Research Article

On the Syntactic Monoids Associated with a Class of Synchronized Codes

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A complete code C over an alphabet A is called *synchronized* if there exist $x, y \in C^*$ such that $xA^* \cap A^*y \subseteq C^*$. In this paper we describe the syntactic monoid $\text{Syn}(C^+)$ of C^+ for a complete synchronized code C over A such that C^+ , the semigroup generated by C , is a single class of its syntactic congruence P_{C^+} . In particular, we prove that, for such a code C , either $C = A$ or $\text{Syn}(C^+)$ is isomorphic to a special submonoid of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$, where $\mathcal{T}^l(I)$ and $\mathcal{T}^r(\Lambda)$ are the full transformation semigroups on the nonempty sets I and Λ , respectively.

1. Introduction

Theory of codes is an important branch in the field of information science. Many methods, including combinatorics methods, analysis methods, and algebraic methods, are applied to study codes. As a kind of algebraic methods, it is effective to study some kinds of codes by considering syntactic monoids of the semigroups and monoids generated by these codes.

As we have known, prefix codes have fundamental importance in theory of codes. Many authors are devoted to the investigation of prefix codes by using several methods (cf. [1–4]). In particular, Petrich et al. [4] investigated maximal prefix codes by considering the syntactic monoids of the semigroups generated by them in 1996. They described the semigroup structure of the syntactic monoid $\text{Syn}(C^+)$ of C^+ , the semigroup generated by a maximal prefix code C for which C^+ is a single class of the syntactic congruence P_{C^+} .

On the other hand, synchronized codes are also important both in theory and in applications. Many interesting results are obtained on this class of codes in the text of Berstel et al. [1]. Recently, Liu [3] investigated synchronized codes by algebraic methods and obtained an algebraic characterization of complete synchronized codes (see Lemma 5 in this paper). Furthermore, Liu [3, 5] also studied some generalizations of synchronized codes.

In this paper, by using the algebraic characterization of complete synchronized codes obtained in Liu [3] and some techniques developed in Petrich et al. [4], we give a description of the syntactic monoid $\text{Syn}(C^+)$ of C^+ for a complete synchronized code C over an alphabet A such that C^+ is a single class of its syntactic congruence P_{C^+} .

2. Preliminaries

A semigroup S is a *left zero semigroup* if $ab = a$ for any $a, b \in S$. Dually, we have *right zero semigroups*. A *rectangular band* is a semigroup which is isomorphic to a direct product of a left zero semigroup and a right zero semigroup. For rectangular bands, we have the following obvious result.

Lemma 1 (see [6]). *Let S be a rectangular band. Then $axa = a$ for any $a, x \in S$. As a consequence, $aSa = \{a\}$ for any $a \in S$. In particular, if S has an identity e , then $S = \{e\}$.*

An *ideal* of a semigroup S is a nonempty subset I of S satisfying that the union of IS and SI is contained in I . Recall that the unique minimum ideal (with respect to set inclusion) of a semigroup S (if exists) is called *the kernel of S* . For the kernel of a semigroup, we have the following.

Lemma 2 (see [6]). *If the kernel of a semigroup S only consists of idempotents, then this kernel is a rectangular band.*

Let S be a semigroup. A function λ (resp., ρ) on S is a *left translation* (resp., *right translation*) of S if $\lambda(xy) = (\lambda x)y$ (resp., $(xy)\rho = x(y\rho)$) for all $x, y \in S$. A left translation λ and a right translation ρ are *linked* if $x(\lambda y) = (x\rho)y$ in which case the pair (λ, ρ) is a *bitranslation* of S . Denote the set of left translations and that of right translations on S by $\Lambda(S)$ and $P(S)$, respectively. Clearly, $\Lambda(S)$ forms a monoid under the usual composition of functions: $(\lambda\lambda')x = \lambda(\lambda'x)$ for all $x \in S$. Dually, $P(S)$ forms a monoid under the usual composition of functions: $x(\rho\rho') = (x\rho)\rho'$ for all $x \in S$. The set of all bitranslations of S forms a submonoid of the direct product $\Lambda(S) \times P(S)$, which is called *the translation hull* of S , to be denoted by $\Omega(S)$.

Let s be an element of a semigroup S . Then the function λ_s defined by $\lambda_s x = sx$ for all $x \in S$ is the *inner left translation induced by s* . Dually, we have *inner right translation ρ_s induced by s* . Finally, the pair $\pi_s = (\lambda_s, \rho_s)$ is the *inner bitranslation induced by s* . The set $\Pi(S)$ of all inner bitranslations is the *inner part* of $\Omega(S)$. From Corollary III.1.7 in Petrich [7], $\Pi(S)$ is an ideal of $\Omega(S)$.

In the sequel, the set of all transformations on a set Q written and composed as right (resp., left) operators is denoted by $\mathcal{T}^r(Q)$ (resp., $\mathcal{T}^l(Q)$). The identity mapping on a set Q is denoted by ι_Q . If $q \in Q$, then $\langle q \rangle$ denotes the constant function on Q whose value is q . Clearly, $\mathcal{T}^l(Q)$ and $\mathcal{T}^r(Q)$ are semigroups with their own compositions and $\mathcal{T}_0^l(Q) = \{\langle q \rangle \mid q \in Q\}$ (as left operators) and $\mathcal{T}_0^r(Q) = \{\langle q \rangle \mid q \in Q\}$ (as right operators) are subsemigroups of $\mathcal{T}^l(Q)$ and $\mathcal{T}^r(Q)$, respectively.

On the translation hull of a rectangular band, we have the following results which can be found in Section III.7 in Petrich and Reilly [6].

Lemma 3 (see [6]). *Let $I \times \Lambda$ be a rectangular band.*

- (1) $\Omega(I \times \Lambda) \cong \mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$.
- (2) $I \times \Lambda \cong \Pi(I \times \Lambda) \cong \mathcal{T}_0^l(I) \times \mathcal{T}_0^r(\Lambda)$.
- (3) $\mathcal{T}_0^l(I) \times \mathcal{T}_0^r(\Lambda)$ is the kernel of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$.

Let S be a semigroup and J an ideal of S . Then S is called an *extension* of J . From Definition III.5.4 in Petrich [7], an extension S of J is called *dense* if for each congruence ρ on S , the fact that the restriction of ρ to J is the equality relation on J implies that ρ is the equality relation on S .

On dense extensions of rectangular bands, the following results can be obtained as a special case from Theorem III.1.12 and Corollary III.5.5 in [7] and can also be proved easily.

Lemma 4. *If S is a dense extension of a rectangular band J , then the following semigroup homomorphism*

$$\tau : S \longrightarrow \Omega(J), \quad s \longmapsto s\tau = (\lambda^s, \rho^s) \quad (1)$$

is injective, where for each $s \in S$,

$$\lambda^s : J \longrightarrow J, \quad j \longmapsto sj, \quad \rho^s : J \longrightarrow J, \quad j \longmapsto js. \quad (2)$$

Clearly in this case, S is isomorphic to $S\tau$ which contains $\Pi(J)$ as an ideal.

*Let S be a semigroup and let S^1 be the semigroup obtained from S by adjoining an identity if necessary. The syntactic congruence P_L determined by a subset L of S is the following relation on S : $\{(x, y) \in S \times S \mid uxv \in L \text{ if and only if } uyv \in L \text{ for all } u, v \in S^1\}$. In particular, if $L = \{x\}$, we call $P_{\{x\}}$ the syntactic congruence determined by x and denote it by P_x . Moreover, x is called *disjunctive* in S if P_x is the equality relation on S . It is easy to see that the relation P_L saturates L for every subset L of S ; that is, L is a union of some P_L -classes for every subset L of S .*

Let A be an alphabet, let A^* be the free monoid generated by A , and let 1 be the identity of A^* . For any $L \subseteq A^*$, the quotient monoid A^*/P_L is called *the syntactic monoid of L* , to be denoted by $\text{Syn}(L)$. A nonempty set C of $A^+ = A^* \setminus \{1\}$ is called a *code* over A if the fact that

$$x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_n, \quad x_i, y_j \in C, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n, \quad (3)$$

implies that $m = n$ and $x_i = y_i, i = 1, 2, \dots, n$.

A submonoid S of a monoid M is called *stable* in M if the fact that $u, wu, uw \in S$ implies that $w \in S$ for all $u, v, w \in M$. It is well known that the monoid C^* generated by a code C over A is stable in A^* (see Proposition 2.2.5 in [1]).

A code C over A is called *complete* if, for any $w \in A^*$, there exist $u, v \in A^*$ such that $uwv \in C^*$, where C^* is the monoid generated by C . Recall that a complete code C over A is said to be *synchronized* if $xA^* \cap A^*y \subseteq C^*$ for some $x, y \in C^*$ (see details in Proposition 10.1.14 of [1]). On complete synchronized codes, Liu [3] obtained the following algebraic characterizations recently.

Lemma 5 (see [3]). *A complete code C over A is synchronized if and only if the kernel of $\text{Syn}(C^*)$ is a rectangular band.*

3. A Characterization of Complete Synchronized Codes

This section gives a characterization of complete synchronized codes by using the syntactic monoid $\text{Syn}(C^+)$ of C^+ , the semigroup generated by a code C over A . To this aim, we need several lemmas.

Lemma 6. *Let C be a code over A .*

- (1) *For any $x, y \in A^+$, $xP_{C^+}y$ if and only if $xP_{C^*}y$.*
- (2) *The P_{C^+} -class containing 1 is $\{1\}$.*
- (3) *$\text{Syn}(C^+) \setminus \{1P_{C^+}\}$ is a subsemigroup of $\text{Syn}(C^+)$.*

Proof. (1) This follows from the fact that $uxv \in C^+$ if and only if $uxv \in C^*$ for any $x \in A^+$ and $u, v \in A^*$.

(2) Let $w \in A^*$ and $wP_{C^+}1$. Since $1 \notin C^+$ and C^+ is a union of some P_{C^+} -classes, it follows that $w \notin C^+$. On the other hand, for any $c \in C$, we have $cwP_{C^+}c$ and $wcP_{C^+}c$, whence $cw, wc, c \in C^+$ by the fact that C^+ is a union of some P_{C^+} -classes. Since C is a code over A , C^* is stable. Therefore, $w \in C^*$. This implies that $w = 1$. Thus, the P_{C^+} -class containing 1 is $\{1\}$.

(3) This follows from (2). $\square$

Lemma 7. Let C be a code over A . If the P_{C^+} -class containing 1 is $\{1\}$, then $\text{Syn}(C^*) \cong \text{Syn}(C^+)$. Otherwise, $\text{Syn}(C^*) \cong \text{Syn}(C^+) \setminus \{1P_{C^+}\}$.

Proof. If the P_{C^+} -class containing 1 is $\{1\}$, by items (1) and (2) in Lemma 6, $P_{C^+} = P_{C^*}$ in this case, and so $\text{Syn}(C^*)$ is isomorphic to $\text{Syn}(C^+)$.

If the P_{C^+} -class containing 1 is not $\{1\}$, then there exists $\alpha \in A^+$ such that $1P_{C^+}\alpha$. Moreover, $\text{Syn}(C^+) \setminus \{1P_{C^+}\}$ is a subsemigroup of $\text{Syn}(C^+)$ by item (3) of Lemma 6. In the sequel, we show that the following

$$\begin{aligned} \psi : \text{Syn}(C^*) &\longrightarrow \text{Syn}(C^+) \setminus \{1P_{C^+}\}, \\ (xP_{C^*})\psi &= \begin{cases} \alpha P_{C^+}, & \text{if } x = 1, \\ xP_{C^+}, & \text{if } x \neq 1, \end{cases} \end{aligned} \quad (4)$$

is a semigroup isomorphism. We first show that ψ is well defined. In fact, let $x, y \in A^*$ and $xP_{C^*} = yP_{C^*}$. We divide the discussion into the following four cases.

- (i) $x = y = 1$. In this case, $(xP_{C^*})\psi = (yP_{C^*})\psi = \alpha P_{C^+}$.
- (ii) $x = 1, y \neq 1$. In this case, $(xP_{C^*})\psi = \alpha P_{C^+}$ and $(yP_{C^*})\psi = yP_{C^+}$. Observe that $\alpha P_{C^+} = xP_{C^+}y$; it follows that $\alpha P_{C^+} = yP_{C^+}$ whence $\alpha P_{C^+} = yP_{C^+}$ from item (1) in Lemma 6. This implies that $(xP_{C^*})\psi = (yP_{C^*})\psi$.
- (iii) $x \neq 1, y = 1$. This is the dual of case (ii).
- (iv) $x \neq 1, y \neq 1$. This follows from item (1) in Lemma 6.

By similar methods, we can show that ψ is injective, and the surjectivity of ψ is obvious.

On the other hand, for any $xP_{C^*}, yP_{C^*} \in \text{Syn}(C^*)$, we assert that

$$[(xP_{C^*})(yP_{C^*})]\psi = (xP_{C^*})\psi(yP_{C^*})\psi \quad (5)$$

and so ψ is a semigroup morphism. In fact, we have the following cases.

- (a) $x = y = 1$. In this case,

$$\begin{aligned} [(xP_{C^*})(yP_{C^*})]\psi &= (1P_{C^*})\psi = \alpha P_{C^+}, \\ (xP_{C^*})\psi(yP_{C^*})\psi &= (\alpha P_{C^+})(\alpha P_{C^+}) = \alpha^2 P_{C^+}. \end{aligned} \quad (6)$$

Observe that $1P_{C^*}\alpha, \alpha^2 P_{C^*}\alpha$. This implies that $\alpha^2 P_{C^*}\alpha$ by item (1) in Lemma 6 whence $[(xP_{C^*})(yP_{C^*})]\psi = (xP_{C^*})\psi(yP_{C^*})\psi$.

- (b) $x = 1, y \neq 1$. In this case,

$$\begin{aligned} [(xP_{C^*})(yP_{C^*})]\psi &= (yP_{C^*})\psi = yP_{C^+}, \\ (xP_{C^*})\psi(yP_{C^*})\psi &= (\alpha P_{C^+})(yP_{C^+}) = (\alpha y)P_{C^+}. \end{aligned} \quad (7)$$

Observe that $1P_{C^*}\alpha, yP_{C^*}\alpha y$. This implies that $yP_{C^*}\alpha y$ by item (1) in Lemma 6 again whence $[(xP_{C^*})(yP_{C^*})]\psi = (xP_{C^*})\psi(yP_{C^*})\psi$.

- (c) $x \neq 1, y = 1$. This is the dual of case (b).
- (d) $x \neq 1, y \neq 1$. This is obvious. $\square$

Lemma 8. If M is a monoid with identity 1 and $M \setminus \{1\}$ is a subsemigroup of M , then M has a kernel if and only if $M \setminus \{1\}$ has a kernel. If this is the case, the two kernels are equal.

Proof. Observe that

$$\begin{aligned} \{N \subseteq M \mid N \text{ is an ideal of } M\} \\ = \{N \subseteq M \setminus \{1\} \mid N \text{ is an ideal of } M \setminus \{1\}\} \cup \{M\}; \end{aligned} \quad (8)$$

the result follows. $\square$

Combining Lemmas 5, 6, 7, and 8, we have the following result.

Theorem 9. A complete code C over A is a synchronized code if and only if the kernel of $\text{Syn}(C^+)$ is a rectangular band.

4. Main Results

Let I and Λ be two nonempty sets and $\emptyset \neq K \subseteq \mathcal{I}^l(I) \times \mathcal{I}^r(\Lambda)$. Assume that p_1 and p_2 are the projections onto the first and second components of $\mathcal{I}^l(I) \times \mathcal{I}^r(\Lambda)$, respectively. For $(i, \lambda), (i_0, \lambda_0) \in I \times \Lambda$, we denote

$$\Omega_{(i_0, \lambda_0)}^{(i, \lambda)} = \{(F, \Phi) \in p_1(K) \times p_2(K) \mid (Fi, \lambda\Phi) = (i_0, \lambda_0)\}. \quad (9)$$

Theorem 10. Let C be a complete synchronized code over an alphabet A such that $C \neq A$ and C^+ is a P_{C^+} -class. Then $\text{Syn}(C^+)$ is isomorphic to a submonoid K of $\mathcal{I}^l(I) \times \mathcal{I}^r(\Lambda)$ for some nonempty sets I and Λ with $(|I|, |\Lambda|) \neq (1, 1)$, and the following conditions hold:

- (1) $\mathcal{I}_0^r(I) \times \mathcal{I}_0^r(\Lambda) \subseteq K$,
- (2) $K \setminus \{(i_1, i_\Lambda)\}$ is a subsemigroup of K ,
- (3) there exists $(i_0, \lambda_0) \in I \times \Lambda$ such that the identity $\Omega_{(i_0, \lambda_0)}^{(i, \lambda)} = \Omega_{(i_0, \lambda_0)}^{(j, \mu)}$ implies that $(i, \lambda) = (j, \mu)$ for all $(i, \lambda), (j, \mu) \in I \times \Lambda$ and the identity $(Fi_0, \lambda_0\Phi) = (i_0, \lambda_0)$ implies that $(F, \Phi) \in \{(i_1, i_\Lambda), (\langle i_0 \rangle, \langle \lambda_0 \rangle)\}$ for all $(F, \Phi) \in K$.

The above submonoid K will be called (i_0, λ_0) -submonoid of $\mathcal{I}^l(I) \times \mathcal{I}^r(\Lambda)$.

Proof. Let $M = \text{Syn}(C^+)$ and denote the set of P_{C^+} -classes with representatives from T by $\bar{T}$ for $T \subseteq A^*$. Since C^+ is a P_{C^+} -class, we can also let $e = \overline{C^+}$. Obviously, e is an idempotent in M .

We first assert that $\{e, \bar{1}\}$ is stable and e is disjunctive in M . Let $w \in A^*$ and $c \in C$. Then $e = \bar{c}$. If $\bar{w}\bar{1}, \bar{1}\bar{w} \in \{e, \bar{1}\}$, then $\bar{w} \in \{e, \bar{1}\}$ obviously. Moreover, let $\bar{w}e, e\bar{w} \in \{e, \bar{1}\}$. Since the P_{C^+} -class containing 1 is $\{1\}$ by Lemma 6, we have $e = \bar{w}e = e\bar{w}$. This implies that $\bar{c} = \bar{w}\bar{c} = \bar{c}\bar{w}$. Since C^+ is a single P_{C^+} -class, we have $wc, cw, c \in C^+$. Because C^* is stable, it follows that $w \in C^*$. This implies that $\bar{w} \in \{e, \bar{1}\}$. On the other hand, let $p, q \in A^*$ and $\bar{p}P_e\bar{q}$ in M . Then for all $s, t \in A^*$, $\overline{spt} = e = \overline{C^+}$

if and only if $\overline{sqt} = e = \overline{C^+}$. Since C^+ is a single P_{C^+} -class, it follows that $spt \in C^+$ if and only if $sqt \in C^+$, and so $pP_{C^+}q$. Thus $\overline{p} = \overline{q}$.

Now, let $J = MeM$. We assert that J is the kernel of M . In fact, J is an ideal of M clearly. Moreover, since C is complete, there exist $u, v \in A^*$ such that $uwv \in C^+$ for all $w \in A^*$. Therefore, there exist $m, n \in M$ such that $mxn = e$ for all $x \in M$. Now, let I be an ideal of M and $x \in I$. Then $uxv = e$ for some $u, v \in M$ whence $J = MeM = MuxvM \subseteq I$. Thus, J is the least ideal of M and so is the kernel of M . By Theorem 9, J is a rectangular band.

If $J = \{e\}$, then we have $em = me = e$ for any $m \in M$. This implies that $\overline{c}\overline{w} = \overline{w}\overline{c} = \overline{c}$ for all $c \in C$ and $w \in A^*$. Since C^+ is a single P_{C^+} -class, it follows that $wc, cw, c \in C^+$. Because C^* is stable, we have $w \in C^*$. Therefore, $C^* = A^*$ and hence $C = A$. A contradiction. Thus, $J \neq \{e\}$. Now let ρ be a congruence on M whose restriction to J is the identity relation on J . Assume that $x \in M$ and xpe . Then $xepex$, where $e, xe, ex \in J$ and thus $e = xe = ex$. Since $\{e, \overline{1}\}$ is stable in M , it follows that $x \in \{e, \overline{1}\}$. If $x = \overline{1}$, then $\overline{1}pe$ and for any $y \in J$, we obtain $ypye, ypey$, and $ye, y, ey \in J$, whence $y = ye = ey$. Thus, J is a rectangular band with the identity e . And hence, $J = \{e\}$ by Lemma 1. A contradiction. It follows that $x = e$ and $\{e\}$ is a ρ -class in M . Now, let $x, y \in M$ and xpy . Then for any $u, v \in M$, we have $uxvpyuv$. Observe that $\{e\}$ is a ρ -class, it follows that $uxv = e$ if and only if $uyv = e$. Therefore, $xP_e y$ in M . By the disjointness of e in M , we have $x = y$. We conclude that ρ is the equality relation on M . Thus, M is a dense extension of J .

By Lemma 4, M is isomorphic to a subsemigroup of $\Omega(J)$ containing the inner part $\Pi(J)$ as an ideal. Since J is a rectangular band, J is isomorphic to the product $I \times \Lambda$ of a left zero semigroup I and a right zero semigroup Λ . Observe that $J \neq \{e\}$, $|J| = |I \times \Lambda| > 1$, whence $(|I|, |\Lambda|) \neq (1, 1)$. By Lemma 3, $\Omega(I \times \Lambda) \cong \mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$, and in this isomorphism $\Pi(I \times \Lambda) \cong \mathcal{T}_0^l(I) \times \mathcal{T}_0^r(\Lambda)$. Therefore, M is isomorphic to a subsemigroup K of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$ and K contains $\mathcal{T}_0^l(I) \times \mathcal{T}_0^r(\Lambda)$ as an ideal. Furthermore, since M is a monoid, K has an identity. Let (F, Φ) be the identity of K . Then for any $(\langle i \rangle, \langle \lambda \rangle) \in \mathcal{T}_0^l(I) \times \mathcal{T}_0^r(\Lambda)$,

$$(F, \Phi)(\langle i \rangle, \langle \lambda \rangle) = (\langle i \rangle, \langle \lambda \rangle)(F, \Phi) = (\langle i \rangle, \langle \lambda \rangle), \quad (10)$$

whence $(Fi, \lambda\Phi) = (i, \lambda)$. This implies that $(F, \Phi) = (i_I, i_\Lambda)$. Thus, K is a submonoid of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$. Since $M \setminus \{1\}$ is a subsemigroup of M , it follows that $K \setminus \{(i_I, i_\Lambda)\}$ is a subsemigroup of K . Thus, Conditions (1) and (2) hold.

Since J is the kernel of M and e is an idempotent in J , it follows that the image of e in K is the form $a = (\langle i_0 \rangle, \langle \lambda_0 \rangle)$ for some $i_0 \in I$ and $\lambda_0 \in \Lambda$. Since e is disjunctive in M , a is disjunctive in K . Let $(i, \lambda), (j, \mu) \in I \times \Lambda$ and $\Omega_{(i_0, \lambda_0)}^{(i, \lambda)} = \Omega_{(i_0, \lambda_0)}^{(j, \mu)}$. Then,

$$(Fi, \lambda\Phi) = (i_0, \lambda_0) \quad \text{iff} \quad (Fj, \mu\Phi) = (i_0, \lambda_0) \quad (11)$$

for any $(F, \Phi) \in p_1(M) \times p_2(M)$. This implies that

$$(\langle Fi \rangle, \langle \lambda\Phi \rangle) = a \quad \text{iff} \quad (\langle Fj \rangle, \langle \mu\Phi \rangle) = a, \quad (12)$$

for any $(F, \Phi) \in p_1(M) \times p_2(M)$. Thus,

$$(F, \Psi)(\langle i \rangle, \langle \lambda \rangle)(G, \Phi) = a \quad \text{iff} \quad (F, \Psi)(\langle j \rangle, \langle \mu \rangle)(G, \Phi) = a, \quad (13)$$

for any $(F, \Psi), (G, \Phi) \in K$. This shows that $(\langle i \rangle, \langle \lambda \rangle)P_a(\langle j \rangle, \langle \mu \rangle)$ and so $(\langle i \rangle, \langle \lambda \rangle) = (\langle j \rangle, \langle \mu \rangle)$ since a is disjunctive in K . Thus $(i, \lambda) = (j, \mu)$.

Finally, let $(F, \Phi) \in K$ and $(Fi_0, \lambda_0\Phi) = (i_0, \lambda_0)$. Then,

$$(F, \Phi)(\langle i_0 \rangle, \langle \lambda_0 \rangle) = (\langle Fi_0 \rangle, \langle \lambda_0\Phi \rangle) = (\langle i_0 \rangle, \langle \lambda_0 \rangle). \quad (14)$$

Dually, we have $(\langle i_0 \rangle, \langle \lambda_0 \rangle)(F, \Phi) = (\langle i_0 \rangle, \langle \lambda_0 \rangle)$. Since e is stable in M , it follows that a is stable in K . Hence, $(F, \Phi) \in \{(\langle i_0 \rangle, \langle \lambda_0 \rangle), (i_I, i_\Lambda)\}$. Therefore, Condition (3) is also satisfied. $\square$

We end our paper by giving an example to illustrate our result.

Example 11. Let $A = \{a, b\}$ and $C = b^*a^+b$. Then $C^+ = A^*ab$. It is routine to check that C is a synchronized code and C^+ is a single P_L -class. Moreover, we can also check that $\text{Syn}(C^+) = \{A^*ab, A^*bb, A^*a, b, \overline{1}\}$.

On the other hand, let $I = \{1\}$ and $\Lambda = \{1, 2, 3\}$. Denote

$$M_1 = \{(\langle 1 \rangle, \langle 1 \rangle), (\langle 1 \rangle, \langle 2 \rangle), (\langle 1 \rangle, \langle 3 \rangle), (\langle 1 \rangle, \alpha), (i_I, i_\Lambda)\}, \quad (15)$$

where

$$\alpha : \Lambda \longrightarrow \Lambda, 1 \longmapsto 2, 2 \longmapsto 2, 3 \longmapsto 1. \quad (16)$$

Then M_1 is a $(1, 1)$ -submonoid of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$. In fact, observe that $\alpha^2 = \langle 2 \rangle$, M_1 is submonoid of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$, and $M_1 \setminus \{(i_I, i_\Lambda)\}$ is a subsemigroup of M_1 . Let $\lambda, \mu \in \Lambda$. If $\lambda = 1$, $\mu = 2$ or $\lambda = 1$, $\mu = 3$, then there exists $\Phi = i_\Lambda \in p_2(M_1)$ such that $\lambda\Phi = 1$, $\mu\Phi = \mu$. If $\lambda = 2$, $\mu = 3$, then there exists $\Phi = \alpha \in p_2(M_1)$ such that $\lambda\Phi = 2$, $\mu\Phi = 1$. Furthermore, if $(F, \Phi) \in M_1$ with $F1 = 1$, $1\Phi = 1$, then $F = \langle 1 \rangle = i_I$ and $\Phi = \langle 1 \rangle$ or $\Phi = i_\Lambda$. Thus, M_1 is a $(1, 1)$ -submonoid of $\mathcal{T}^l(I) \times \mathcal{T}^r(\Lambda)$. It is easy to see that

$$\begin{aligned} \psi : \text{Syn}(C^+) &\longrightarrow M_1, \overline{A^*ab} \longmapsto (\langle 1 \rangle, \langle 1 \rangle), \\ \overline{A^*bb} &\longmapsto (\langle 1 \rangle, \langle 2 \rangle), \overline{A^*a} \longmapsto (\langle 1 \rangle, \langle 3 \rangle), \\ \overline{b} &\longmapsto (\langle 1 \rangle, \alpha), \overline{1} \longmapsto (i_I, i_\Lambda) \end{aligned} \quad (17)$$

is an isomorphism from $\text{Syn}(C^+)$ onto M_1 .

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