Research Article

Bounds for Combinations of Toader Mean and Arithmetic Mean in Terms of Centroidal Mean

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The authors find the greatest value \( \lambda \) and the least value \( \mu \), such that the double inequality
\[
C(\lambda a + (1-\lambda) b, \lambda b + (1-\lambda) a) < A(a, b) + (1-A) T(a, b) < C(\mu a + (1-\mu) b, \mu b + (1-\mu) a)
\]
holds for all \( \alpha \in (0, 1) \) and \( a, b > 0 \) with \( a \neq b \), where \( C(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)} \), \( A(a, b) = \frac{(a + b)}{2} \), and \( T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \) denote, respectively, the centroidal, arithmetic, and Toader means of the two positive numbers \( a \) and \( b \).

1. Introduction

In [1], Toader introduced a mean
\[
T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta
\]
\[
= \begin{cases} 
2a E^\sqrt{\frac{1-(b/a)^2}{\pi}}, & a > b, \\
2b E^\sqrt{\frac{1-(a/b)^2}{\pi}}, & a < b, \\
 a, & a = b,
\end{cases}
\]
(1)

where
\[
E(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2 \theta\right)^{1/2} d\theta,
\]
(2)

for \( r \in [0, 1] \) is the complete elliptic integral of the second kind.

In recent years, there have been plenty of literature, such as [2–6], dedicated to the Toader mean.

For \( p \in \mathbb{R} \) and \( a, b > 0 \), the centroidal mean \( C(a, b) \) and \( p \)th power mean \( M_p(a, b) \), are, respectively, defined by
\[
C(a, b) = \frac{2(a^2 + ab + b^2)}{3(a + b)},
\]
\[
M_p(a, b) = \begin{cases} 
\left(\frac{a^p + a^p}{2}\right)^{1/p}, & p \neq 0, \\
\sqrt{ab}, & p = 0,
\end{cases}
\]
(3)

In [7], Vuorinen conjectured that
\[
M_{3/2}(a, b) < T(a, b),
\]
(4)

for all \( a, b > 0 \) with \( a \neq b \). This conjecture was verified by Qiu and Shen [8] and by Barnard et al. [9], respectively.

In [10], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:
\[
T(a, b) < M_{\log 2/\log(\pi/2)}(a, b),
\]
(5)

for all \( a, b > 0 \) with \( a \neq b \).

Chu et al. [5] proved that the double inequality
\[
C(\alpha a + (1-\alpha) b, \alpha b + (1-\alpha) a) < T(a, b) < C(\beta a + (1-\beta) b, \beta b + (1-\beta) a)
\]
(6)
holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/4$ and $\beta \geq 1/2 + \sqrt{\pi - \pi^2/(2\pi)}$.

Very recently, Hua and Qi [11] proved that the double inequality

$$aC(a,b) + (1 - \alpha) A(a,b) < T(a,b)$$

$$< \beta C(a,b) + (1 - \beta) A(a,b)$$

is valid for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/4$ and $\beta \geq 1/2 + \sqrt{\pi - \pi^2/(2\pi)}$.

Very recently, Hua and Qi [11] proved that the double inequality

$$\frac{1}{2} \leq \frac{1}{2} + \sqrt{\pi - \pi^2/(2\pi)}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/4$ and $\beta \geq 1/2 + \sqrt{\pi - \pi^2/(2\pi)}$.

For positive numbers $a,b > 0$ with $a \neq b$, let

$$J(x) = C(xa + (1 - x)b, xb + (1 - x)a)$$

be on $[1/2,1]$. It is not difficult to directly verify that $J(x)$ is continuous and strictly increasing on $[1/2,1]$.

The main purpose of the paper is to find the greatest value $\lambda$ and the least value $\mu$, such that the double inequality $C(\lambda a + (1 - \lambda b), \lambda b + (1 - \lambda)a) < aC(a,b) + (1 - \alpha)T(a,b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)$ holds for all $\alpha \in (0,1)$ and $a,b > 0$ with $a \neq b$. As applications, we also present new bounds for the complete elliptic integral of the second kind.

2. Preliminaries and Lemmas

In order to establish our main result, we need several formulas and Lemmas below.

For $0 < r < 1$ and $r' = \sqrt{1 - r^2}$, Legendre’s complete elliptic integrals of the first and second kinds are defined in [12,13] by

$$K(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2 \theta\right)^{-1/2} d\theta,$$

$$E(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2 \theta\right)^{1/2} d\theta,$$

$$K(0) = \frac{\pi}{2}, \quad K(1) = \infty,$$

$$E(0) = \frac{\pi}{2}, \quad E(1) = 1,$$ (9)

respectively.

For $0 < r < 1$, the formulas

$$\frac{dK}{dr} = K - r^2 K,$$

$$\frac{dE}{dr} = E - r^2 E,$$

$$d\left(\frac{E - r^2 K}{r}\right) = rK,$$

$$\frac{d(\frac{E - r^2 K}{r})}{dr} = rK,$$

$$E\left(\frac{2\sqrt{r}}{1 + r}\right) = E - r^2 K,$$ (10)

were presented in [14, Appendix E, pages 474-475].

Lemma 1 (see [14, Theorem 3.21(1), 3.43 exercises 13(a)]). The function $(E - r^2 K)/r^2$ is strictly increasing from $(0, 1)$ to $(\pi/4, 1)$, and the function $2E - r^2 K$ is increasing from $(0, 1)$ to $(\pi/2, 2)$.

Lemma 2. Let $u, \alpha \in (0, 1)$ and

$$f_{u,\alpha}(r) = \frac{1}{3} ur^2$$

$$- (1 - \alpha) \left(\frac{2}{\pi} (2E(r) - (1 - r^2)K(r)) - 1\right).$$

Then, $f_{u,\alpha} > 0$, for all $r \in (0, 1)$ if and only if $u \geq 3(1 - \alpha)(4/\pi - 1)$, and $f_{u,\alpha} < 0$, for all $r \in (0, 1)$ if and only if $u \leq 3(1 - \alpha)/4$.

Proof. From (II), one has

$$f_{u,\alpha}(0^+) = 0,$$ (12)

$$f_{u,\alpha}(1^-) = \frac{1}{3} u - (1 - \alpha) \left(\frac{4}{\pi} - 1\right),$$ (13)

$$f_{u,\alpha}'(r) = \frac{2}{3} r [u - 3(1 - \alpha) g(r)],$$ (14)

where $g(r) = (1/\pi)((E - r^2 K)/r^2)$.

We divide the proof into four cases.

Case 1 ($u \geq 3(1 - \alpha)/\pi$). From (14) and Lemma 1 together with the monotonicity of $g(r)$, we clearly see that $f_{u,\alpha}(r)$ is strictly increasing on $(0, 1)$. Therefore, $f_{u,\alpha}(r) > 0$, for all $r \in (0, 1)$.

Case 2 ($u \leq 3(1 - \alpha)/4$). From (14) and Lemma 1 together with the monotonicity of $g(r)$, we obtain that $f_{u,\alpha}(r)$ is strictly decreasing on $(0, 1)$. Therefore, $f_{u,\alpha}(r) < 0$, for all $r \in (0, 1)$.

Case 3 ($3(1 - \alpha)/4 < u \leq 3(1 - \alpha)(4/\pi - 1)$). From (13) and (14) together with the monotonicity of $g(r)$, we see that there exists $\lambda \in (0, 1)$, such that $f_{u,\alpha}(r)$ is strictly increasing in $(0, \lambda)$ and strictly decreasing in $[\lambda, 1)$ and

$$f_{u,\alpha}(1^-) \leq 0.$$ (15)

Therefore, making use of (12) and inequality (15) together with the piecewise monotonicity of $f_{u,\alpha}(r)$ leads to the conclusion that there exists $0 < \lambda < \eta < 1$, such that $f_{u,\alpha}(r) > 0$ for $r \in (0, \eta)$ and $f_{u,\alpha}(r) < 0$ for $r \in (\eta, 1)$.

Case 4 ($3(1 - \alpha)(4/\pi - 1) < u < 3(1 - \alpha)/\pi$). Equation (13) leads to

$$f_{u,\alpha}(1^-) \geq 0.$$ (16)

From (13) and (14) together with the monotonicity of $g(r)$, we clearly see that there exists $\lambda \in (0, 1)$, such that $f_{u,\alpha}(r)$ is strictly increasing in $(0, \lambda]$ and strictly decreasing in $[\lambda, 1)$. Therefore, $f_{u,\alpha}(r) > 0$ for $r \in (0, 1)$ follows from (12) and (16) together with the piecewise monotonicity of $f_{u,\alpha}(r)$. □
3. Main Results

Now, we are in a position to state and prove our main results.

**Theorem 3.** If $\alpha \in (0, 1)$ and $\lambda, \mu \in (1/2, 1)$, then the double inequality

\[
C(\lambda a + (1 - \lambda) b, \lambda b + (1 - \lambda) a) < \alpha A(a, b) + (1 - \alpha) T(a, b) < C(\mu a + (1 - \mu) b, \mu b + (1 - \mu) a)
\]

holds for all $a, b > 0$ with $a \neq b$ if and only if

\[
\lambda \leq 1 + \frac{\sqrt{3(1 - \alpha)}}{4}, \\
\mu \geq 1 + \frac{1 + \sqrt{3(1 - \alpha)}(4\pi - 1)}{4}.
\]

**Proof.** Since $A(a, b), T(a, b),$ and $C(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $p \in (1/2, 1)$, $t = b/a \in (0, 1)$, and $r = (1 - t)/(1 + t)$. Then,

\[
C(pa + (1 - p)b, pb + (1 - p)a) = a \left\{ \frac{2}{3} \left( \left( p + (1 - p) \frac{b}{a} \right)^2 + \left( p + (1 - p) \frac{b}{a} \right) \left( \frac{b}{a} + 1 - p \right) + \left( \frac{b}{a} + 1 - p \right)^2 \right) \left( 1 + \frac{b}{a} \right)^{-1} \right. - a \left[ \frac{1 + (b/a)}{2} - \frac{1}{2} \left( \frac{b}{a} \right)^2 \right] - (1 - \alpha) \frac{2}{\pi} \mathcal{E} \left( 1 - \left( \frac{b}{a} \right)^2 \right) \right. \\
- \left. a \left\{ \frac{1}{2} \left( (p + (1 - p) t)^2 + (p + (1 - p) t) (pt + 1 - p) + (pt + 1 - p)^2 \right) \left( 1 + t \right)^{-1} - \frac{1 + t}{2} - (1 - \alpha) \frac{2}{\pi} \mathcal{E} \left( \sqrt{1 - t^2} \right) \right\} \right.
\]

\[
= a \left\{ \frac{1}{3} (1 - 2p)^2 r^2 + 3 \left( 1 + r \right) - \frac{1}{1 + r} \right. - (1 - \alpha) \frac{2}{\pi} \frac{\mathcal{E} - r^2 \mathcal{K}}{1 + r} \right\}.
\]

(19)

Therefore, Theorem 3 follows easily from Lemma 2 and (19).

Let $\alpha = 1/4$, $\lambda = 7/8$, $\mu = (1/2)(1 + (3\sqrt{4/\pi - 1/2}))$. Then, from Theorem 3, we get new bounds for the complete elliptic integral $\mathcal{E}(r)$ of the second kind in terms of elementary functions as follows.

**Corollary 4.** For $r \in (0, 1)$ and $r' = \sqrt{1 - r^2}$, one has

\[
\pi \left[ \frac{5 + 6r' + 5r'^2}{8 + (1 + r')^2} \right] < \mathcal{E}(r) < \pi \left[ \frac{r' + (2/\pi) \left( 1 - r' \right)^2}{1 + r'} \right].
\]

(20)

4. Remarks

**Remark 5.** In the recent past, the complete elliptic integrals have attracted the attention of numerous mathematicians. In [4], it was established that

\[
\frac{\pi}{2} \left[ \frac{1 + r'^2}{2} + \frac{1 + r'}{4} \right] < \mathcal{E}(r) < \pi \left( \frac{4 - \pi}{\sqrt{2 - 1} \pi} \right) \frac{1 + r'^2}{2} + \frac{1 - r'^2}{2} \mathcal{E}(r),
\]

(21)

for all $r \in (0, 1)$.

Guo and Qi [15] proved that

\[
\frac{\pi}{2} - \frac{1}{2} \log \frac{(1 + r)^{1-r}}{(1 - r)^{1+r}} < \mathcal{E}(r) < \frac{\pi - 1}{2} + \frac{1 - r^2}{4r} \log \frac{1 + r}{1 - r},
\]

(22)

for all $r \in (0, 1)$.

Yin and Qi [16] presented that

\[
\frac{\pi \sqrt{6 + 2\sqrt{1 - r^2 - 3r^2}}}{2 \sqrt{2}} \leq \mathcal{E}(r) \leq \frac{\pi}{2} \frac{\sqrt{10 - 2\sqrt{1 - r^2 - 5r^2}}}{2 \sqrt{2}},
\]

(23)

for all $r \in (0, 1)$.

It was pointed out in [4] that the bounds in (21) for $\mathcal{E}(r)$ are better than the bounds in (22) for some $r \in (0, 1)$. 
Remark 6. The lower bound in (20) for $E_\mathcal{C}(r)$ is better than the lower bound in (21). Indeed,

$$
\frac{5 + 6x + 5x^2}{8(1 + x)} = \frac{3x^2 + 2x + 3 - 2\sqrt{2} \left( 1 + x^2 \right)^{1/2}}{8(1 + x)},
$$

(24)

for all $x \in (0,1)$.

Remark 7. The following equivalence relations for $x \in (0,1)$ show that the lower bound in (20) for $E_\mathcal{C}(r)$ is better than the lower bound in (23):

$$
\frac{5 + 6x + 5x^2}{8(1 + x)} > \frac{\sqrt{6 + 2x - 3 \left( 1 - x^2 \right)}}{2\sqrt{2}}
$$

$\iff$

$$
\left( 5x^2 + 6x + 5 \right)^2 > 8(x + 1)^2 \left( 3x^2 + 2x + 3 \right)
$$

(25)

$\iff (x - 1)^4 > 0.$

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References

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