Research Article
On the Stability of One-Dimensional Wave Equation

Soon-Mo Jung

Mathematics Section, College of Science and Technology, Hongik University, Sejong 339-701, Republic of Korea

Correspondence should be addressed to Soon-Mo Jung; smjung@hongik.ac.kr

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The terminologies, the generalized Hyers-Ulam stability, and the Hyers-Ulam stability can also be applied to the case of other functional equations, differential equations, and various integral equations.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \), for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \), for all \( x \in G_1 \)?

The case of approximately additive functions was solved by Hyers [2] under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. Indeed, he proved that each solution of the inequality \( \|f(x + y) - f(x) - f(y)\| \leq \varepsilon \), for all \( x \) and \( y \), can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, \( f(x + y) = f(x) + f(y) \), is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)
\]

and proved Hyers’ theorem. That is, Rassias proved the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation. Since then, the stability of several functional equations has been extensively investigated [4–9].

We prove the generalized Hyers-Ulam stability of the one-dimensional wave equation, \( u_{tt} = c^2 u_{xx} \), in a class of twice continuously differentiable functions.

Given a real number \( c > 0 \), the partial differential equation

\[
u_{tt}(x, t) - c^2 u_{xx}(x, t) = 0
\]

is called the (one-dimensional) wave equation, where \( u_{tt}(x, t) \) and \( u_{xx}(x, t) \) denote the second time derivative and the second space derivative of \( u(x, t) \), respectively.

Let \( \varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty) \) be a function. If, for each twice continuously differentiable function \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) satisfying

\[
\|u_{tt}(x, t) - c^2 u_{xx}(x, t)\| \leq \varphi(x, t) \quad (x, t \in \mathbb{R}) ,
\]

there exist a solution \( u_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) of the (one-dimensional) wave equation (2) and a function \( \Phi : \mathbb{R} \times \mathbb{R} \to [0, \infty) \) such that

\[
\|u(x, t) - u_0(x, t)\| \leq \Phi(x, t) \quad (x, t \in \mathbb{R}) ,
\]

where \( \Phi(x, t) \) is independent of \( u(x, t) \) and \( u_0(x, t) \), then we say that the wave equation (2) has the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability).

In this paper, using an idea from [10], we prove the generalized Hyers-Ulam stability of the (one-dimensional) wave equation (2).
2. Generalized Hyers-Ulam Stability

In the following theorem, using the d'Alembert method (method of characteristic coordinates), we prove the generalized Hyers-Ulam stability of the (one-dimensional) wave equation (2).

Theorem 1. Let a function \( \varphi : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty) \) be given such that the double integral

\[
\int_0^b \int_0^a \varphi \left( \frac{\mu + \gamma}{2}, \frac{\mu - \gamma}{2c} \right) d\mu d\nu
\]

exists for all \( a, b \in \mathbb{R} \). If a twice continuously differentiable function \( u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) satisfies the inequality

\[
\left| u_{tt} (x, t) - c^2 u_{xx} (x, t) \right| \leq \varphi (x, t)
\]

for all \( x, t \in \mathbb{R} \), then there exists a solution \( u_0 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) of the wave equation (2) which satisfies

\[
\left| u (x, t) - u_0 (x, t) \right| \leq \frac{1}{4c^2} \int_0^{x-ct} \int_0^{x+ct} \varphi \left( \frac{\mu + \gamma}{2}, \frac{\mu - \gamma}{2c} \right) d\mu d\nu
\]

for all \( x, t \in \mathbb{R} \).

Proof. Let us define a function \( \nu : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) by

\[
\nu (w, z) := u \left( \frac{w + z}{2}, \frac{w - z}{2c} \right).
\]

If we set \( w = x + ct \) and \( z = x - ct \), then we have \( u(x, t) = \nu(w, z) \) and

\[
\begin{align*}
& u_t (x, t) = \nu_w (w, z) \frac{\partial w}{\partial t} + \nu_z (w, z) \frac{\partial z}{\partial t} \\
& \quad = cv_{ww} (w, z) + cv_{zz} (w, z),
& u_{tt} (x, t) = c^2 v_{www} (w, z) + c^2 v_{zzz} (w, z).
\end{align*}
\]

Hence, it follows from (13) and the last equalities that

\[
\left| u \left( \frac{w + z}{2}, \frac{w - z}{2c} \right) - u \left( \frac{w}{2}, \frac{w}{2c} \right) \right| \leq \frac{1}{4c^2} \int_0^{x-ct} \int_0^{x+ct} \varphi \left( \frac{\mu + \gamma}{2}, \frac{\mu - \gamma}{2c} \right) d\mu d\nu,
\]

for all \( w, z \in \mathbb{R} \). If we set \( w = x + ct \) and \( z = x - ct \) in the last inequality, then we obtain

\[
\left| u (x, t) - u_0 (x, t) \right| \leq \frac{1}{4c^2} \int_0^{x-ct} \int_0^{x+ct} \varphi \left( \frac{\mu + \gamma}{2}, \frac{\mu - \gamma}{2c} \right) d\mu d\nu,
\]

for all \( x, t \in \mathbb{R} \), where we set

\[
\begin{align*}
u_0 (x, t) := u & \left( \frac{x + ct}{2}, \frac{x - ct}{2c} \right) + u \left( \frac{x - ct}{2}, \frac{x + ct}{2c} \right) - u (0, 0).
\end{align*}
\]
By some tedious calculations, we get

$$
\frac{\partial}{\partial t} u_0(x, t) = \frac{c}{2} u_x \left( \frac{x^2}{2} + \frac{ct}{2} + \frac{t^2}{2} \right) + \frac{1}{2} u_t \left( \frac{x^2}{2} + \frac{ct}{2} + \frac{t^2}{2} \right),
$$

$$
\frac{\partial^2}{\partial t^2} u_0(x, t) = \frac{c^2}{4} u_{xx} \left( \frac{x^2}{2} + \frac{ct}{2} + \frac{t^2}{2} \right) + \frac{c}{2} u_{xt} \left( \frac{x^2}{2} + \frac{ct}{2} + \frac{t^2}{2} \right) + \frac{1}{2} u_{tt} \left( \frac{x^2}{2} + \frac{ct}{2} + \frac{t^2}{2} \right),
$$

for all \( x, t \in \mathbb{R} \). Hence, we know that

$$
\frac{\partial^2}{\partial t^2} u_0(x, t) - c^2 \frac{\partial^2}{\partial x^2} u_0(x, t) = 0,
$$

(18)

for any \( x, t \in \mathbb{R} \); that is, \( u_0(x, t) \) is a solution of the wave equation (2).

**Corollary 2.** Given a constant \( \alpha > 0 \), let a function \( \varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty) \) be given as

$$
\varphi(x, t) = a e^{-x^2 - ct^2}.
$$

(20)

If a twice continuously differentiable function \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) satisfies inequality (6), for all \( x, t \in \mathbb{R} \), then there exists a solution \( u_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) of the wave equation (2) which satisfies

$$
|u(x, t) - u_0(x, t)| \leq \frac{\alpha \pi}{8c^2} \operatorname{erf} \left( \frac{x - ct}{\sqrt{2}} \right) \operatorname{erf} \left( \frac{x + ct}{\sqrt{2}} \right),
$$

(21)

for all \( x, t \in \mathbb{R} \).

**Proof.** Since

$$
\left| \int_{0}^{b} \int_{a}^{c} \varphi \left( \frac{\mu + \nu}{2}, \frac{\mu - \nu}{2c} \right) d\mu d\nu \right| = \left| \int_{0}^{b} \int_{a}^{c} e^{-x^2 - ct^2} d\mu d\nu \right| = \alpha \left| \int_{0}^{b} e^{-x^2} d\nu \right| \left| \int_{a}^{c} e^{-ct^2} d\mu \right|
$$

$$
\leq 2 \alpha \pi \left( \frac{2}{\sqrt{\pi}} \int_{0}^{b} e^{-x^2} d\nu \right) \left( \frac{2}{\sqrt{\pi}} \int_{a}^{c} e^{-ct^2} d\mu \right)
$$

$$
\leq \frac{\alpha \pi}{2} \left( \operatorname{erf} \left( \frac{b}{\sqrt{2}} \right) \operatorname{erf} \left( \frac{a}{\sqrt{2}} \right) \right) < \infty,
$$

for all \( a, b \in \mathbb{R} \), in view of Theorem 1, we conclude that the statement of this corollary is true. \( \square \)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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