Research Article

On k th-Order Slant Weighted Toeplitz Operator

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Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $0 < \beta_n/\beta_{n+1} \leq 1$ when $n \geq 0$ and $0 < \beta_n/\beta_{n-1} \leq 1$ when $n \leq 0$. A k th-order slant weighted Toeplitz operator on $L^2(\beta)$ is given by $U_\phi = W_k M_\phi$, where M_ϕ is the multiplication on $L^2(\beta)$ and W_k is an operator on $L^2(\beta)$ given by $W_k e_{nk}(z) = (\beta_n/\beta_{nk})e_n(z)$, $\{e_n(z) = z^k/\beta_k\}_{k \in \mathbb{Z}}$ being the orthonormal basis for $L^2(\beta)$. In this paper, we define a k th-order slant weighted Toeplitz matrix and characterise U_ϕ in terms of this matrix. We further prove some properties of U_ϕ using this characterisation.

1. Introduction

Toeplitz operators and slant Toeplitz operators [1] have been found immensely useful, especially in the study of prediction theory [2], wavelet analysis [3], and solution of differential equations [4]. Originally, these operators were defined and studied on the usual H^2 and L^2 spaces. During the past few decades, different generalisations of these spaces, like the weighted function spaces H_w^P and L_w^P and the weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ have developed. Shields [5] has made a systematic study of the multiplication operator on $L^2(\beta)$. Lauric [6] has discussed the Toeplitz operator on $H^2(\beta)$. Motivated by these studies, we introduced and studied the notion of a slant weighted Toeplitz operator [7] on $L^2(\beta)$. In this paper, we study a k th-order slant weighted Toeplitz operator on the space $L^2(\beta)$. We begin with the following preliminaries.

Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $0 < (\beta_n/\beta_{n+1}) \leq 1$ when $n \geq 0$ and $0 < (\beta_n/\beta_{n-1}) \leq 1$ when $n \leq 0$.

Throughout the paper, we assume that for a fixed integer $k \geq 2$, $\beta_k/\beta_n \leq M < \infty$. Consider the following spaces [5, 6]:

$$\begin{aligned} L^2(\beta) &= \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \right. \\ &\quad \left. \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}, \\ H^2(\beta) &= \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \right. \\ &\quad \left. \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}. \end{aligned} \quad (1)$$

Then, $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space [6] with an orthonormal basis given by $\{e_k(z) = z^k/\beta_k\}_{k \in \mathbb{Z}}$ and with an inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2. \quad (2)$$

Also, $H^2(\beta)$ is a subspace of $L^2(\beta)$. Further, the space

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid \phi L^2(\beta) \subseteq L^2(\beta), \exists c \in \mathbb{R} \right. \\ \left. \text{such that } \|\phi f\|_\beta \leq c \|f\|_\beta \forall f \in L^2(\beta) \right\} \quad (3)$$

is a Banach space with respect to the norm defined by

$$\|\phi\|_\infty = \inf \{ c \mid \|\phi f\|_\beta \leq c \|f\|_\beta \forall f \in L^2(\beta) \}. \quad (4)$$

The mapping $P : L^2(\beta) \rightarrow H^2(\beta)$ is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. For a given $\phi \in L^\infty(\beta)$, the induced weighted multiplication operator [5] is denoted by M_ϕ and is given by $M_\phi : L^2(\beta) \rightarrow L^2(\beta)$ such that

$$M_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_{n-k} \beta_n e_n(z). \quad (5)$$

If we put $\phi_1(z) = z$, then $M_{\phi_1} = M_z$ is the operator defined as $M_z e_k(z) = w_k e_{k+1}(z)$, where $w_k = \beta_{k+1}/\beta_k$ for all $k \in \mathbb{Z}$, and it is known as a weighted shift [5].

The slant weighted Toeplitz operator [7] A_ϕ is an operator on $L^2(\beta)$ defined as $A_\phi : L^2(\beta) \rightarrow L^2(\beta)$ such that $A_\phi e_k(z) = (1/\beta_k) \sum_{n=-\infty}^{\infty} a_{2n-k} \beta_n e_n(z)$.

If $W : L^2(\beta) \rightarrow L^2(\beta)$ is such that

$$W e_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z), \quad W e_{2n-1}(z) = 0 \quad \forall n \in \mathbb{Z}, \quad (6)$$

then A_ϕ can be alternately defined by

$$A_\phi(f) = W M_\phi(f) = W(\phi f) \quad \forall f \in L^2(\beta). \quad (7)$$

2. kth-Order Slant Weighted Toeplitz Operator

Suppose that the operator $W_k : L^2(\beta) \rightarrow L^2(\beta)$ is such that

$$W_k e_n(z) = \begin{cases} \frac{\beta_{n/k}}{\beta_n} e_{n/k}(z) & \text{if } n \text{ is divisible by } k, \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

Then the matrix of W_k is

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{\beta_0}{\beta_0} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \frac{\beta_1}{\beta_k} & 0 & \cdots & \cdots \\ \cdots & \underbrace{0 & 0 & 0 & 0}_{k\text{-times}} & \cdots & \cdots & 0 & \frac{\beta_2}{\beta_{2k}} & \cdots \\ \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \quad (9)$$

The adjoint of W_k is given by

$$W_k^* e_n(z) = \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \quad \forall n \in \mathbb{Z}. \quad (10)$$

Definition 1 (see [8]). For an integer $k \geq 2$, the k th order slant weighted Toeplitz operator $U_\phi : L^2(\beta) \rightarrow L^2(\beta)$ is such that $U_\phi(f) = W_k M_\phi(f)$ for all $f \in L^2(\beta)$. Thus, $U_\phi e_j(z) = (1/\beta_j) \sum_{n=-\infty}^{\infty} a_{nk-j} \beta_n e_n(z)$.

The (i, j) th entry of the matrix of U_ϕ is given by

$$\langle U_\phi e_j, e_i \rangle = \left\langle \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{nk-j} \beta_n e_n(z), e_i(z) \right\rangle \\ = a_{ik-j} \frac{\beta_i}{\beta_j}. \quad (11)$$

Hence, the matrix of U_ϕ with respect to this basis is as follows:

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-k+1} \frac{\beta_{-1}}{\beta_{-1}} & a_{-k} \frac{\beta_1}{\beta_0} & a_{-k-1} \frac{\beta_{-1}}{\beta_1} & a_{-k-2} \frac{\beta_{-1}}{\beta_2} & \cdots \\ \cdots & a_1 \frac{\beta_0}{\beta_{-1}} & a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & \cdots \\ \cdots & a_{k+1} \frac{\beta_1}{\beta_{-1}} & a_k \frac{\beta_1}{\beta_0} & a_{k-1} \frac{\beta_1}{\beta_1} & a_{k-2} \frac{\beta_1}{\beta_2} & \cdots \\ \cdots & a_{2k+1} \frac{\beta_2}{\beta_{-1}} & a_{2k} \frac{\beta_2}{\beta_0} & a_{2k-1} \frac{\beta_2}{\beta_1} & a_{2k-2} \frac{\beta_2}{\beta_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \quad (12)$$

Theorem 2. The mapping $\phi \rightarrow U_\phi$ is linear and one to one.

Proof. For linearity, consider

$$U_{\alpha\phi+\beta\psi} = W_k M_{\alpha\phi+\beta\psi} \\ = W_k (M_{\alpha\phi} + M_{\beta\psi}) \\ = W_k M_{\alpha\phi} + W_k M_{\beta\psi} \\ = \alpha W_k M_\phi + \beta W_k M_\psi \\ = \alpha U_\phi + \beta U_\psi. \quad (13)$$

For checking one-to-one, let $U_\phi = U_\psi$.

Then, $U_\phi - U_\psi = 0$. By linearity, we get $U_{\phi-\psi} = 0$.

Hence,

$$U_{\phi-\psi} e_n(z) = 0, \quad n \in \mathbb{Z}, \\ \implies W_k M_{\phi-\psi} e_n(z) = 0, \quad n \in \mathbb{Z}, \\ \implies W_k (\phi - \psi) e_n(z) = 0, \quad n \in \mathbb{Z}. \quad (14)$$

This implies that either $\phi - \psi = 0$ or the degree of $(\phi - \psi) e_n(z)$ is not divisible by k . But since this is true for all $n \in \mathbb{Z}$, the second possibility is ruled out. Hence, $\phi - \psi = 0$ or $\phi = \psi$. $\square$

Theorem 3. Consider the following: (i) $M_z W_k = W_k M_{z^k}$; (ii) $M_{z^m} W_k = W_k M_{z^{km}}$, $m \in \mathbb{Z}$.

Proof. (i) It is sufficient to prove that

$$M_z W_k e_n(z) = W_k M_{z^k} e_n(z) \quad \forall n \in \mathbb{Z}. \quad (15)$$

Suppose that n is not a multiple of k . Then,

$$M_z W_k e_n(z) = M_z 0 = 0 = W_k M_{z^k} e_n(z). \quad (16)$$

Now, when $n = pk$ (multiple of k),

$$\begin{aligned} M_z W_k e_n(z) &= M_z \frac{\beta_p}{\beta_{pk}} e_p(z) \\ &= \frac{\beta_{p+1}}{\beta_{pk}} e_{p+1}(z). \end{aligned} \quad (17)$$

On the other hand,

$$\begin{aligned} W_k M_{z^k} e_n(z) &= W_k M_{z^k} e_{pk}(z) \\ &= W_k \frac{\beta_{(p+1)k}}{\beta_{pk}} e_{(p+1)k}(z) \\ &= \frac{\beta_{p+1}}{\beta_{pk}} e_{p+1}(z). \end{aligned} \quad (18)$$

Hence from (17) and (18), we get that

$$M_z W_k e_n(z) = W_k M_{z^k} e_n(z) \quad \text{whenever } n \text{ is a multiple of } k. \quad (19)$$

We therefore conclude that for all $n \in \mathbb{Z}$,

$$\begin{aligned} M_z W_k e_n(z) &= W_k M_{z^k} e_n(z) \\ \implies M_z W_k &= W_k M_{z^k}. \end{aligned} \quad (20)$$

(ii) We prove the result by induction on m .

For $m = 1$, the result is true by part (i).

Suppose that the result is true for $m = l$. Then,

$$M_{z^l} W_k = W_k M_{z^{lk}}. \quad (21)$$

Now

$$\begin{aligned} M_{z^{(l+1)k}} W_k &= M_{z^l} M_{z^k} W_k \\ &= M_{z^l} W_k M_{z^{lk}} \\ &= W_k M_{z^{lk}} M_{z^{lk}} \\ &= W_k M_{z^{(l+1)k}}. \end{aligned} \quad (22)$$

Hence by induction, the result is true for all $m \in \mathbb{Z}^+$.

For $m = -1$ and $1 \leq j \leq k-1$,

$$M_{z^m} W_k e_{nk+j}(z) = M_{z^{-1}} 0 = 0 = W_k M_{z^{-k}} e_{nk+j}(z). \quad (23)$$

For $m = -1$ and $j = nk$ (a multiple of k),

$$\begin{aligned} M_{z^m} W_k e_j(z) &= M_{z^{-1}} W_k e_{nk}(z) \\ &= M_{z^{-1}} \frac{\beta_n}{\beta_{nk}} e_n(z) \\ &= \frac{\beta_n}{\beta_{nk}} M_{z^{-1}} e_n(z) \\ &= \frac{\beta_n}{\beta_{nk}} \frac{\beta_{n-1}}{\beta_n} e_{n-1}(z) \\ &= \frac{\beta_{n-1}}{\beta_{nk}} e_{n-1}(z). \end{aligned} \quad (24)$$

On the other hand,

$$\begin{aligned} W_k M_{z^{-1}} e_{nk}(z) &= W_k \left(z^{-1} \frac{z^{nk}}{\beta_{nk}} \right) \\ &= W_k \frac{z^{(n-1)k}}{\beta_{nk}} \frac{\beta_{(n-1)k}}{\beta_{(n-1)k}} \\ &= \frac{\beta_{(n-1)k}}{\beta_{nk}} W_k e_{(n-1)k}(z) \\ &= \frac{\beta_{(n-1)k}}{\beta_{nk}} \frac{\beta_{n-1}}{\beta_{(n-1)k}} e_{n-1}(z) \\ &= \frac{\beta_{n-1}}{\beta_{nk}} e_{n-1}(z). \end{aligned} \quad (25)$$

From (23), (24), and (25), we conclude that

$$M_{z^{-1}} W_k = W_k M_{z^{-k}}. \quad (26)$$

Hence, the result is true for $m = -1$ also. Therefore, by using induction, we can prove it for all negative integers m . The case when $m = 0$ is trivially true.

Hence the theorem is true. $\square$

Corollary 4. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. Then,

$$M_{\phi(z)} W_k = W_k M_{\phi(z^k)}. \quad (27)$$

Theorem 5. Consider the following: $M_{\phi(z)} U_{\psi(z)} = U_{\phi(z^k)\psi(z)}$.

Proof. The proof is as follows:

$$\begin{aligned} M_{\phi(z)} U_{\psi(z)} &= M_{\phi(z)} W_k M_{\psi(z)} \\ &= W_k M_{\phi(z^k)} M_{\psi(z)} \\ &= W_k M_{\phi(z^k)\psi(z)} \\ &= U_{\phi(z^k)\psi(z)}. \end{aligned} \quad (28)$$

$\square$

3. Matrix Characterisation of U_ϕ

Definition 6. Let $w_n = \beta_{n+1}/\beta_n$ for all $n \in \mathbb{Z}$. Then, the k th order slant weighted Toeplitz matrix corresponding to the

weight sequence $\langle w_n \rangle$ is a bilaterally infinite matrix $\langle \lambda_{ij} \rangle$ such that

$$\lambda_{i+1,j+k} = \frac{w_i}{w_j w_{j+1} \cdots w_{j+k-1}} \lambda_{i,j}, \quad i, j = 0, \pm 1, \pm 2, \dots \quad (29)$$

We have proved earlier [8] that U is a k th order slant weighted Toeplitz operator if and only if $M_z U = U M_{z^k}$.

Next, we prove a characterisation of the k th order slant weighted Toeplitz operator in terms of the matrix previously defined.

Theorem 7. A necessary and sufficient condition that an operator U on $L^2(\beta)$ is a k th order slant weighted Toeplitz operator is that its matrix with respect to the orthonormal basis $\{e_k(z) = z^k/\beta_k\}_{k \in \mathbb{Z}}$ is a k th order slant weighted Toeplitz matrix.

Proof. Suppose that U is a k th order slant weighted Toeplitz operator. Then, the corresponding matrix $\langle \lambda_{ij} \rangle$ is given by

$$\lambda_{ij} = \langle Ue_j, e_i \rangle = a_{ki-j} \frac{\beta_i}{\beta_j}. \quad (30)$$

Further,

$$\begin{aligned} \lambda_{i+1,j+k} &= \langle Ue_{j+k}, e_{i+1} \rangle \\ &= a_{ki+k-j-k} \frac{\beta_{i+1}}{\beta_{j+k}} \\ &= a_{ki-j} \frac{\beta_{i+1}}{\beta_{j+k}} \\ &= \frac{\beta_{i+1}}{\beta_{j+k}} \frac{\beta_j}{\beta_i} \lambda_{i,j} \\ &= \frac{w_i}{w_j w_{j+1} \cdots w_{j+k-1}} \lambda_{i,j}, \end{aligned} \quad (31)$$

where $w_n = \beta_{n+1}/\beta_n$ for all $n \in \mathbb{Z}$. Thus, the matrix of U is a k th order slant weighted Toeplitz matrix. Conversely, we assume that U is an operator on $L^2(\beta)$ whose matrix is a k th order slant weighted Toeplitz matrix. This means that for all $i, j \in \mathbb{Z}$, we have

$$\langle Ue_j, e_i \rangle = \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_i} \langle Ue_{j+k}, e_{i+1} \rangle. \quad (32)$$

Now,

$$\begin{aligned} \langle M_z Ue_j, e_i \rangle &= \langle Ue_j, M_z^* e_i \rangle \\ &= \langle Ue_j, w_{i-1} e_{i-1} \rangle \\ &= w_{i-1} \langle Ue_j, e_{i-1} \rangle \\ &= w_{i-1} \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_{i-1}} \\ &\quad \times \langle Ue_{j+k}, e_i \rangle \quad (\text{using (32)}) \\ &= \langle U M_{z^k} e_j, e_i \rangle, \quad \forall i, j \in \mathbb{Z}. \end{aligned} \quad (33)$$

Hence $M_z U = U M_{z^k}$. Therefore, we conclude that U is a k th order slant weighted Toeplitz operator. $\square$

Next, consider the operator $S : L^2(\beta) \rightarrow L^2(\beta)$ given by $Se_j = (1/w_j)e_{j+1}$. Then, $S^* e_j = (1/w_{j-1})e_{j-1}$.

Now, S is bounded as $\langle w_n \rangle$ is always positive and bounded.

Lemma 8. Consider the following: $S^* = M_z^{-1}$.

Proof. The proof is as follows:

$$\begin{aligned} S^* M_z e_j &= S^* w_j e_{j+1} = \frac{w_j}{w_j} e_j = e_j, \\ M_z S^* e_j &= M_z \frac{1}{w_{j-1}} e_{j-1} = \frac{w_{j-1}}{w_{j-1}} e_j = e_j. \end{aligned} \quad (34)$$

$\square$

Theorem 9. A bounded operator U on $L^2(\beta)$ is a k th order slant weighted Toeplitz operator if and only if $U = M_z^{-1} U M_{z^k}$, where M_z and M_{z^k} are the multiplication operators induced by z and z^k , respectively.

Proof. Let U be a k th order slant weighted Toeplitz operator on $L^2(\beta)$. Then, from (32) we get that

$$\begin{aligned} \langle Ue_j, e_i \rangle &= \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_i} \langle Ue_{j+k}, e_{i+1} \rangle \\ &= \langle U M_{z^k} e_j, S e_i \rangle \\ &= \langle S^* U M_{z^k} e_j, e_i \rangle \\ &= \langle M_z^{-1} U M_{z^k} e_j, e_i \rangle, \quad i, j = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (35)$$

Thus, $U = M_z^{-1} U M_{z^k}$.

For the converse part, we assume that U is a bounded operator on $L^2(\beta)$ satisfying $U = M_z^{-1} U M_{z^k}$ for some fixed integer $k \geq 2$. Then, for all $i, j \in \mathbb{Z}$,

$$\begin{aligned} \langle Ue_j, e_i \rangle &= \langle M_z^{-1} U M_{z^k} e_j, e_i \rangle \\ &= \langle S^* U M_{z^k} e_j, e_i \rangle \\ &= \langle U M_{z^k} e_j, S e_i \rangle \frac{w_j w_{j+1} \cdots w_{j+k-1}}{w_i} \langle Ue_{j+k}, e_{i+1} \rangle. \end{aligned} \quad (36)$$

The previous equation shows that the matrix corresponding to U is a k th order slant weighted Toeplitz matrix. Hence, by Theorem 7, U is a k th order slant weighted Toeplitz operator. $\square$

Corollary 10. For a fixed integer $k \geq 2$, the set of all k th order slant weighted Toeplitz operators is weakly closed and hence strongly closed.

Proof. We assume that for each positive integer n , U_n is a k th order slant weighted Toeplitz operator and $U_n \rightarrow U$ weakly. Then, for $f, g \in L^2(\beta)$, we get that $\langle U_n f, g \rangle \rightarrow \langle U f, g \rangle$.

From the previous theorem, this implies that

$$\begin{aligned}
 \langle M_z^{-1} U_n M_{z^k} f, g \rangle &= \langle S^* U_n M_{z^k} f, g \rangle \\
 &= \langle U_n M_{z^k} f, Sg \rangle \\
 &= \langle U_n z^k \cdot f, Sg \rangle \longrightarrow \langle U z^k \cdot f, Sg \rangle \\
 &= \langle M_z^{-1} U M_{z^k} f, g \rangle.
 \end{aligned} \quad (37)$$

But, for each n , $U_n = M_z^{-1} U_n M_{z^k}$.

Hence, $U = M_z^{-1} U M_{z^k}$. Therefore, U is a k th order slant weighted Toeplitz operator. $\square$

4. Compression to $H^2(\beta)$

Definition 11. The compression of a k th order slant weighted Toeplitz operator on $H^2(\beta)$ is denoted by V_ϕ and defined as $V_\phi e_j(z) = (1/\beta_j) \sum_{n=0}^{\infty} a_{nk-j} \beta_n e_n(z)$. $j = 0, 1, 2, \dots$

Alternatively,

$$\begin{aligned}
 V_\phi &= P U_\phi|_{H^2(\beta)} \\
 &= P W_k M_\phi|_{H^2(\beta)} \\
 &= W_k P M_\phi|_{H^2(\beta)} \quad (\text{since } P \text{ reduces } W_k \text{ [8]}) \\
 &= W_k T_\phi,
 \end{aligned} \quad (38)$$

where $T_\phi : H^2(\beta) \rightarrow H^2(\beta)$ is the Toeplitz operator on $H^2(\beta)$ induced by $\phi \in L^\infty(\beta)$. The matrix of V_ϕ is unilaterally infinite and has the form

$$\begin{bmatrix}
 a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & \cdots \\
 a_k \frac{\beta_1}{\beta_0} & a_{k-1} \frac{\beta_1}{\beta_1} & a_{k-2} \frac{\beta_1}{\beta_2} & \cdots \\
 a_{2k} \frac{\beta_2}{\beta_0} & a_{2k-1} \frac{\beta_2}{\beta_1} & a_{2k-2} \frac{\beta_2}{\beta_2} & \cdots \\
 \vdots & \vdots & \vdots & \ddots
 \end{bmatrix} \quad (39)$$

for a given $\phi = \sum_{n=-\infty}^{\infty} a_n z^n$.

Further, we observe the following.

(i) The mapping $\phi \rightarrow V_\phi$ is linear and one to one.

(ii) $T_\phi W_k = P M_\phi W_k = P W_k M_\phi(z^k) = P U_\phi(z^k) = V_\phi(z^k)$.

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