Research Article

Some Endpoint Results for $\beta$-Generalized Weak Contractive Multifunctions

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We introduce $\beta$-generalized weak contractive multifunctions and give some results about endpoints of the multifunctions. Also, we give some results about role of a point in the existence of endpoints.

1. Introduction

Let $(X, d)$ be a metric space, $CB(X)$ the collection of all nonempty bounded and closed subsets of $X$, and $H$ the Hausdorff metric with respect to $d$; that is, $H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$ for all $A, B \in CB(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$. Let $T : X \to 2^X$ be a multifunction. An element $x \in X$ is said to be a fixed point of $T$ whenever $x \in T(x)$. Also, an element $x \in X$ is said to be an endpoint of $T$ whenever $T(x) = \{x\}$ [1]. We say that $T$ has the approximate endpoint property whenever $\inf_{x \in X} \sup_{y \in T(x)} d(x, y) = 0$ [1]. Let $f : X \to X$ be a mapping. We say that $f$ has the approximate endpoint property whenever $\inf_{x \in X} \sup_{y \in f(x)} d(x, y) = 0$ [1]. Also, the function $g : \mathbb{R} \to \mathbb{R}$ is called upper semicontinuous whenever $\lim_{n \to \infty} g(\lambda_n) \leq g(\lambda)$ for all sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ with $\lambda_n \to \lambda$ [2]. In 2010, Amini-Harandi defined the concept of approximate endpoint property for multifunctions and proved the following result (see [1]).

Theorem 1. Let $\psi : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function such that $\psi(t) < t$ and $\lim_{t \to \infty} (t - \psi(t)) > 0$ for all $t > 0$. $(X, d)$ a complete metric space, and $T : X \to CB(X)$ a multifunction satisfying $H(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$. Then $T$ has a unique endpoint if and only if $T$ has the approximate endpoint property.

Then Moradi and Khojasteh introduced the concept of $\beta$-generalized weak contractive multifunctions and improved Theorem 1 by providing the following result [3].

Theorem 2. Let $\psi : [0, \infty) \to [0, \infty)$ be an upper semicontinuous function such that $\psi(t) < t$ and $\lim_{t \to \infty} (t - \psi(t)) > 0$ for all $t > 0$. $(X, d)$ a complete metric space, and $T : X \to CB(X)$ a generalized weak contractive multifunction; that is, $T$ satisfies $H(T(x), T(y)) \leq \psi(N(x, y))$ for all $x, y \in X$, where $N(x, y) = \max\{d(x, y), d(x, T(x)), d(y, T(y)), (d(x, T(y)) + d(y, T(x)))/2\}$. Then $T$ has a unique endpoint if and only if $T$ has the approximate endpoint property.

In this paper, we introduce $\beta$-generalized weak contractive multifunctions and by adding some conditions to assumptions of the results, we give some results about endpoints of $\beta$-generalized weak contractive multifunctions. In 2012, the technique of $\alpha$-$\psi$-contractive mappings was introduced by Samet et al. [4]. Later, some authors used it for some subjects in fixed point theory (see for example [5–8]) or generalized it by using the method of $\beta$-$\psi$-contractive multifunctions (see e.g., [9–12]).

Let $(X, d)$ be a metric space and $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping. A multifunction $T : X \to 2^X$ is called $\beta$-generalized weak contraction whenever there...
exists a nondecreasing, upper, semicontinuous function \( \psi : [0, +\infty) \to [0, +\infty) \) such that \( \psi(t) < t \) for all \( t > 0 \) and

\[
\beta(Tx, Ty) H(Tx, Ty) \leq \psi(N(x, y))
\]

for all \( x, y \in X \). We say that \( T \) is \( \beta \)-admissible whenever \( \beta(A, B) \geq 1 \) implies that \( \beta(Tx, Ty) \geq 1 \) for all \( x \in A \), and \( y \in B \), where \( A \) and \( B \) are subsets of \( X \). We say that \( T \) has the property \( (R) \) whenever for each convergent sequence \( \{x_n\} \) in \( X \) with \( x_n \to x \) and \( \beta(Tx_{n-1}, Tx_n) \geq 1 \) for all \( n \geq 1 \), we have \( \beta(Tx_m, Tx_n) \geq 1 \). One can find idea of the property \( (R) \) for mappings in [13]. We say that \( T \) has the property \( (K) \) whenever for each sequence \( \{x_n\} \) in \( X \) with \( \beta(Tx_{n-1}, Tx_n) \geq 1 \) for all \( n \geq 1 \), there exists a natural number \( k \) such that \( \beta(Tx_m, Tx_n) \geq 1 \) for all \( m > n \geq k \). Finally, we say that \( T \) has the property \( (H) \) whenever for each \( \epsilon > 0 \), there exists \( x \in X \) such that \( \sup_{a \in T} d(z, a) < \epsilon \) implies that for every \( x \in X \) there exists \( y \in X \) such that \( H(Tx, Ty) = \sup_{a \in T} d(y, b) \). A multifunction \( T : X \to \mathcal{X} \) is called lower semicontinuous at \( x_0 \in X \) whenever for each sequence \( \{x_n\} \) in \( X \) with \( x_n \to x_0 \) and every \( y \in Tx_0 \), there exists a sequence \( \{y_n\} \) in \( X \) with \( y_n \to y \) in \( X \).

### 2. Main Results

Now, we are ready to state and prove our main results.

**Theorem 3.** Let \((X, d)\) be a complete metric space, \( \beta : 2^X \times 2^X \to [0, \infty) \) a mapping, and \( T : X \to CB(X) \) a \( \beta \)-admissible, \( \beta \)-generalized weak contractive multifunction which has the properties \((R), (K), \) and \((H)\). Suppose that there exist a subset \( A \) of \( X \) and \( x_0 \in A \) such that \( \beta(A, Tx_0) \geq 1 \). Then \( T \) has an endpoint if and only if \( T \) has the approximate endpoint property.

**Proof.** It is clear that if \( T \) has an endpoint, then \( T \) has the approximate endpoint property. Conversely, suppose that \( T \) has the approximate endpoint property. Choose \( A \subset X \) and \( x_0 \in A \) such that \( \beta(A, Tx_0) \geq 1 \). Since \( T \) has the approximate endpoint property, for each \( \epsilon > 0 \), there exists \( z \in X \) such that \( \sup_{a \in T} d(z, a) < \epsilon \). Now by using the condition \( (H) \), choose \( x_1 \in Tx_0 \) such that \( H(Tx_0, Tx_1) = \sup_{a \in T} d(x_1, a) \). Also, choose \( x_2 \in Tx_1 \) such that \( H(Tx_1, Tx_2) = \sup_{a \in T} d(x_2, a) \), and by continuing this process, we find a sequence \( \{x_n\} \) in \( X \) such that \( x_n \in Tx_{n-1} \) and

\[
\beta(Tx_{n-1}, Tx_n) H(Tx_{n-1}, Tx_n) \leq \psi(N(x_{n-1}, x_n))
\]

for all \( n \geq 1 \). Since \( \beta(A, Tx_0) \geq 1 \) and \( T \) is \( \beta \)-admissible, \( \beta(Tx_0, Tx_1) \geq 1 \). By using induction, it is easy to see that \( \beta(Tx_{n-1}, Tx_n) \geq 1 \) for all \( n \geq 1 \). Thus, we obtain

\[
d(x_n, x_{n+1}) \leq \sup_{a \in T} d(x_n, a) = H(Tx_{n-1}, Tx_n) \\
\leq \beta(Tx_{n-1}, Tx_n) H(Tx_{n-1}, Tx_n) \\
\leq \psi(N(x_{n-1}, x_n))
\]

for all \( n \geq 1 \). If \( N(x_{n-1}, x_n) = d(x_{n-1}, x_n) \), then

\[
d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)).
\]

If \( N(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1}) \), then

\[
d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, Tx_{n-1})) \leq \psi(d(x_{n-1}, x_n)).
\]

If \( N(x_{n-1}, x_n) = d(x_n, Tx_n) \), then

\[
d(x_n, x_{n+1}) \leq \psi(d(x_n, Tx_n)) \leq \psi(d(x_n, x_{n+1})),
\]

and so \( d(x_n, x_{n+1}) = 0 \). Thus, \( d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \). If

\[
N(x_{n-1}, x_n) = \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2} \\
\leq \frac{d(x_n, x_{n+1})}{2} \\
\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \\
\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},
\]

then \( d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \) (other case implies that \( d(x_{n-1}, x_{n+1}) = 0 \)). Thus,

\[
d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n))
\]

for all \( n \geq 1 \). We claim that \( \psi(0) = 0 \). If \( \psi(0) > 0 \), then \( \psi^2(0) \geq \psi(0) > 0 \) because \( \psi \) is nondecreasing. On the other hand, since \( \psi(t) < t \) for all \( t > 0 \), we have \( \psi^2(0) < \psi(0) \) which is a contradiction. Hence, \( \psi(0) = 0 \). Let \( d_n = d(x_n, x_{n+1}) \) for all \( n \). If there exists a natural number \( n_0 \) such that \( d_{n_0} = 0 \), then it is easy to see that \( d_{n} = 0 \) for all \( n \geq n_0 \). and so \( \lim_{n \to \infty} d_n = 0 \). Now suppose that \( d_n \neq 0 \) for all \( n \). In this case, we have \( d_n \leq \psi(d_{n-1}) < d_{n-1} \) for all \( n \). Hence, \( \{d_n\} \) is a decreasing sequence, and so there exists \( d \geq 0 \) such that \( \lim_{n \to \infty} d_n = d \). If \( d > 0 \), then \( d_n > 0 \) for all \( n \), and so \( d_n \leq \psi(d_{n-1}) < d_{n-1} \) for all \( n \). Since \( \psi \) is upper and semicontinuous, we obtain \( d = \lim_{n \to \infty} d_n \leq \lim_{n \to \infty} \psi(d_{n-1}) \leq \psi(\lim_{n \to \infty} d_n) = \psi(d) < d \) which is a contradiction. Thus, \( \lim_{n \to \infty} d_n = 0 \). Now, we prove that \( \{x_n\} \) is a Cauchy sequence. If \( \{x_n\} \) is not a Cauchy sequence, then there exist \( \epsilon > 0 \) and natural numbers \( m_k, n_k \) such that \( m_k > n_k \geq k \) and \( d(x_{m_k}, x_{n_k}) \geq \epsilon \) for all \( k \geq 1 \). Also, we choose \( m_k \) as small as possible such that

\[
d(x_{m_k-1}, x_{n_k}) < \epsilon.
\]
Thus, $\varepsilon \leq d(x_m, x_{n_k}) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{n_k}) \leq d_{m-1} + \varepsilon$ for all $k$. Hence, $\lim_{k \to \infty} d(x_m, x_{n_k}) = \varepsilon$. Since $T$ has the property (K), we obtain

$$d(x_m, x_{n_k}) \leq d(x_m, x_{m_k}) + d(x_{m_k}, x_{n_k})$$

for all $k$. Since $\lim_{k \to \infty} d(x_m, x_{n_k}) = \varepsilon$, $\lim_{k \to \infty} N(x_m, x_{n_k}) = \varepsilon$. In fact,

$$d(x_m, x_{n_k}) \leq N(x_m, x_{n_k})$$

and so $\varepsilon = \lim_{k \to \infty} d(x_m, x_{n_k}) \leq \lim_{k \to \infty} N(x_m, x_{n_k}) \leq \varepsilon$. Since $\psi$ is upper semicontinuous, by using (*) we obtain

$$\varepsilon = \lim_{k \to \infty} d(x_m, x_{n_k})$$

$$\leq \lim_{k \to \infty} \psi(N(x_m, x_{n_k})) \leq \psi(\varepsilon) < \varepsilon,$$

which is a contradiction, and so $\{x_n\}$ is a Cauchy sequence. Choose $x^* \in X$ such that $x_n \to x^*$. Now, note that

$$H([x_n], Tx_n) = \max \left\{ d(x_n, Tx_n), \sup_{y \in Tx_n} d(x_n, y) \right\}$$

$$= H(Tx_{n-1}, Tx_n)$$

for all $n$, and so

$$H([x_n], Tx_n) = \frac{1}{2} d(x, y)$$

for all $x, y \in \mathbb{R}$. Thus, $T$ is a $\beta$-generalized weak contractive multifunction.

**Example 4.** Let $X = \mathbb{R}$. Define $T : X \to CB(X)$ by $Tx_n = [x, x+2]$ for all $x \in X$. Suppose that $\psi : [0, +\infty) \to [0, +\infty)$ is an arbitrary upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. If $x = 0$ and $y = 2$, then $H(Tx, Ty) = H([0, 2], [2, 4]) = 2$ and $N(x, y) = 2$. Hence,

$$H(Tx, Ty) = 2 \psi(2) = \psi(N(x, y)).$$

Thus, $T$ is not a generalized weak contractive multifunction. Now, suppose that $\psi(t) = t/2$ for all $t \geq 0$ and define $\beta : 2^X \times 2^X \to [0, +\infty)$ by $\beta(A, B) = 1/2$ for all subsets $A$ and $B$ of $X$. Then, we have

$$\beta(Tx, Ty) H(Tx, Ty) = \frac{1}{2} d(x, y)$$

for all $x, y \in \mathbb{R}$. Thus, $T$ is a $\beta$-generalized weak contractive multifunction.
Next example shows that there are multifunctions which satisfy the conditions of Theorem 3, while they are not generalized weak contractive multifunctions.

Example 5. Let $X = [0, 9/2]$ and let $d(x, y) = |x - y|$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{x/2\} & 0 \leq x \leq 1 \\ \{x - 3/2\} & 1 < x \leq 3/2 \\ \{0\} & 3/2 < x \leq 9/2 \end{cases}$$ (19)

If $x = 1$ and $y = 3/2$, then

$$H(Tx, Ty) = H\left(\left\{\frac{1}{2}\right\}, \left\{\frac{3}{2}\right\}\right) = 4 > 3$$ (20)

$$= N(x, y) > \psi(N(x, y)),$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is an arbitrary upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Thus, $T$ is not a generalized weak contractive multifunction.

Now, we show that $T$ satisfies all conditions of Theorem 3. For this aim, define $\psi(t) = t/2$ and $\beta(A, B) = 1$ whenever $A$ and $B$ are subsets of $[0, 1]$, and $\beta(A, B) = 0$ otherwise. First suppose that $x \notin [0, 1]$ or that $y \notin [0, 1]$. If $x, y \in (3/2, 9/2)$, then $Tx, Ty \subset [0, 1]$ and $\beta(Tx, Ty) = 1$. But, $H(Tx, Ty) = 0$, and so $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(N(x, y))$. If $x \in (1, 3/2)$ or $y \in (1, 3/2)$, then $Tx \notin [0, 1]$, or $Ty \notin [0, 1]$ and $\beta(Tx, Ty) = 0$. Hence, $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(N(x, y))$. Now, suppose that $x, y \in [0, 1]$. In this case, we have $\beta(Tx, Ty) \geq 1$, $H(Tx, Ty) = H([x/2], [y/2]) = (1/2)d(x, y)$, and $N(x, y) = \max[d(x, y), x/2, y/2, (d(x, y)/2 + d(y, x/2))/2]$. Thus, $d(x, y) \leq N(x, y)$, and so

$$\beta(Tx, Ty)H(Tx, Ty) = \frac{1}{2}d(x, y) \leq \psi(d(x, y)) \leq \psi(N(x, y)).$$ (21)

Therefore, $T$ is a $\beta$-generalized weak contractive multifunction. Now, we show that $T$ is $\beta$-admissible. If $\beta(A, B) \geq 1$, then $A, B \subset [0, 1]$, and so $Tx = \{x/2\} \in [0, 1]$ and $Ty = \{y/2\} \in [0, 1]$ for all $x \in A$ and $y \in B$. Thus, $\beta(Tx, Ty) \geq 1$ for all $x \in A$ and $y \in B$. Now, suppose $A = [0, 1/2]$ and $x_0 = 1/4$. Then, $Tx_0 = \{1/8\} \in [0, 1]$ and $[0, 1/2] \subset [0, 1]$. Hence, $\beta(A, T(x_0)) \geq 1$. Now, we show that $T$ satisfies the condition $(H)$. First note that, for each $\varepsilon > 0$, there exists $z \in X$ such that $\sup_{x \in Tz}d(z, a) < \varepsilon$. Now, we show that for each $x \in X$ there exists $y \in Tz$ such that $H(Tx, Ty) = \sup_{b \in Tz}d(y, b)$. If $0 \leq x \leq 1$, then $Tx = \{x/2\}, T(x/2) = \{x/4\}$, and

$$H\left(Tx, T\left(\frac{x}{2}\right)\right) = H\left(\left\{\frac{x}{2}\right\}, \left\{\frac{x}{4}\right\}\right) = \frac{x}{4} = \sup_{b \in T(x/2)}d\left(\frac{x}{2}, b\right).$$ (22)

Since for $1 < x \leq 3/2$ we have $5/2 < 4x - (3/2) \leq 9/2$, $T(4x - (3/2)) = \{0\}$. Thus,

$$H\left(Tx, T\left(\frac{4x - 3}{2}\right)\right) = H\left(\left\{\frac{4x - 3}{2}\right\}, \{0\}\right) = 4x - 3 = \sup_{b \in T(4x - 3/2)}d\left(\frac{4x - 3}{2}, b\right).$$ (23)

If $3/2 < x \leq 9/2$, then $Tx = \{0\}$ and $T(0) = \{0\}$. Hence,

$$H\left(Tx, T(0)\right) = H\left(\{0\}, \{0\}\right) = 0 = \sup_{b \in T(0)}d(0, b).$$ (24)

It is easy to check that $T$ satisfies the conditions $(R)$ and $(K)$. Note that, 0 is the endpoint of $T$.

Now, we add an assumption to obtain uniqueness of endpoint. In this way, we introduce a new notion. Let $X$ be a set and $\beta : 2^X \times 2^X \rightarrow [0, +\infty)$ a map. We say that the set $X$ has the property $(G_\beta)$ whenever $\beta(A, B) \geq 1$ for all subsets $A$ and $B$ of $X$ with $A \notin B$ or $B \notin A$.

**Corollary 6.** Let $(X, d)$ be a complete metric space, $\beta : 2^X \times 2^X \rightarrow [0, +\infty)$ a mapping, and $T : X \rightarrow CB(X)$ a $\beta$-admissible, $\beta$-generalized weak contractive multifunction which has the properties $(R)$, $(K)$, and $(H)$. Suppose that there exist a subset $A$ of $X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. If $T$ has the approximate endpoint property and $X$ has the property $(G_\beta)$, then $T$ has a unique endpoint.

**Proof.** By using Theorem 3, $T$ has a endpoint. If $T$ has two distinct endpoints $x^*$ and $y^*$, then $\beta(Tx^*, Ty^*) = \beta(\{x^*\}, \{y^*\}) \geq 1$ because $X$ has the property $(G_\beta)$. Hence,

$$d(\langle x^*, y^* \rangle) \leq H(Tx^*, Ty^*) \leq \beta(\langle x^*, y^* \rangle)H(Tx^*, Ty^*) \leq \psi(\langle x^*, y^* \rangle) < N(\langle x^*, y^* \rangle) = d(\langle x^*, y^* \rangle),$$

which is a contradiction. Thus, $T$ has a unique endpoint.

In Example 5, $T$ has a unique endpoint, while $X$ does not have the property $(G_\beta)$. Also, $T$ has the property $(R)$, while $T$ is not lower semicontinuous. To see this, consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} 1 - \frac{1}{n} & n = 2k \\ 1 + \frac{1}{n} & n = 2k - 1 \end{cases}$$ (26)

for $k \geq 1$ and put $y = 1/2$ and $x_0 = 1$. Then $x_n \to 1$ and $y \in T(x_0) = \{1/2\}$. Let $\{y_n\}$ be an arbitrary sequence in $X$ such that $y_n \in Tx_n$ for all $n \geq 1$. Then, $y_{2k-1} \in Tx_{2k-1}$ and $y_{2k} \in Tx_{2k}$ for all $k$. But, $y_{2k-1} = 4x_{2k-1} - (3/2)$ for sufficiently large $k$ and $y_{2k} = x_{2k}/2$ for all $k$ since $y_{2k-1} \to 5/2, y_{2k} \to 1/2$. This implies that $T$ is not lower semicontinuous.
Corollary 7. Let \((X, d)\) be a complete metric space, \(\beta : 2^X \times 2^X \to [0, \infty)\) a mapping, and \(T : X \to CB(X)\) a \(\beta\)-admissible multifunction which has the properties (R), (K), and (H). Suppose that \(X\) has the property (G\_3), and there exist a subset \(A\) of \(X\), \(x_0 \in A\) and \(k \in \{0, 1\}\) such that \(\beta(A, Tx_0) \geq 1\) and \(\beta(Tx, Ty)H(Tx, Ty) \leq kN(x, y)\) for all \(x, y \in X\). Then \(T\) has a unique endpoint if and only if \(T\) has the approximate endpoint property.

**Proof.** It is sufficient that we define \(\psi(t) = kt\) for all \(t \geq 0\). Then, Theorem 3 and Corollary 6 guarantee the result. \(\square\)

It has been proved that lower semicontinuity of the multifunction \(T\) and the property \((R)\) are independent conditions [9]. We can replace lower semicontinuity of the multifunction instead of the property \((R)\) to obtain the next result. Its proof is similar to the proof of Theorem 3.

Theorem 8. Let \((X, d)\) be a complete metric space, \(\beta : 2^X \times 2^X \to [0, \infty)\) a mapping, and \(T : X \to CB(X)\) a lower semicontinuous, \(\beta\)-admissible, \(\beta\)-generalized weak contractive multifunction which has the properties (K) and (H). Suppose that there exist a subset \(A\) of \(X\) and \(x_0 \in A\) such that \(\beta(A, Tx_0) \geq 1\). Then \(T\) has the approximate endpoint property if and only if \(T\) has an endpoint.

Corollary 9. Let \((X, d)\) be a complete metric space, \(\beta : 2^X \times 2^X \to [0, \infty)\) a mapping, and \(T : X \to CB(X)\) a lower semicontinuous, \(\beta\)-admissible, \(\beta\)-generalized weak contractive multifunction which has the properties (K) and (H). Suppose that there exist a subset \(A\) of \(X\) and \(x_0 \in A\) such that \(\beta(A, Tx_0) \geq 1\). If \(T\) has the approximate endpoint property and \(X\) has the property (G\_3), then \(T\) has a unique endpoint.

Corollary 10. Let \((X, d)\) be a complete metric space, \(\beta : 2^X \times 2^X \to [0, \infty)\) a mapping, and \(T : X \to CB(X)\) a \(\beta\)-admissible multifunction which has the properties (R), (K), and (H). Suppose that \(X\) has the property (G\_3), and there exist a subset \(A\) of \(X\), \(x_0 \in A\) and \(k \in \{0, 1\}\) such that \(\beta(A, Tx_0) \geq 1\) and \(\beta(Tx, Ty)H(Tx, Ty) \leq kN(x, y)\) for all \(x, y \in X\). If \(T\) has the approximate endpoint property, then \(\text{Fix}(T) = \text{End}(T) = \{x\}\).

**Proof.** If we put \(\psi(t) = kt\), then, by using Theorem 2.10 in [9], \(T\) has a fixed point. Since \(T\) has the approximate endpoint property, by using Corollary 7, \(T\) has a unique endpoint such \(x\). Let \(y \in \text{Fix}(T)\). If \(Tx = Ty\), then \(y = x\). If \(Tx \neq Ty\), then \(\beta(Tx, Ty) \geq 1\) because \(X\) has the property (G\_3). Also, we have

\[
d(x, y) \leq H([x], Ty) = H(Tx, Ty) \leq \beta(Tx, Ty)H(Tx, Ty) \leq kN(x, y).
\]

But, \(N(x, y) = \max[d(x, y), d(x, Tx), d(y, Ty), (d(x, y) + d(y, Ty))/2] = d(x, y)\). Thus, \(d(x, y) = 0\), and so \(\text{Fix}(T) = \text{End}(T) = \{x\}\).

Next corollary shows us the role of a point in the existence of endpoints.

Corollary 11. Let \((X, d)\) be a complete metric space, \(x^* \in X\) a fixed element, and \(T : X \to CB(X)\) a multifunction such that \(T\) has the property (H) and \(x^* \in Tx \cap Ty\) for all subsets \(A\) and \(B\) of \(X \times x^* \in A \cap B\) and all \(x \in A\) and \(y \in B\). Assume that \(H(Tx, Ty) \leq \psi(N(x, y))\) for all \(x, y \in X\) with \(x^* \in Tx \cap Ty\), where \(\psi : [0, \infty) \to [0, \infty)\) is a nondecreasing upper semicontinuous function such that \(\psi(t) < t\) for all \(t > 0\). Suppose that there exist a subset \(A_0\) of \(X\) and \(x_0 \in A_0\) such that \(x^* \in A_0 \cap Tx_0\). Assume that for each convergent sequence \(\{x_n\}\) in \(X\) with \(x_n \to x^* \in Tx_0\) for all \(n \geq 1\), one has \(x^* \in Tx_0 \cap X\). Also, for each sequence \(\{x_n\}\) in \(X\) with \(x^* \in Tx_{n-1} \cap Tx_n\) for all \(n \geq 1\), there exists a natural number \(k\) such that \(x^* \in Tx_{n-1} \cap Tx_n\) for all \(m > n \geq k\). Then \(T\) has an endpoint if and only if \(T\) has the approximate endpoint property.

**Proof.** It is sufficient we define \(\beta : 2^X \times 2^X \to [0, \infty)\) by \(\beta(A, B) = 1\) whenever \(x^* \in A \cap B\) and \(\beta(A, B) = 0\) otherwise, and then we use Theorem 3. \(\square\)

Corollary 12. Let \((X, d)\) be a complete metric space, \(x^* \in X\) a fixed element and \(T : X \to CB(X)\) a lower semicontinuous multifunction such that \(T\) has the property (H) and \(x^* \in Tx \cap Ty\) for all subsets \(A\) and \(B\) of \(X\) with \(x^* \in A \cap B\) and all \(x \in A\) and \(y \in B\). Assume that

\[
H(Tx, Ty) \leq \psi(N(x, y))
\]

(28)

for all \(x, y \in X\) with \(x^* \in Tx \cap Ty\), where \(\psi : [0, \infty) \to [0, \infty)\) is a nondecreasing upper semicontinuous function such that \(\psi(t) < t\) for all \(t > 0\). Suppose that there exist a subset \(A_0\) of \(X\) and \(x_0 \in A_0\) such that \(x^* \in A_0 \cap Tx_0\). Assume that for each convergent sequence \(\{x_n\}\) in \(X\) with \(x_n \to x^* \in Tx_0\) for all \(n \geq 1\), we have \(x^* \in Tx_0 \cap X\). Then \(T\) has an endpoint if and only if \(T\) has the approximate endpoint property.

**Proof.** It is sufficient to define \(\beta : 2^X \times 2^X \to [0, \infty)\) by \(\beta(A, B) = 1\) whenever \(x^* \in A \cap B\) and \(\beta(A, B) = 0\) otherwise, and then we use Theorem 8. \(\square\)

Let \((X, d, \leq)\) be an ordered metric space. Define the order \(\leq\) on arbitrary subsets \(A\) and \(B\) of \(X\) by \(A \leq B\) if and only if for each \(a \in A\) there exists \(b \in B\) such that \(a \leq b\). It is easy to check that \((CB(X), \leq)\) is a partially ordered set.

Theorem 13. Let \((X, d, \leq)\) be a complete ordered metric space and \(T\) a closed and bounded valued multifunction on \(X\) such that \(T\) has the property (H) and \(Tx \leq Ty\) for all subsets \(A\) and \(B\) of \(X\) with \(A \leq B\) and all \(x \in A\) and \(y \in B\). Assume that \(H(Tx, Ty) \leq \psi(N(x, y))\) for all \(x, y \in X\) with \(Tx \leq Ty\), where \(\psi : [0, \infty) \to [0, \infty)\) is a nondecreasing upper semicontinuous function such that \(\psi(t) < t\) for all \(t > 0\). Suppose that there exist a subset \(A_0\) of \(X\) and \(x_0 \in A_0\) such that \(A_0 \leq Tx_0\). Assume that for each convergent sequence \(\{x_n\}\) in \(X\) with \(x_n \to x^* \in Tx_0\) for all \(n \geq 1\), one has \(x^* \leq Tx_0\). Also, for each sequence \(\{x_n\}\) in \(X\) with \(Tx_{n-1} \leq Tx_n\) for all \(n \geq 1\), there exists a natural number \(k\) such that \(Tx_m \leq Tx_n\) for all \(m > n \geq k\). Then \(T\) has an endpoint if and only if \(T\) has the approximate endpoint property.
Proof. Define $\beta(A, B) = 1$ whenever $A \preceq B$ and $\beta(A, B) = 0$ otherwise, and then we use Theorem 3.

Corollary 14. Let $(X, d, \preceq)$ be a complete ordered metric space and $T$ a closed and bounded valued multifunction on $X$ such that $T$ has the property (H) and $Tx \preceq Ty$ for all subsets $A$ and $B$ of $X$ with $A \preceq B$, all $x \in A$, and $y \in B$. Assume that $H(Tx, Ty) \leq \psi(N(x, y))$ for all $x, y \in X$ with $Tx \preceq Ty$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Then $T$ has the approximate endpoint property.

Proof. Define $\beta(A, B) = 1$ whenever $A \preceq B$ and $\beta(A, B) = 0$ otherwise, and then we use Corollary 6.

Let $(X, d)$ be a metric space and $T : X \to 2^X$ a multifunction. We say that $T$ is an $H$-multifunction whenever for each $x \in X$ there exists $y \in Tx$ such that $H(Tx, Ty) = \sup_{b \in Ty} d(y, b)$. It is obvious that each $H$-multifunction is a multifunction which has the property (H). Thus, one can conclude similar results to above ones for $H$-multifunctions. Here, we provide some ones. Although by considering $H$-multifunctions we restrict ourselves, we obtain strange results with respect to above ones. One can prove the following by reading exactly the proofs of similar above results.

Theorem 15. Let $(X, d)$ be a complete metric space, $\beta : 2^X \times 2^X \to [0, +\infty)$ a mapping, and $T : X \to CB(X)$ a $\beta$-admissible, $\beta$-generalized weak contractive $H$-multifunction which has the properties (R) and (K). Suppose that there exist a subset $A$ of $X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then $T$ has an endpoint, and so $T$ has the approximate endpoint property.

Theorem 16. Let $(X, d)$ be a complete metric space, $\beta : 2^X \times 2^X \to [0, +\infty)$ a mapping, and $T : X \to CB(X)$ a lower semicontinuous, $\beta$-admissible, and $\beta$-generalized weak contractive $H$-multifunction which has the property (K). Suppose that there exist a subset $A$ of $X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then $T$ has an endpoint, and so $T$ has the approximate endpoint property.

The next result is a consequence of Theorem 15.

Corollary 17. Let $(X, d)$ be a complete metric space, $x^* \in X$ a fixed element, and $T : X \to CB(X)$ an $H$-multifunction such that $x^* \in Tx \cap Ty$ for all subsets $A$ and $B$ of $X$ with $x^* \in A \cap B$, all $x \in A$, and $y \in B$. Assume that $H(Tx, Ty) \leq \psi(N(x, y))$ for all $x, y \in X$ with $x^* \in Tx \cap Ty$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Then $T$ has an endpoint, and so $T$ has the approximate endpoint property.

The next result is a consequence of Theorem 16.

Corollary 18. Let $(X, d, \preceq)$ be a complete ordered metric space and $T$ a closed and bounded valued lower semicontinuous $H$-multifunction on $X$ such that $Tx \preceq Ty$ for all subsets $A$ and $B$ of $X$ with $A \preceq B$, all $x \in A$, and $y \in B$. Assume that $H(Tx, Ty) \leq \psi(N(x, y))$ for all $x, y \in X$ with $Tx \preceq Ty$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$. Suppose that there exist a subset $A$ of $X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then $T$ has an endpoint, and so $T$ has the approximate endpoint property.


