Hindawi Publishing Corporation
The Scientific World Journal
Volume 2013, Article ID 978754, 3 pages
http://dx.doi.org/10.1155/2013/978754

Research Article

On the Stability of One-Dimensional Wave Equation

Soon-Mo Jung

Mathematics Section, College of Science and Technology, Hongik University, Sejong 339-701, Republic of Korea

Correspondence should be addressed to Soon-Mo Jung; smjung@hongik.ac.kr

Received 5 August 2013; Accepted 16 September 2013

Academic Editors: K. Ammari, I. Canak, and M. M. Cavalcanti

We prove the generalized Hyers-Ulam stability of the one-dimensional wave equation, $u_{tt} = c^2 u_{xx}$, in a class of twice continuously differentiable functions.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

> *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$, for all $x \in G_1$?*

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \left(\|x\|^p + \|y\|^p\right) \tag{1}$$

and proved Hyers' theorem. That is, Rassias proved the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation. Since then, the stability of several functional equations has been extensively investigated [4–9].

The terminologies, the generalized Hyers-Ulam stability, and the Hyers-Ulam stability can also be applied to the case of other functional equations, differential equations, and various integral equations.

Given a real number $c > 0$, the partial differential equation

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \tag{2}$$

is called the (one-dimensional) wave equation, where $u_{tt}(x,t)$ and $u_{xx}(x,t)$ denote the second time derivative and the second space derivative of $u(x,t)$, respectively.

Let $\varphi : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be a function. If, for each twice continuously differentiable function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfying

$$\left|u_{tt}(x,t) - c^2 u_{xx}(x,t)\right| \le \varphi(x,t) \quad (x,t \in \mathbb{R}), \tag{3}$$

there exist a solution $u_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of the (one-dimensional) wave equation (2) and a function $\Phi : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ such that

$$\left|u(x,t) - u_0(x,t)\right| \le \Phi(x,t) \quad (x,t \in \mathbb{R}), \tag{4}$$

where $\Phi(x,t)$ is independent of $u(x,t)$ and $u_0(x,t)$, then we say that the wave equation (2) has the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability).

In this paper, using an idea from [10], we prove the generalized Hyers-Ulam stability of the (one-dimensional) wave equation (2).

2. Generalized Hyers-Ulam Stability

In the following theorem, using the d'Alembert method (method of characteristic coordinates), we prove the generalized Hyers-Ulam stability of the (one-dimensional) wave equation (2).

Theorem 1. *Let a function $\varphi : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be given such that the double integral*

$$\int_0^b \int_0^a \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu \tag{5}$$

exists for all $a, b \in \mathbb{R}$. If a twice continuously differentiable function $u : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ satisfies the inequality

$$\left|u_{tt}(x,t) - c^2 u_{xx}(x,t)\right| \le \varphi(x,t) \tag{6}$$

for all $x, t \in \mathbb{R}$, then there exists a solution $u_0 : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ of the wave equation (2) which satisfies

$$\begin{aligned} &\left|u(x,t) - u_0(x,t)\right| \\ &\quad \le \frac{1}{4c^2}\left|\int_0^{x-ct}\int_0^{x+ct} \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu\right| \end{aligned} \tag{7}$$

for all $x, t \in \mathbb{R}$.

Proof. Let us define a function $v : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ by

$$v(w,z) := u\left(\frac{w+z}{2}, \frac{w-z}{2c}\right). \tag{8}$$

If we set $w = x + ct$ and $z = x - ct$, then we have $u(x,t) = v(w,z)$ and

$$\begin{aligned} u_t(x,t) &= v_w(w,z)\frac{\partial w}{\partial t} + v_z(w,z)\frac{\partial z}{\partial t} \\ &= c v_w(w,z) - c v_z(w,z), \\ u_{tt}(x,t) &= c v_{ww}(w,z)\frac{\partial w}{\partial t} + c v_{wz}(w,z)\frac{\partial z}{\partial t} \\ &\quad - c v_{zw}(w,z)\frac{\partial w}{\partial t} - c v_{zz}(w,z)\frac{\partial z}{\partial t} \\ &= c^2 v_{ww}(w,z) - 2c^2 v_{wz}(w,z) + c^2 v_{zz}(w,z), \\ u_x(x,t) &= v_w(w,z)\frac{\partial w}{\partial x} + v_z(w,z)\frac{\partial z}{\partial x} \\ &= v_w(w,z) + v_z(w,z), \\ u_{xx}(x,t) &= v_{ww}(w,z)\frac{\partial w}{\partial x} + v_{wz}(w,z)\frac{\partial z}{\partial x} \\ &\quad + v_{zw}(w,z)\frac{\partial w}{\partial x} + v_{zz}(w,z)\frac{\partial z}{\partial x} \\ &= v_{ww}(w,z) + 2v_{wz}(w,z) + v_{zz}(w,z), \end{aligned} \tag{9}$$

for all $x, t \in \mathbb{R}$. Hence, we have

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = -4c^2 v_{wz}(w,z), \tag{10}$$

for any $x, t \in \mathbb{R}$. Thus, it follows from inequality (6) that

$$\left|v_{wz}(w,z)\right| \le \frac{1}{4c^2}\varphi\left(\frac{w+z}{2}, \frac{w-z}{2c}\right), \tag{11}$$

for any $w, z \in \mathbb{R}$.

Therefore, we get

$$\begin{aligned} &-\frac{1}{4c^2}\int_0^z\int_0^w \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu \\ &\quad \le \int_0^z\int_0^w v_{wz}(\mu,\nu)\, d\mu d\nu \\ &\quad \le \frac{1}{4c^2}\int_0^z\int_0^w \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu \end{aligned} \tag{12}$$

or equivalently

$$\begin{aligned} &\left|v(w,z) - v(w,0) - v(0,z) + v(0,0)\right| \\ &\quad \le \frac{1}{4c^2}\left|\int_0^z\int_0^w \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu\right|, \end{aligned} \tag{13}$$

for all $w, z \in \mathbb{R}$.

On account of (8), we get

$$\begin{aligned} v(w,z) &= u\left(\frac{w+z}{2}, \frac{w-z}{2c}\right), \qquad v(w,0) = u\left(\frac{w}{2}, \frac{w}{2c}\right), \\ v(0,z) &= u\left(\frac{z}{2}, -\frac{z}{2c}\right), \qquad v(0,0) = u(0,0). \end{aligned} \tag{14}$$

Hence, it follows from (13) and the last equalities that

$$\begin{aligned} &\left|u\left(\frac{w+z}{2}, \frac{w-z}{2c}\right) - u\left(\frac{w}{2}, \frac{w}{2c}\right) - u\left(\frac{z}{2}, -\frac{z}{2c}\right) + u(0,0)\right| \\ &\quad \le \frac{1}{4c^2}\left|\int_0^z\int_0^w \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu\right|, \end{aligned} \tag{15}$$

for all $w, z \in \mathbb{R}$.

If we set $w = x + ct$ and $z = x - ct$ in the last inequality, then we obtain

$$\begin{aligned} &\left|u(x,t) - u_0(x,t)\right| \\ &\quad \le \frac{1}{4c^2}\left|\int_0^{x-ct}\int_0^{x+ct} \varphi\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2c}\right) d\mu d\nu\right|, \end{aligned} \tag{16}$$

for all $x, t \in \mathbb{R}$, where we set

$$\begin{aligned} u_0(x,t) &:= u\left(\frac{x}{2} + \frac{c}{2}t, \frac{x}{2c} + \frac{t}{2}\right) \\ &\quad + u\left(\frac{x}{2} - \frac{c}{2}t, -\frac{x}{2c} + \frac{t}{2}\right) - u(0,0). \end{aligned} \tag{17}$$

By some tedious calculations, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}u_0(x,t)\\
&\quad=\frac{c}{2}u_x\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{2}u_t\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)\\
&\qquad-\frac{c}{2}u_x\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{2}u_t\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right),\\
&\frac{\partial^2}{\partial t^2}u_0(x,t)\\
&\quad=\frac{c^2}{4}u_{xx}\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)+\frac{c}{2}u_{xt}\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)\\
&\qquad+\frac{1}{4}u_{tt}\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)+\frac{c^2}{4}u_{xx}\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right)\\
&\qquad-\frac{c}{2}u_{xt}\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{4}u_{tt}\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right),\\
&\frac{\partial}{\partial x}u_0(x,t)\\
&\quad=\frac{1}{2}u_x\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{2c}u_t\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)\\
&\qquad+\frac{1}{2}u_x\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right)-\frac{1}{2c}u_t\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right),\\
&\frac{\partial^2}{\partial x^2}u_0(x,t)\\
&\quad=\frac{1}{4}u_{xx}\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{2c}u_{xt}\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)\\
&\qquad+\frac{1}{4c^2}u_{tt}\left(\frac{x}{2}+\frac{ct}{2},\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{4}u_{xx}\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right)\\
&\qquad-\frac{1}{2c}u_{xt}\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right)+\frac{1}{4c^2}u_{tt}\left(\frac{x}{2}-\frac{ct}{2},-\frac{x}{2c}+\frac{t}{2}\right),
\end{aligned}\tag{18}$$

for all $x,t\in\mathbb{R}$. Hence, we know that

$$\frac{\partial^2}{\partial t^2}u_0(x,t)-c^2\frac{\partial^2}{\partial x^2}u_0(x,t)=0,\tag{19}$$

for any $x,t\in\mathbb{R}$; that is, $u_0(x,t)$ is a solution of the wave equation (2). □

Corollary 2. *Given a constant $\alpha>0$, let a function $\varphi:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be given as*

$$\varphi(x,t)=\alpha e^{-x^2-c^2t^2}.\tag{20}$$

If a twice continuously differentiable function $u:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ satisfies inequality (6), for all $x,t\in\mathbb{R}$, then there exists a solution $u_0:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ of the wave equation (2) which satisfies

$$\left|u(x,t)-u_0(x,t)\right|\le\frac{\alpha\pi}{8c^2}\left|\operatorname{erf}\left(\frac{x-ct}{\sqrt{2}}\right)\operatorname{erf}\left(\frac{x+ct}{\sqrt{2}}\right)\right|,\tag{21}$$

for all $x,t\in\mathbb{R}$.

Proof. Since

$$\begin{aligned}
&\left|\int_0^b\int_0^a\varphi\left(\frac{\mu+\nu}{2},\frac{\mu-\nu}{2c}\right)d\mu d\nu\right|\\
&\quad=\left|\int_0^b\int_0^a\alpha e^{-\mu^2/2-\nu^2/2}d\mu d\nu\right|\\
&\quad=\alpha\left|\int_0^b e^{-\nu^2/2}d\nu\right|\left|\int_0^a e^{-\mu^2/2}d\mu\right|\\
&\quad=2\alpha\frac{\pi}{4}\left|\left(\frac{2}{\sqrt{\pi}}\int_0^{b/\sqrt{2}}e^{-\nu^2}d\nu\right)\left(\frac{2}{\sqrt{\pi}}\int_0^{a/\sqrt{2}}e^{-\mu^2}d\mu\right)\right|\\
&\quad=\frac{\alpha\pi}{2}\left|\operatorname{erf}\left(\frac{b}{\sqrt{2}}\right)\operatorname{erf}\left(\frac{a}{\sqrt{2}}\right)\right|<\infty,
\end{aligned}\tag{22}$$

for all $a,b\in\mathbb{R}$, in view of Theorem 1, we conclude that the statement of this corollary is true. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2013R1A1A2005557).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of USA*, vol. 27, pp. 222–224, 1941.

[3] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, pp. 297–300, 1978.

[4] G. L. Forti, "Hyers-Ulam stability of functional equations in several variables," *Aequationes Mathematicae*, vol. 50, no. 1-2, pp. 143–190, 1995.

[5] P. Găvrută, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[6] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations of Several Variables*, Birkhauser, Boston, Mass, USA, 1998.

[7] D. H. Hyers and T. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[8] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, vol. 48 of *Springer Optimization and Its Applications*, Springer, New York, NY, USA, 2011.

[9] T. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[10] B. Hegyi and S.-M. Jung, "On the stability of Laplace's equation," *Applied Mathematics Letters*, vol. 26, pp. 549–552, 2013.