Research Article

On Satnoianu-Wu’s Inequality

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1. Introduction

For \( n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) and \( x_i \in \mathbb{R}_+ = (0, \infty) \) with \( i = 1, 2, \ldots, n \), let

\[
A_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad G_n(x) = \left( \prod_{i=1}^{n} x_i \right)^{1/n}, \quad H_n(x) = \frac{n}{\sum_{i=1}^{n} (1/x_i)}.
\]

These quantities are, respectively, called the arithmetic, geometric, and harmonic means of a positive sequence \( x = (x_1, x_2, \ldots, x_n) \). For more information on the theory of means, see the monograph [1] or the papers [2–5] and plenty of references therein.

For convenience, in what follows, we use the notation \( x^q = (x_1^q, x_2^q, \ldots, x_n^q) \) for

\[
0 < m \leq x_1^q, x_2^q, \ldots, x_n^q \leq M,
\]

and \( q \in \mathbb{R} = (-\infty, \infty) \).

In [6], Satnoianu posed the following conjecture.

Conjecture 1. For \( n \geq 2, \ x_i > 0, \) and \( \lambda \geq n^{-1} - 1 \), it is valid that

\[
\sum_{i=1}^{n} \left( \frac{x_i^{n-1}}{x_i^{n-1} + \lambda \prod_{k=1, k \neq i}^{n} x_k} \right)^{1/(n-1)} \geq \frac{n}{1 + \lambda}^{1/(n-1)}.
\]

This conjecture has been solved and researched in [7–10]. Among them, Wu obtained in [10] the following result.

\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.
\]

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 3 (see [II, page 8]). Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). We say that \( x \) is majorized by \( y \) (in symbols \( x < y \)) if

\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.
\]
for $k = 1, 2, \ldots, n - 1$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $x$ and $y$ in a descending order.

**Definition 4** (see [12]). Let $\Omega \subseteq \mathbb{R}_+^n$.

1. The set $\Omega$ is said to be geometrically convex if $(x_1^y, y_1^x, \ldots, x_n^y, y_n^x) \in \Omega$ for every $x, y \in \Omega$ and $\lambda \in [0, 1]$.
2. A function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur-geometrically convex on $\Omega$ if $\ln x = (\ln x_1, \ldots, \ln x_n) < \ln y = (\ln y_1, \ldots, \ln y_n)$ implies $\varphi(x) < \varphi(y)$ for every $x, y \in \Omega$.
3. A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-geometrically concave on $\Omega$ if $\ln x = (\ln x_1, \ldots, \ln x_n) < \ln y = (\ln y_1, \ldots, \ln y_n)$ implies $\varphi(x) \geq \varphi(y)$ for every $x, y \in \Omega$.

**Lemma 5** (see [12]). Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric and geometrically convex set with inner points and $\varphi : \Omega \to \mathbb{R}_+$ a symmetric and differentiable function in $\Omega$. Then $\varphi$ is a Schur-geometrically convex (or Schur-geometrically concave, resp.) function on $\Omega$ if and only if

$$\ln(x_1 - \ln x_2) \left[ x_1 \frac{\partial \varphi(x)}{\partial x_1} - x_2 \frac{\partial \varphi(x)}{\partial x_2} \right] \geq 0 \quad (or \leq 0, \text{ resp.}), \quad x \in \Omega.$$  

**Lemma 6** (see [1, page 4, Bernoulli’s inequality]). The inequality

$$(1 + t)^r \geq 1 + rt \quad (8)$$

holds for $r \geq 1$ and $t \geq -1$ or for $r \leq 0$ and $t > -1$. If $0 < r < 1$ and $t \geq -1$, inequality (8) is reversed.

## 3. Main Results

Now we start off to demonstrate our main results.

**Theorem 7.** Let $m, M$ be defined as in (2) and $n \geq 2$, let $\alpha, \beta, x_i \in \mathbb{R}$, for $i = 1, 2, \ldots, n$, and let $p, q \in \mathbb{R}$ with $p \neq 0$. If $(1 - p) m \beta \geq 2p M \alpha$, then

$$\sum_{i=1}^{n} \left[ \frac{x_i^q}{\alpha x_i^q + \beta A_n(x^n)} \right]^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}.$$  

If $(1 - p) M \beta \leq 2p M \alpha$, inequality (9) is reversed.

**Proof.** For $M' > m' > 0$, let $f(\mathbf{u} = \sum_{i=1}^{n} (1 + u_i)^p, \quad \mathbf{u} \in [m', M']^n).$

Then

$$\frac{\partial f(\mathbf{u})}{\partial u_i} = (1 + u_i)^{r-1},$$

$$\frac{\partial}{\partial u_i} \left[ \frac{\partial f(\mathbf{u})}{\partial u_i} \right] = (1 + u_i)^{r-2} (1 + ru_i),$$

for $1 \leq i \leq n$. When $r > 0$, or when $r < 0$ and $1 + r M' < 0$, we have

$$\ln(u_1 - \ln u_2) \left[ \frac{\partial f(\mathbf{u})}{\partial u_1} - \frac{\partial f(\mathbf{u})}{\partial u_2} \right] \geq 0, \quad \mathbf{u} \in [m', M']^n;$$

when $r < 0$ and $1 + r M' > 0$, inequality (17) is reversed.
Using Lemma 5, we have the following conclusions:

1. If \( r > 0 \), or if \( r < 0 \) and \( 1 + rm' < 0 \), the function \( f(u) \) is Schur-geometrically convex on \([m', M']\); 
2. If \( r < 0 \) and \( 1 + rm' > 0 \), the function \( f(u) \) is Schur-geometrically concave on \([m', M']\).

By the fact that
\[
\ln G_n(u) = (\ln u_1, \ln u_2, \ldots, \ln u_n), \quad u \in \mathbb{R}_+^n,
\]
and by Definition 4, if \( r > 0 \), or if \( r < 0 \) and \( 1 + rm' < 0 \), we have
\[
\sum_{i=1}^{n} (1 + u_i)^r \geq n(1 + G_n(u))^r, \quad u \in [m', M']^n; 
\]
if \( r < 0 \) and \( 1 + rm' > 0 \), inequality (19) is reversed.

Letting \( u_i = \beta G_n(x_i)/\alpha x_i \) for \( i = 1, 2, \ldots, n \), one has
\[
\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta A_n(x_i)} \right)^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}. 
\]

**Remark 12.** When \( n \geq 2 \) and \( p > 1 \), from Lemma 6 and (4), it follows that
\[
\beta \geq \left( n^{\max\{p,1\}} - 1 \right) \alpha > p(n-1)\alpha. 
\]

**Theorem 13.** Let \( n \geq 2, \alpha, \beta, x_i \in \mathbb{R}_+ \) for \( i = 1, 2, \ldots, n \), and \( p, q \in \mathbb{R} \) with \( p \neq 0 \). If \( p \geq 1 \), then
\[
\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta H_n(x_i)} \right)^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}; 
\]
if \( p \leq -1 \), inequality (22) is reversed.

**Proof.** Since \((1 + r)^p \) is a convex (or concave, resp.) function on \( \mathbb{R}_+ \) for \( r \geq 1 \) or \( r < 0 \) (or for \( 0 < r \leq 1 \), resp.), by Jensen's inequality, if \( r \geq 1 \) or \( r < 0 \), we have
\[
\sum_{i=1}^{n} (1 + u_i)^r \geq n(1 + A_n(u))^r, \quad u \in \mathbb{R}_+^n; 
\]
if \( 0 < r \leq 1 \), inequality (22) is reversed.

Letting \( u_i = \beta H_n(x_i)/\alpha x_i \) and \( r = -1/p \) shows \( A_n(u) = \beta/\alpha \). Further from (23), we obtain inequality (22). The proof of Theorem 13 is complete.

**Remark 14.** It is clear that inequalities (9) and (22) both generalize inequality (3).

**Corollary 15.** Let \( n \geq 2, \alpha, \beta, x_i \in \mathbb{R}_+ \) for \( i = 1, 2, \ldots, n \), \( p, q \in \mathbb{R} \) with \( p \neq 0 \), and \( m, M \) defined as in (2).

1. When \(-1 \leq p < 0\), one has
\[
\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta A_n(x_i)} \right)^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}. 
\]

2. When \( 0 < p < 1 \) and \((1 - p)m\beta > 2pM\alpha \), one has
\[
\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta H_n(x_i)} \right)^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}. 
\]

3. When \( p > 0 \) and \( m\beta > pM\alpha \), one has
\[
\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta A_n(x_i)} \right)^{1/p} \leq \frac{n}{(\alpha + \beta)^{1/p}}. 
\]

4. When \( p > 0 \) and \( M\beta < mp\alpha \), one has
\[
\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta H_n(x_i)} \right)^{1/p} \leq \frac{n}{(\alpha + \beta)^{1/p}}. 
\]

**Proof.** This follows from utilizing the well-known harmonic-geometric-arithmetic mean inequality
\[
H_n(x) \leq G_n(x) \leq A_n(x) 
\]
Corollary 16. Under the conditions of Corollary 15 and when $M = (n - 1)m$,

(1) if $0 < p < 1$ and $(1 - p)\beta > 2p(n - 1)\alpha$, one has

$$\sum_{i=1}^{n} \left[ \frac{x_i^{\alpha}}{ax_i^{\alpha} + \beta H_n(x^\alpha)} \right]^{1/p} \geq \sum_{i=1}^{n} \left[ \frac{x_i^{\alpha}}{ax_i^{\alpha} + \beta H_n(x^\alpha)} \right]^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}.$$ (29)

(2) if $p > 0$ and $\beta > p(n - 1)\alpha$, one has

$$\sum_{i=1}^{n} \left[ \frac{x_i^{\alpha}}{ax_i^{\alpha} + \beta H_n(x^\alpha)} \right]^{1/p} \geq \sum_{i=1}^{n} \left[ \frac{x_i^{\alpha}}{ax_i^{\alpha} + \beta A_n(x^\alpha)} \right]^{1/p} \geq \frac{n}{(\alpha + \beta)^{1/p}}.$$ (30)

(3) if $p > 0$ and $(n - 1)\beta < p\alpha$, one has

$$\sum_{i=1}^{n} \left[ \frac{x_i^{\alpha}}{ax_i^{\alpha} + \beta A_n(x^\alpha)} \right]^{1/p} \leq \sum_{i=1}^{n} \left[ \frac{x_i^{\alpha}}{ax_i^{\alpha} + \beta G_n(x^\alpha)} \right]^{1/p} \leq \frac{n}{(\alpha + \beta)^{1/p}}.$$ (31)

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