Research Article

Analysis of a Fractional-Order Couple Model with Acceleration in Feelings

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A fractional-order nonlinear dynamical model of couple has been introduced. Upper bounds are obtained for a fractional-order nonlinear dynamical model. Also different from other models, a model with the order 2α is discussed. We are expecting an acceleration in feelings; that is why we increase the order of the derivative between $1 < 2\alpha \leq 2$. Stability analysis of the fractional-order nonlinear dynamical model of involving two persons is studied using the fractional Routh-Hurwitz criteria. By using stability analysis on fractional-order system, we obtain sufficient condition on the parameters for the locally asymptotic stability of equilibrium points. Finally, numerical simulations are presented to verify the obtained results.

1. Introduction

The first noninteger order differentiation and integration notion was considered in 1695 by Leibniz and L'Hôpital. In a letter to L'Hôpital in 1695, Leibniz raised the following question: "Can the meaning of derivatives with integer order be generalized to derivatives with noninteger orders?" L'Hôpital was somewhat curious about that question and replied by another question to Leibniz: "What if the order will be $1/2$?" After the letter was answered by Leibniz, fractional order in the concept of derivative was formed [1].

There are lots of topics on fractional modeling, but in recent decades the study of interpersonal relationships has begun to be popular. Interpersonal relationships appear in many contexts, such as in family, kinship, acquaintance, work, and clubs [2]. Mathematical modeling in interpersonal relationships is very important for capturing the dynamics of people, but there are few models in this area and models have been limited to integer order differential equations. Another interesting dynamic is marriage. Marriage has been studied scientifically for the past sixty years [3]. Researchers are trying to understand why some couples divorce, but others do not, and why, among those who remain married, some are happy and some are miserable with one another [4].

Since experiments in these areas are difficult to generate, mathematical models may play a role in explanation of the dynamics of a couple and behavioral features.

Recently, a fractional-order system for the dynamics of love affair between a couple has been considered [5]. In this paper, different from [5], a model with the order 2α is discussed. We are expecting an acceleration in feelings; that is why we increase the order of the derivative between $1 < 2\alpha \leq 2$. Also, upper bounds are discussed for the system.

We begin by giving the definitions and properties of fractional-order integrals and derivatives [6].

2. Preliminaries and Definitions

The three most common definitions for fractional derivative can be given as the Grünwald-Letnikov definition, the Riemann-Liouville definition, and the Caputo definition.

Definition 1. The Riemann-Liouville type fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Grünwald-Letnikov definition is given as

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t-jh). \quad (2)$$

Definition 3. The Riemann-Liouville type fractional derivative of order $\alpha > 0$ of a function $f: (0, \infty) \rightarrow R$ is defined by

$$D^\alpha f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (3)$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α .

Definition 4. The Caputo type fractional derivative of order $\alpha > 0$ of a function $f: (0, \infty) \rightarrow R$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau, \quad (4)$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α .

Some properties of the Caputo derivative and the Riemann-Liouville derivative formulas are given below:

$$\begin{aligned} {}^C D_t^\alpha ({}_C D_t^m f(t)) &= {}^C D_t^m ({}_C D_t^\alpha f(t)) = {}^C D_t^{\alpha+m} f(t), \\ f^{(s)}(0) &= 0, \quad s = n, n+1, \dots, m \\ m &= 0, 1, 2, \dots; \quad n-1 < \alpha < n, \\ {}_a D_t^m ({}_a D_t^\alpha f(t)) &= {}_a D_t^\alpha ({}_a D_t^m f(t)) = {}_a D_t^{\alpha+m} f(t), \\ f^{(s)}(0) &= 0, \quad s = 0, 1, 2, \dots, m \\ m &= 0, 1, 2, \dots; \quad n-1 < \alpha < n. \end{aligned} \quad (5)$$

We see that, contrary to the Riemann-Liouville approach, in the case of the Caputo derivative, there are no restrictions on the values $f^{(s)}(0)$ ($s = 0, 1, \dots, n-1$).

3. Equilibrium Points and Their Locally Asymptotic Stability

In this section, we consider a fractional-order nonlinear two-dimensional system as follows:

$$\begin{aligned} D^{2\alpha} x_1(t) &= -\alpha_1 x_1 + \beta_1 x_2 (1 - \varepsilon x_2^2) + A_1, \\ D^{2\alpha} x_2(t) &= -\alpha_2 x_2 + \beta_2 x_1 (1 - \varepsilon x_1^2) + A_2, \\ x_1(0) &= 0, \quad x_2(0) = 0, \end{aligned} \quad (6)$$

where $D^{2\alpha}$ is the fractional derivative of order $1 < 2\alpha \leq 2$. $\alpha_i > 0$, α_i , β_i , and A_i ($i = 1, 2$) are real constants. These parameters are oblivion, reaction, and attraction constants. In the equations above, we assume that feelings decay exponentially fast in the absence of partners. The parameters specify the romantic style of individuals 1 and 2. In the beginning of relationships, because they have no feelings towards each other, initial conditions are considered zero.

We note that, with zero initial conditions, the following equation is valid:

$$D^\alpha (D^\alpha x(t)) = D^{2\alpha} (x(t)). \quad (7)$$

In that case, the system can be considered as follows:

$$\begin{aligned} D^{2\alpha} x_1(t) &= D^\alpha (D^\alpha x_1(t)), \\ D^{2\alpha} x_2(t) &= D^\alpha (D^\alpha x_2(t)), \\ x_1(0) &= 0, \quad x_2(0) = 0. \end{aligned} \quad (8)$$

Let us make the following changes of variables:

$$\begin{aligned} x_1 &= y_1, & D^\alpha x_1 &= y_2, \\ x_2 &= y_3, & D^\alpha x_2 &= y_4. \end{aligned} \quad (9)$$

Then, transformed system is given below:

$$\begin{aligned} D^\alpha y_1(t) &= y_2, \\ D^\alpha y_2(t) &= -\alpha_1 y_1 + \beta_1 y_3 (1 - \varepsilon y_3^2) + A_1, \\ D^\alpha y_3(t) &= y_4, \\ D^\alpha y_4(t) &= -\alpha_2 y_3 + \beta_2 y_1 (1 - \varepsilon y_1^2) + A_2, \end{aligned} \quad (10)$$

with initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 0, \quad (11)$$

where $0.5 < \alpha \leq 1$, $\alpha_i > 0$, α_i , β_i , and A_i ($i = 1, 2$) are real constants.

Let $\alpha \in (0.5, 1]$ and consider the system

$$\begin{aligned} D^\alpha y_1(t) &= f_1(y_1, y_2, y_3, y_4), \\ D^\alpha y_2(t) &= f_2(y_1, y_2, y_3, y_4), \\ D^\alpha y_3(t) &= f_3(y_1, y_2, y_3, y_4), \\ D^\alpha y_4(t) &= f_4(y_1, y_2, y_3, y_4), \end{aligned} \quad (12)$$

with the initial values

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 0. \quad (13)$$

Here,

$$\begin{aligned} f_1(y_1, y_2, y_3, y_4) &= y_2, \\ f_2(y_1, y_2, y_3, y_4) &= -\alpha_1 y_1 + \beta_1 y_3 (1 - \varepsilon y_3^2) + A_1, \\ f_3(y_1, y_2, y_3, y_4) &= y_4, \\ f_4(y_1, y_2, y_3, y_4) &= -\alpha_2 y_3 + \beta_2 y_1 (1 - \varepsilon y_1^2) + A_2. \end{aligned} \quad (14)$$

To evaluate the equilibrium points, let

$$D^\alpha y_i(t) = 0 \implies f_i(y_1^*, y_2^*, y_3^*, y_4^*) = 0, \quad i = 1, 2, 3, 4, \quad (15)$$

from which we can get the equilibrium points $K_0 = (0, 0, 0, 0)$ for $A_1 = A_2 = 0$ and $K_1 = (y_1^*, y_2^*, y_3^*, y_4^*)$.

The Jacobian matrix $J(K_1)$ for the system given in (14) is

$$J(K_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\alpha_1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & -\alpha_2 & 0 \end{bmatrix}, \quad (16)$$

where

$$a = \beta_1 (1 - 3\varepsilon(y_3^*)^2), \quad b = \beta_2 (1 - 3\varepsilon y_1^{*2}). \quad (17)$$

To discuss the local stability of the equilibrium $K_1 = (y_1^*, y_2^*, y_3^*, y_4^*)$ of the system given by (14), we consider the linearized system at K_1 . The characteristic equation of the linearized system is of the form

$$P(\lambda) = \lambda^4 + (\alpha_2 + \alpha_1)\lambda^2 + (\alpha_1\alpha_2 - ab) = 0. \quad (18)$$

If λ^2 is taken as k , we have the following reduced equation:

$$P(\lambda) = k^2 + a_1k + a_2 = 0, \quad (19)$$

where

$$\begin{aligned} a_1 &= (\alpha_2 + \alpha_1), \\ a_2 &= (\alpha_1\alpha_2 - ab). \end{aligned} \quad (20)$$

According to the fractional Routh-Hurwitz criteria, we have the following theorem.

Theorem 5. If $a_1 > 0$ and $a_2 > 0$, then the equilibrium point $K_1 = (y_1^*, y_2^*, y_3^*, y_4^*)$ is locally asymptotically stable for all $\alpha \in (0, 1)$.

Proof. $K_1 = (y_1^*, y_2^*, y_3^*, y_4^*)$ equilibrium of the system given by (12) is asymptotically stable if all of the eigenvalues, $k_i, i = 1, 2$, of $J(K_1)$, satisfy the following condition (negative real part) [7, 8]:

$$|\arg \lambda_i| > \frac{\alpha\pi}{2}. \quad (21)$$

For $n = 2$, the Routh-Hurwitz criteria are just $a_1 > 0$ and $a_2 > 0$. The characteristic polynomial $P(\lambda) = k^2 + a_1k + a_2 = 0$ satisfies eigenvalues as below:

$$k_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}. \quad (22)$$

Now, suppose that a_1 and a_2 are positive. It is easy to see that if the roots are real, they are both negative, and if they are complex conjugates, they have a negative real part.

Next, to prove the converse, suppose that the roots are either negative or have a negative real part. Then, it follows that $a_1 > 0$. If the roots are complex conjugates, $0 < a_1^2 < 4a_2$, which implies that a_2 is also positive. If the roots are real, then since both of the roots are negative, it follows that $a_2 > 0$. $\square$

Theorem 6. Let $a_2 = (\alpha_1\alpha_2 - ab)$ be as given in (20). If $a_2 < 0$, then the positive equilibrium point $K_1 = (y_1^*, y_2^*, y_3^*, y_4^*)$ of the system given in (12) is unstable.

Proof. If $a_2 < 0$, from Descartes' rule of signs, it is clear that the characteristic equation $P(\lambda)$ has at least one positive real root. So, the equilibrium point $K_1 = (y_1^*, y_2^*, y_3^*, y_4^*)$ of the system given in (12) is unstable. $\square$

4. Analysis of a Model with Upper Bounds

In this section, we consider fractional-order system with the order α between $0 < \alpha < 1$:

$$\begin{aligned} D^\alpha x_1(t) &= -\alpha_1 x_1 + \beta_1 x_2 (1 - \varepsilon x_2^2) + A_1, \\ D^\alpha x_2(t) &= -\alpha_2 x_2 + \beta_2 x_1 (1 - \varepsilon x_1^2) + A_2, \\ x_1(0) &= 0, \quad x_2(0) = 0. \end{aligned} \quad (23)$$

A detailed analysis of this model is given in [5]. With the help of the following lemmas, upper bounds are discussed for the system.

Before giving our results, we give some useful lemmas [9, 10].

Lemma 7. Let α, β, γ , and p be positive constants. Then,

$$\begin{aligned} \int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds \\ = t^\theta B[p(\gamma-1)+1, p(\beta-1)+1], \quad t \in R_+, \end{aligned} \quad (24)$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1} (1-s)^{\eta-1} ds$ ($\Re \xi > 0, \Re \eta > 0$) ve $\theta = p(\beta + \gamma - 2) + 1$.

Lemma 8. Let u, v , and $f_i \in C(I, R_+)$, $i = 1, 2$, with f_i be nondecreasing; let $\varphi_{ij} \in C(I \times I, R_+)$ be nondecreasing in a variable t for every s fixed ($i = 1, 2$). If

$$\begin{aligned} u(t) &\leq f_1(t) + \int_0^t [\varphi_{11}(t, s) u(s) + \varphi_{12}(t, s) v(s)] ds, \\ v(t) &\leq f_2(t) + \int_0^t [\varphi_{21}(t, s) u(s) + \varphi_{22}(t, s) v(s)] ds, \\ t &\in I, \end{aligned} \quad (25)$$

then, for $t \in I$, one has

$$\begin{aligned} u(t) &\leq \left[f_1(t) + f_2(t) \int_0^t \varphi_{12}(t, s) \Phi_2(s) ds \right] \\ &\times \exp \left\{ \int_0^t \varphi_{11}(t, s) ds \right. \\ &\quad \left. + \int_0^t \varphi_{12}(t, s) \Phi_2(s) \right. \\ &\quad \left. \times \left(\int_0^s \varphi_{21}(s, \tau) \Phi_1(\tau) d\tau \right) ds \right\}, \end{aligned}$$

$$\begin{aligned}
v(t) \leq & \left[f_2(t) + f_1(t) \int_0^t \varphi_{21}(t,s) \Phi_1(s) ds \right] \\
& \times \exp \left\{ \int_0^t \varphi_{22}(t,s) ds \right. \\
& + \int_0^t \varphi_{21}(t,s) \Phi_1(s) \\
& \left. \times \left(\int_0^s \varphi_{12}(s,\tau) \Phi_2(\tau) d\tau \right) ds \right\}, \quad (26)
\end{aligned}$$

where $\Phi_i(t) := \exp \int_0^t \varphi_{ii}(t,s) ds$, $i = 1, 2$.

Let $\alpha \in (0, 1]$ and consider the system

$$\begin{aligned}
D^\alpha x_1(t) &= f_1(t, x_1, x_2), \\
D^\alpha x_2(t) &= f_2(t, x_1, x_2), \quad (27)
\end{aligned}$$

with the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$. Here, $f_1(t, x_1, x_2) = -\alpha_1 x_1 + \beta_1 x_2(1 - \varepsilon x_2^2) + A_1$ and $f_2(t, x_1, x_2) = -\alpha_2 x_2 + \beta_2 x_1(1 - \varepsilon x_1^2) + A_2$. Now, upper bounds for a fractional-order nonlinear system are discussed with the following theorem.

Theorem 9. Let f_1 and $f_2 \in C(I \times R^2, R)$ and satisfy the following inequality:

$$\begin{aligned}
|f_1(t, x_1, x_2)| &\leq \alpha_1(t) |x_1| + \beta_1(t) |x_2|, \\
|f_2(t, x_1, x_2)| &\leq \beta_2(t) |x_1| + \alpha_2(t) |x_2|, \quad (28)
\end{aligned}$$

where α_i and $\beta_i \in C(I, R_+)$ ($i, j = 1, 2$) and $x_1, x_2 \in R$. Then, one has the following upper bounds for system of fractional order:

$$\begin{aligned}
& |x_1(t)| \\
& \leq t^{\alpha-1} \\
& \times \exp \left\{ \frac{1}{q} k^*(t) \right. \\
& \times \left[\int_0^t \alpha_1^q(s) ds \right. \\
& + \int_0^t \beta_1^q(s) \Psi_2(s) \\
& \left. \left. \times \left(k^*(s) \int_0^s \beta_2^q(\tau) \Psi_1(\tau) d\tau \right) ds \right] \right\},
\end{aligned}$$

$$\begin{aligned}
& |x_2(t)| \\
& \leq t^{\alpha-1} \\
& \times \exp \left\{ \frac{1}{q} k^*(t) \right. \\
& \times \left[\int_0^t \alpha_2^q(s) ds \right. \\
& + \int_0^t \beta_2^q(s) \Psi_1(s) \\
& \left. \left. \times \left(k^*(s) \int_0^s \beta_1^q(\tau) \Psi_2(\tau) d\tau \right) ds \right] \right\}, \quad (29)
\end{aligned}$$

for $t > 0$, where

$$\begin{aligned}
p &= \frac{1+4\alpha}{1+3\alpha}, \quad q = \frac{1+4\alpha}{\alpha}, \\
k^*(t) &= \frac{t^{q\alpha-1} B^{q/p} [p(\alpha-1)+1, p(\alpha-1)+1]}{\Gamma^q(\alpha)}, \quad (30) \\
\Psi_1(t) &= \exp \left(k^*(t) \int_0^t \alpha_1^q(s) ds \right), \\
\Psi_2(t) &= \exp \left(k^*(t) \int_0^t \alpha_2^q(s) ds \right).
\end{aligned}$$

Proof. Since f_1 and f_2 are assumed to be continuous functions, every solution of the initial value problem (IVP) given by (23) is also a solution of the following integral system for $0 < \alpha < 1$:

$$\begin{aligned}
x_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_1(\tau, x_1(\tau), x_2(\tau)) d\tau, \\
x_2(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_2(\tau, x_1(\tau), x_2(\tau)) d\tau. \quad (31)
\end{aligned}$$

Moreover, every solution of integral system is a solution of the IVP [11]. Now, we derive from (28) and (33) the following:

$$\begin{aligned}
\beta(t) &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\alpha-1} [\alpha_1 \beta(\tau) + \beta_1 \gamma(\tau)] d\tau, \\
\gamma(t) &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\alpha-1} [\beta_2 \beta(\tau) + \alpha_2 \gamma(\tau)] d\tau, \quad (32)
\end{aligned}$$

where

$$\beta(t) = |x_1(t)| t^{1-\alpha}, \quad \gamma(t) = |x_2(t)| t^{1-\alpha}. \quad (33)$$

Using Hölder's inequality for $(1/p) + (1/q) = 1$ with $p = (1+4\alpha)/(1+3\alpha)$ in (32), we get the inequality below:

$$\begin{aligned}
\beta(t) &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} k(t) \left(\int_0^t [\alpha_1^q \beta^q(\tau) + \beta_1^q \gamma^q(\tau)] d\tau \right)^{1/q}, \\
\gamma(t) &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} k(t) \left(\int_0^t [\beta_2^q \beta^q(\tau) + \alpha_2^q \gamma^q(\tau)] d\tau \right)^{1/q}, \quad (34)
\end{aligned}$$

where $k(t) = (\int_0^t (t-\tau)^{p(\alpha-1)} \tau^{p(\alpha-1)} d\tau)^{1/p}$. Then, we have

$$\beta^q(t) \leq \frac{t^{q(1-\alpha)}}{\Gamma^q(\alpha)} k^q(t) \int_0^t [\alpha_1^q \beta^q(\tau) + \beta_1^q \gamma^q(\tau)] d\tau, \quad (35)$$

$$\gamma^q(t) \leq \frac{t^{q(1-\alpha)}}{\Gamma^q(\alpha)} k^q(t) \int_0^t [\beta_2^q \beta^q(\tau) + \alpha_2^q \gamma^q(\tau)] d\tau.$$

By Lemma 7, the following inequality is obtained for $k(t)$:

$$k(t) = t^{(2\alpha-2)+(1/p)} B^{1/p} [p(\alpha-1)+1, p(\alpha-1)+1]. \quad (36)$$

By using the last relation in (35), we obtain

$$\beta^q(t) \leq k^*(t) \int_0^t [\alpha_1^q \beta^q(\tau) + \beta_1^q \gamma^q(\tau)] d\tau, \quad (37)$$

$$\gamma^q(t) \leq k^*(t) \int_0^t [\beta_2^q \beta^q(\tau) + \alpha_2^q \gamma^q(\tau)] d\tau,$$

where

$$k^*(t) = \frac{t^{q\alpha-1} B^{q/p} [p(\alpha-1)+1, p(\alpha-1)+1]}{\Gamma^q(\alpha)}. \quad (38)$$

Since

$$p(\alpha-1)+1 = \frac{1+4\alpha}{1+3\alpha}(\alpha-1)+1 = \frac{4\alpha^2}{1+3\alpha} > 0, \quad (39)$$

$$q\alpha-1 = \left(\frac{1+4\alpha}{\alpha}\right)\alpha-1 = 4\alpha > 0,$$

we have the following inequality:

$$0 < B[p(\alpha-1)+1, p(\alpha-1)+1] < +\infty \quad (40)$$

and the function $k^*(t)$ is nondecreasing on I . Now, with an application of Lemma 8 to (37) combining with (33), upper bounds for a fractional-order nonlinear system are obtained. $\square$

5. Numerical Simulation

In nonlinear dynamic systems, predictability can be possible with stability. Also relationship development would be predictable given the right parameters. In this model, parameters provide the condition for the locally asymptotic stability of equilibrium points by using stability analysis on fractional-order transformed system.

In this paper, we focus on couple dynamics depending on the parameters as below. Many scenarios are possible. But in this model, secure or cautious lover (individual 1 retreats from his own feelings but is encouraged by that of individual 2 ($\alpha_1 < 0$ and $\beta_1 > 0$)) and hermit (individual 1 retreats from his own feelings and that of individual 2 ($\alpha_2 < 0$ and $\beta_2 < 0$)) are considered.

Let

$$\begin{aligned} \alpha_1 &= 0.005, & \alpha_2 &= 0.006, & \beta_1 &= 0.0004, \\ \beta_2 &= -0.0001, & \varepsilon &= 0.001, & A_1 &= 0.02, \\ A_2 &= 0.03, & 2\alpha &= 1.6. \end{aligned} \quad (41)$$

FIGURE 1: The graphs of $y_3(t)$ (above) and $y_1(t)$ (below) with the order 2α .

Now, we consider the system

$$D^{2\alpha} x_1 = -0.005x_1 + 0.0004x_2 (1 - 0.001x_2^2) + 0.02, \quad (42)$$

$$D^{2\alpha} x_2 = -0.0001x_1 (1 - 0.001x_1^2) - 0.006x_2 + 0.03.$$

Let the initial conditions be

$$x_1(0) = 0, \quad x_2(0) = 0. \quad (43)$$

After the system is transformed, the following system is obtained with the order of $\alpha = 0.8$:

$$f_1(y_1, y_2, y_3, y_4) = y_2,$$

$$\begin{aligned} f_2(y_1, y_2, y_3, y_4) &= -0.005y_1 \\ &\quad + 0.0004y_3 (1 - 0.001y_3^2) + 0.02, \end{aligned} \quad (44)$$

$$f_3(y_1, y_2, y_3, y_4) = y_4,$$

$$\begin{aligned} f_4(y_1, y_2, y_3, y_4) &= -0.006y_3 \\ &\quad - 0.0001y_1 (1 - 0.001y_1^2) + 0.03. \end{aligned}$$

Let the initial conditions be

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 0. \quad (45)$$

Positive equilibrium point for the problem (44) and (45) is calculated as

$$y_1^* = 4.38469, \quad y_2^* = 0, \quad y_3^* = 4.92833, \quad y_4^* = 0. \quad (46)$$

The approximate solutions $y_1(t)$ and $y_3(t)$ (resp., govern the feelings (x_1) of A to B and the feelings (x_2) of B to A) are displayed in Figure 1 for $2\alpha = 1.6$ with acceleration in feelings. Figure 2 shows the asymptotic approximation of $(y_1(t), y_2(t), y_3(t), y_4(t))$ to the equilibrium point $(4.38469, 0, 4.92833, 0)$ for $\alpha = 0.8$. For the numerical solution of the system, we use the predictor corrector method [12].

We have demonstrated via numerical simulations that the fractional-order nonlinear couple model (42) and (43) can

FIGURE 2: Approximation to the equilibrium point with the order α .

exhibit asymptotic behavior in the presence of nonlinearity for an appropriate set of model parameters. We have observed that the model approaches the equilibrium points asymptotically.

6. Conclusions

In this paper, stability analysis of the fractional-order nonlinear dynamical model of couple is studied by using the fractional Routh-Hurwitz criteria. By using stability analysis on fractional-order system, sufficient condition on the parameters for the locally asymptotic stability of equilibrium points is obtained. A fractional-order nonlinear dynamical model of couple with the order 2α has been formulated and analyzed. In the discussed model, acceleration is observed in the solution. Also upper bounds for a system with the order α have been obtained.

Finally, we have demonstrated via numerical simulations that a fractional-order nonlinear model of couple can exhibit asymptotic behavior in the presence of an appropriate set of model parameters.

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