in [3–9]).

If there is no statement, the meanings of notations mentioned in this paragraph hold all over the paper. K denotes an arbitrary field, $\bar{K}$ is its algebraic closure, L is an arbitrary algebraic extension of K , and $\text{car}(K)$ is the characteristic of K . Z^+ denotes the set of all positive integers, $[s] = \{z \in Z^+ \mid 1 \leq z \leq s\}$ for some $s \in Z^+$. $M_{m,n}(K)$ denotes the space consisting of all $m \times n$ matrices over K ; $M_n(K) = M_{n,n}(K)$. $r(A)$ is the rank of $A \in M_{m,n}(K)$. E denotes a vector space over K and $\dim(E)$ is the dimension of E . $X \in M_n(K)$ is called ${}_s^2N$ in $M_n(K)$ if there exist square nilpotent N_1 and $N_2 \in M_n(K)$ such that $X = N_1 + N_2$, while X is called an (α, β) composite in $M_n(K)$ if there exist idempotent P_1 and $P_2 \in M_n(K)$ such that $X = \alpha P_1 + \beta P_2$, where $\alpha, \beta \in K \setminus \{0\}$ (Definition 1 in [2]); in particular, X is called ${}_sP$ if X is an $(\alpha, -\alpha)$ composite in $M_n(L)$ for every algebraic extension L of K and arbitrary nonzero $\alpha \in L$ (when $\text{car}(K) = 2$, we still use ${}_sP$ for the meaning of (α, α) composites).

For $X \in M_n(K)$, on the one hand, we will prove that X is ${}_s^2N$ in $M_n(K)$ implies X is ${}_sP$; that is, the form provided by Botha satisfies the condition as in [2]; on the other hand, we will also prove that X is ${}_sP$ implies X is ${}_s^2N$ in $M_n(K)$; that is, we can derive the form provided in [1] from the results obtained in [2]. In fact, the following theorem is the main result of this paper.

Theorem 1 (main theorem). *Suppose K is an arbitrary field and $X \in M_n(K)$; then X is ${}_s^2N$ in $M_n(K)$ if and only if X is ${}_sP$.*

In Section 2, we will state some related theorems and notations from [2] and we will give some necessary corollaries. The proof of Theorem 1 will be carried out in Section 3.

2. More Notations and Necessary Corollaries

Suppose $X \in M_n(K)$ and $X_i \in M_{n_i}(K)$, we denote by $X = X_1 \oplus \cdots \oplus X_s$ the following matrix with $\sum_{i=1}^s n_i = n$:

$$\begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_s \end{pmatrix}. \quad (1)$$

Notation 1 (Notation 2 in [2]). Let $X \in M_n(K)$, $\lambda \in \bar{K}$ and $k \in Z^+$; we denote by

- (1) $j_k(X, \lambda)$ the number of blocks of size k for the eigenvalue λ in the Jordan reduction of X ;
- (2) $n_k(X, \lambda)$ the number of blocks of size greater or equal to k for the eigenvalue λ in the Jordan reduction of X .

Definition 2 (Definition 3 in [2]). Two sequences $(u_k)_{k \geq 1}$ and $(v_k)_{k \geq 1}$ are said to be intertwined if for all $k \in \mathbb{Z}^+$, $v_k \geq u_{k+1}$, and $u_k \geq v_{k+1}$.

Notation 2 (Notation 4 in [2]). Given a monic polynomial, $P = x^n - a_{n-1}x^{n-1} - \dots - a_1x - a_0$, denote the following $C(P)$ by its companion matrix:

$$C(P) = \begin{pmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & 0 & \dots & a_1 \\ 0 & 1 & 0 & \dots & a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} a_{n-1} \quad (2)$$

Theorem 3 (Theorem 1 in [2]). Assume $\text{car}(K) \neq 2$ and let $X \in M_n(K)$. Then X is an $(\alpha, -\alpha)$ composite if and only if all the following conditions hold.

- (1) The sequences $(n_k(X, \alpha))_{k \geq 1}$ and $(n_k(X, -\alpha))_{k \geq 1}$ are intertwined;
- (2) for all $\lambda \in \overline{K} \setminus \{0, \alpha, -\alpha\}$ and for all $k \in \mathbb{Z}^+$, $j_k(X, \lambda) = j_k(X, -\lambda)$.

Theorem 4 (Theorem 5 in [2]). Assume $\text{car}(K) = 2$ and let $X \in M_n(K)$. Then X is an $(\alpha, -\alpha)$ composite if and only if for every $\lambda \in \overline{K} \setminus \{0, \alpha\}$, all blocks in the Jordan reduction of X with respect to λ have an even size.

Suppose $X \in M_n(K)$ is $\pm P$, where $\text{car}(K) \neq 2$. Then X is $(\alpha, -\alpha)$ composite and $(\beta, -\beta)$ composite in $M_n(L)$ for some algebraic extension L of K , where $\alpha, \beta \in L \setminus \{0\}$ with $\alpha \neq \pm \beta$. By Theorem 3, the following statements are true:

- (1) for all $\lambda \in \overline{K} \in \{0, \alpha, -\alpha\}$ and for all $k \in \mathbb{Z}^+$, $j_k(X, \lambda) = j_k(X, -\lambda)$;
- (2) for all $\lambda \in \overline{K} \in \{0, \beta, -\beta\}$ and for all $k \in \mathbb{Z}^+$, $j_k(X, \lambda) = j_k(X, -\lambda)$.

so for all $\lambda \in \overline{K} \setminus \{0\}$ and for all $k \in \mathbb{Z}^+$, $j_k(X, \lambda) = j_k(X, -\lambda)$.

On the other hand, note that for nonzero $\alpha \in \overline{K}$ with $\text{car}(K) \neq 2$, the sequences $(n_k(X, \alpha))_{k \geq 1}$ and $(n_k(X, -\alpha))_{k \geq 1}$ are intertwined if for all $k \in \mathbb{Z}^+$, $j_k(X, \alpha) = j_k(X, -\alpha)$. Then for all $\lambda \in \overline{K} \setminus \{0\}$, $k \in \mathbb{Z}^+$, $j_k(X, \lambda) = j_k(X, -\lambda)$ implies that for every algebraic extension L of K and arbitrary nonzero $\alpha \in L$, X is an $(\alpha, -\alpha)$ composite in $M_n(L)$; that is, X is $\pm P$.

Therefore the following corollary is true.

Corollary 5. Assume $\text{car}(K) \neq 2$ and let $X \in M_n(K)$. Then X is $\pm P$ if and only if for all $\lambda \in \overline{K} \setminus \{0\}$ for all $k \in \mathbb{Z}^+$, $j_k(X, \lambda) = j_k(X, -\lambda)$.

Similarly, we can derive the following corollary from Theorem 4.

Corollary 6. Assume $\text{car}(K) = 2$ and let $X \in M_n(K)$. Then X is $\pm P$ if and only if for every $\lambda \in \overline{K} \setminus \{0\}$, all blocks in the Jordan reduction of X with respect to λ have an even size.

Naturally, we derive the following corollary from the above two corollaries.

Corollary 7. Every nilpotent is $\pm P$.

In fact, arbitrary nilpotent is not only $\pm P$ but also ${}_s^2N$.

Lemma 8. Every nilpotent $N \in M_n(K)$ is ${}_s^2N$.

Proof. For arbitrary field K , let $N \in M_n(K)$ is nilpotent; then N is similar to $N_1 \oplus N_2 \oplus \dots \oplus N_s$, where for every $i \in [s]$, $N_i \in M_{r_i}(K)$, $\sum_{i=1}^s r_i = n$, and both the characteristic polynomial and minimal polynomial of N_i are x^{r_i} . Furthermore, N_i is similar to $C(x^{r_i})$ as follows:

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}_{r_i \times r_i} \quad (3)$$

That is, $C(x^{r_i}) = E_{2,1} + E_{3,2} + \dots + E_{r_i, r_i-1} \in M_{r_i}(K)$.

When r_i is even, $C(x^{r_i}) = \sum_{j=1}^{r_i/2} E_{2j, 2j-1} + \sum_{j=1}^{r_i/2-1} E_{2j+1, 2j}$; when r_i is odd, $C(x^{r_i}) = \sum_{j=1}^{(r_i-1)/2} E_{2j, 2j-1} + \sum_{j=1}^{(r_i-1)/2} E_{2j+1, 2j}$. Note that both $\sum E_{2j, 2j-1}$ and $\sum E_{2j+1, 2j}$ are square nilpotent matrices then $C(x^{r_i})$ is ${}_s^2N$, and N_i is ${}_s^2N$ follows. Hence N is ${}_s^2N$. $\square$

3. Proof of Main Theorem

${}_s^2N \rightarrow \pm P$. Suppose $X \in M_n(K)$ is ${}_s^2N$ in $M_n(K)$; that is, there exist square nilpotent matrices N_1 and $N_2 \in M_n(K)$ such that $X = N_1 + N_2$. It will take two steps to prove X is $\pm P$.

Step 1. If X is nonsingular, then X is $\pm P$.

Since $X = N_1 + N_2$ with $N_1^2 = N_2^2 = 0$, inspect the eigenspaces of N_1 and N_2 . Note that N_1 and N_2 are square nilpotent matrices, their ranks satisfy the following inequality matrices.

$$r(N_1) + r(N_2) \leq n, \quad (4)$$

where equality holds if and only if $r(N_1) = r(N_2) = n/2$.

At first, X is nonsingular implies 0 is not its eigenvalue. Secondly, if the inequality is strict, then intersection of eigenspaces of N_1 and N_2 contains nonzero vectors; that is, there exists nonzero $x \in M_{n,1}(K)$ such that $N_1x = N_2x = 0$, which implies that 0 is one of eigenvalues of X . This is a contradiction. Hence, $r(N_1) + r(N_2) = n$; that is, n is even and N_1 and N_2 are similar but not equal.

Because N_1 is square nilpotent with $r(N_1) = n/2$, we can choose $n/2$ linear independent vectors from the set of its column vectors which can make up a base of eigenspace of N_1 and denote β by the $n \times (n/2)$ matrix consisting of these $n/2$ columns. Correspondingly, we have $n \times (n/2)$ matrix γ with

all columns from the set of columns of N_2 . Because 0 is the only vector in the intersection of eigenspaces of N_1 and N_2 , $n \times n$ matrix $(\beta \ \gamma)$ is nonsingular.

$N_1^2 \gamma = 0$ implies that nonzero column vectors of $N_1 \gamma$ are eigenvectors of N_1 and $N_1(\beta \ \gamma) = (0 \ N_1 \gamma)$ implies $r(N_1 \gamma) = n/2$. Hence; $N_1 \gamma$ and β are equal under certain column transformation; that is, there is an invertible matrix T_1 such that $N_1 \gamma = \beta T_1$. Correspondingly, there is an invertible matrix T_2 such that $N_2 \beta = \gamma T_2$.

Let $\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ be the inverse of $(\beta \ \gamma)$, where γ_1 and γ_2 are $(n/2) \times n$ matrices. Naturally, the following equation is true:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} (\beta \ \gamma) = \begin{pmatrix} \gamma_1 \beta & \gamma_1 \gamma \\ \gamma_2 \beta & \gamma_2 \gamma \end{pmatrix} = \begin{pmatrix} I_{n/2} & 0_{n/2} \\ 0_{n/2} & I_{n/2} \end{pmatrix}. \quad (5)$$

Now, we carry out the same similarity transformation on N_1 and N_2 as follows:

$$\begin{aligned} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} N_1 (\beta \ \gamma) &= \begin{pmatrix} \gamma_1 N_1 \beta & \gamma_1 N_1 \gamma \\ \gamma_2 N_1 \beta & \gamma_2 N_1 \gamma \end{pmatrix}, \\ \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} N_2 (\beta \ \gamma) &= \begin{pmatrix} \gamma_1 N_2 \beta & \gamma_1 N_2 \gamma \\ \gamma_2 N_2 \beta & \gamma_2 N_2 \gamma \end{pmatrix}. \end{aligned} \quad (6)$$

Note that $N_1 \gamma = \beta T_1$ and $N_2 \beta = \gamma T_2$, the above three equations imply that N_1 is similar to $\begin{pmatrix} 0_{n/2} & T_1 \\ 0_{n/2} & 0_{n/2} \end{pmatrix} \in M_n(K)$ and N_2 is similar to $\begin{pmatrix} 0_{n/2} & 0_{n/2} \\ T_2 & 0_{n/2} \end{pmatrix} \in M_n(K)$.

Hence, X is similar to $\begin{pmatrix} 0_{n/2} & T_1 \\ 0_{n/2} & 0_{n/2} \end{pmatrix} + \begin{pmatrix} 0_{n/2} & 0_{n/2} \\ T_2 & 0_{n/2} \end{pmatrix}$. For every algebraic extension L of K and arbitrary nonzero $\alpha \in L$, X is also similar to the following matrix:

$$\alpha \begin{pmatrix} I_{n/2} & \alpha^{-1} T_1 \\ 0_{n/2} & 0_{n/2} \end{pmatrix} - \alpha \begin{pmatrix} I_{n/2} & 0_{n/2} \\ -\alpha^{-1} T_2 & 0_{n/2} \end{pmatrix} \quad (7)$$

That is, X is ${}_{\pm}P$.

Step 2. If X is singular and similar to $Y \oplus N$, where Y is nonsingular and N is nilpotent. Then X is ${}_{\pm}P$.

At first, we need to prove that Y is ${}^2_s N$. Without loss of generality, we assume $X = Y \oplus N$ in the following proof since ${}^2_s N$ holds under similarity transformations.

Let $N_1 = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$, where the order of n_1 is the same for Y and the order of n_4 is the same for N . Then $N_1^2 = 0$ implies the following equations are true:

$$\begin{aligned} n_1^2 + n_2 n_3 &= 0, & n_1 n_2 + n_2 n_4 &= 0 \\ n_3 n_1 + n_4 n_3 &= 0, & n_3 n_2 + n_4^2 &= 0. \end{aligned} \quad (8)$$

Since $(X - N_1)^2 = N_2^2 = 0$, we get the following equations after replacing n_1 with $Y - n_1$ and n_4 with $N - n_4$ in the previous equations:

$$\begin{aligned} (Y - n_1)^2 + n_2 n_3 &= 0, \\ (Y - n_1) n_2 + n_2 (N - n_4) &= 0, \\ n_3 (Y - n_1) + (N - n_4) n_3 &= 0, \\ n_3 n_2 + (N - n_4)^2 &= 0. \end{aligned} \quad (9)$$

We can derive the following equations from the 3rd and 4th equations in the above two sets of equations:

$$Y n_2 + n_2 N = 0, \quad n_3 Y + N n_3 = 0. \quad (10)$$

Note that N is nilpotent, assume its index is r ; that is, $N^{r-1} \neq 0$ and $N^r = 0$. After multiplying the right side of equation $Y n_2 + n_2 N = 0$ by N^{r-1} , we can get $Y n_2 N^{r-1} = 0$. Y is nonsingular implies $n_2 N^{r-1} = 0$. Repeat the operation, we eventually get $n_2 = 0$. Similarly, we can also get $n_3 = 0$.

So N_1 is quasidiagonal and N_2 is also quasidiagonal through similar proof; that is, n_1 and n_4 are square nilpotent same as the corresponding parts of N_2 . Finally, we prove that Y is ${}^2_s N$.

Since Y is ${}_{\pm}P$ by Step 1 and N is ${}_{\pm}P$ by Corollary 7, it is true that X is ${}_{\pm}P$.

${}_{\pm}P \rightarrow {}^2_s N$. Suppose $X \in M_n(K)$ is ${}_{\pm}P$. If X is similar to $Y \oplus N$, where Y is nonsingular and N is nilpotent, then X is ${}_{\pm}P$ if and only if Y is ${}_{\pm}P$ by Corollaries 5, 6, and 7. Without loss of generality, we can assume X is nonsingular. Furthermore, if X is nonsingular and similar to $Y_1 \oplus Y_2$, where all eigenvalues of Y_1 are not in K and all eigenvalues of Y_2 are in K . Then X is ${}_{\pm}P$ if and only if Y_1 is ${}_{\pm}P$ and Y_2 is ${}_{\pm}P$. It will take two steps to prove X is ${}^2_s N$.

Step 3. Suppose $\text{car}(K) \neq 2$ and all eigenvalues of X are not in K ; then for arbitrary nonzero $\alpha \in K$, X is an $(\alpha, -\alpha)$ composite; that is, there exist idempotent matrices P_1 and $P_2 \in M_n(K)$ such that $X = \alpha P_1 - \alpha P_2$.

Let $Q_1^{(0)}$ be the eigenspace of P_1 with respect to 0, $Q_1^{(1)}$ the eigenspace of P_1 with respect to 1, let $Q_2^{(0)}$ be the eigenspace of P_2 with respect to 0, and $Q_2^{(1)}$ the eigenspace of P_2 with respect to 1. Both α and $-\alpha$ are not eigenvalues of X implies that $Q_1^{(0)} \cap Q_2^{(0)} = Q_1^{(1)} \cap Q_2^{(1)} = Q_1^{(0)} \cap Q_2^{(1)} = Q_1^{(1)} \cap Q_2^{(0)} = \{0\}$; then $\dim(Q_1^{(0)}) = \dim(Q_1^{(1)}) = \dim(Q_2^{(0)}) = \dim(Q_2^{(1)}) = n/2$ (otherwise, $\dim(Q_1^{(0)}) \geq n/2$ implies $Q_1^{(0)} \cap Q_2^{(0)} \neq \{0\}$ or $Q_1^{(0)} \cap Q_2^{(1)} \neq \{0\}$, etc.); that is, n is even.

Suppose S and T are $n \times (n/2)$ matrices with $r(S) = r(T) = n/2$ satisfying $P_1 S = 0$ and $P_2 T = T$; then $(S \ T)$ is $n \times n$ nonsingular matrix. Let $\begin{pmatrix} U \\ V \end{pmatrix}$ be its inverse; that is,

$$\begin{pmatrix} U \\ V \end{pmatrix} (S \ T) = \begin{pmatrix} US & UT \\ VS & VT \end{pmatrix} = \begin{pmatrix} I_{n/2} & 0_{n/2} \\ 0_{n/2} & I_{n/2} \end{pmatrix}. \quad (11)$$

Then we carry out the same similarity transformation on P_1 and P_2 as follows:

$$\begin{aligned} \begin{pmatrix} U \\ V \end{pmatrix} P_1 (S \ T) &= \begin{pmatrix} UP_1 S & UP_1 T \\ VP_1 S & VP_1 T \end{pmatrix} = \begin{pmatrix} 0_{n/2} & UP_1 T \\ 0_{n/2} & VP_1 T \end{pmatrix}, \\ \begin{pmatrix} U \\ V \end{pmatrix} P_2 (S \ T) &= \begin{pmatrix} UP_2 S & UP_2 T \\ VP_2 S & VP_2 T \end{pmatrix} = \begin{pmatrix} UP_2 S & 0_{n/2} \\ VP_2 S & I_{n/2} \end{pmatrix}, \end{aligned} \quad (12)$$

where P_1 and P_2 are idempotent implies that $VP_1 T$ and $UP_2 S$ are idempotent and $r(P_1) = r(P_2) = n/2$ implies that $VP_1 T = I_{n/2}$ and $UP_2 S = 0_{n/2}$. Hence, X is similar to the following matrix:

$$\begin{pmatrix} 0_{n/2} & \alpha UP_1 T \\ 0_{n/2} & 0_{n/2} \end{pmatrix} + \begin{pmatrix} 0_{n/2} & 0_{n/2} \\ -\alpha VP_2 S & 0_{n/2} \end{pmatrix}. \quad (13)$$

That is, X is ${}^2_s N$ in $M_n(K)$.

When $\text{car}(K) = 2$, X is (α, α) composite for arbitrary nonzero $\alpha \in K$, we can similarly prove that X is ${}^2_s N$ in $M_n(K)$ replacing $-\alpha$ with α in the previous proof.

Step 4. Suppose $\text{car}(K) \neq 2$ and all eigenvalues of X are in K ; then by Corollary 5, $j_k(X, \alpha) = j_k(X, -\alpha)$ for every $k \in \mathbb{Z}^+$ and arbitrary nonzero $\alpha \in K$.

Moreover, X is similar to $X_1 \oplus \cdots \oplus X_s$, where both the characteristic polynomial and the minimal polynomial of X_i are $[(x - \alpha_i)(x + \alpha_i)]^{r_i} = (x^2 - \alpha_i^2)^{r_i}$ with $2 \sum_{i=1}^s r_i = n$ and $\alpha_i \in K \setminus \{0\}$ is one of eigenvalues of X for every $i \in [s]$. Without loss of generality, we just need to prove X_i is ${}^2_s N$.

Since X_i is similar to $C((x^2 - \alpha_i^2)^{r_i})$ as follows:

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 & a_{2r_i-2} \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad (14)$$

where $(x^2 - \alpha_i^2)^{r_i} = x^{2r_i} - a_{2r_i-2}x^{2r_i-2} - \cdots - a_2x^2 - a_0$. We have $C((x^2 - \alpha_i^2)^{r_i}) = E_{2,1} + \cdots + E_{2r_i,2r_i-1} + a_0E_{1,2r_i} + a_2E_{3,2r_i} + \cdots + a_{2r_i-2}E_{2r_i-1,2r_i} = (E_{2,1} + E_{4,3} + \cdots + E_{2r_i,2r_i-1}) + (E_{3,2} + \cdots + E_{2r_i-1,2r_i-2} + a_0E_{1,2r_i} + a_2E_{3,2r_i} + \cdots + a_{2r_i-2}E_{2r_i-1,2r_i}) = N_1 + N_2$. Obviously, both N_1 and N_2 are square nilpotent matrices; that is, X_i is ${}^2_s N$. Hence, X is ${}^2_s N$ in $M_n(K)$.

When $\text{car}(K) = 2$, all blocks in the Jordan reduction of X with respect to $\alpha \in K \setminus \{0\}$ have an even size by Corollary 6; that is, both the characteristic polynomial and minimal polynomial of every block with respect to α are $(x + \alpha)^{s_i} = ((x + \alpha)^2)^{s_i/2} = (x^2 + \alpha^2)^{s_i/2}$, where s_i is even. Similarly, we can also prove that X is ${}^2_s N$ in $M_n(K)$.

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