Research Article

Approximation of Bivariate Functions via Smooth Extensions

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For a smooth bivariate function defined on a general domain with arbitrary shape, it is difficult to do Fourier approximation or wavelet approximation. In order to solve these problems, in this paper, we give an extension of the bivariate function on a general domain with arbitrary shape to a smooth, periodic function in the whole space or to a smooth, compactly supported function in the whole space. These smooth extensions have simple and clear representations which are determined by this bivariate function and some polynomials. After that, we expand the smooth, periodic function into a Fourier series or a periodic wavelet series or we expand the smooth, compactly supported function into a wavelet series. Since our extensions are smooth, the obtained Fourier coefficients or wavelet coefficients decay very fast. Since our extension tools are polynomials, the moment theorem shows that a lot of wavelet coefficients vanish. From this, with the help of well-known approximation theorems, using our extension methods, the Fourier approximation and the wavelet approximation of the bivariate function on the general domain with small error are obtained.

1. Introduction

In the recent several decades, various approximation tools have been widely developed [1–14]. For example, a smooth periodic function can be approximated by trigonometric polynomials; a square-integrable smooth function can be expanded into a wavelet series and be approximated by partial sum of the wavelet series; and a smooth function on a cube can be approximated well by polynomials. However, for a smooth function on a general domain with arbitrary shape, even if it is infinitely many times differentiable, it is difficult to do Fourier approximation or wavelet approximation. In this paper, we will extend a function on general domain with arbitrary shape to a smooth, periodic function in the whole space or to a smooth, compactly supported function in the whole space. After that, it will be easy to do Fourier approximation or wavelet approximation. For the higher-dimensional case, the method of smooth extensions is similar to that in the two-dimensional case, but the representations of smooth extensions will be too complicated. Therefore, in

this paper, we mainly consider the smooth extension of a bivariate function on a planar domain. By the way, for the one-dimensional case, since the bounded domain is reduced to a closed interval, the smooth extension can be regarded as a corollary of the two-dimensional case.

This paper is organized as follows. In Section 2, we state the main theorems on the smooth extension of the function on the general domain and their applications. In Sections 3 and 4, we give a general method of smooth extensions and complete the proofs of the main theorems. In Section 5, we use our extension method to discuss two important special cases of smooth extensions.

Throughout this paper, we denote $T = [0, 1]^2$ and the interior of T by T° and always assume that Ω is a simply connected domain. We say that $f \in C^q(\Omega)$ if the derivatives $(\partial^{i+j} f / \partial x^i \partial y^j)$ are continuous on Ω for $0 \leq i + j \leq q$. We say that $f \in C^\infty(\Omega)$ if all derivatives $(\partial^{i+j} f / \partial x^i \partial y^j)$ are continuous on Ω for $i, j \in \mathbb{Z}_+$. We say that a function $h(x, y)$ is a γ -periodic function if $h(x + \gamma k, y + \gamma l) = h(x, y)$ $((x, y) \in$

$T; k, l \in \mathbb{Z}$), where γ is an integer. We appoint that $0! = 1$ and the notation $[\alpha]$ is the integral part of the real number α .

2. Main Theorems and Applications

In this section, we state the main results of smooth extensions and their applications in Fourier analysis and wavelet analysis.

2.1. Main Theorems. Our main theorems are stated as follows.

Theorem 1. Let $f \in C^\infty(\Omega)$, where $\Omega \subset T^\circ$ and the boundary $\partial\Omega$ is a piecewise infinitely many time smooth curve. Then for any $r \in \mathbb{Z}_+$ there is a function $F \in C^r(T)$ such that

- (i) $F(x, y) = f(x, y) \ ((x, y) \in \Omega)$;
- (ii) $(\partial^{i+j} F / \partial x^i \partial y^j)(x, y) = 0$ on the boundary ∂T for $0 \leq i + j \leq r$;
- (iii) on the complement $T \setminus \Omega$, $F(x, y)$ can be expressed locally in the forms

$$\sum_{j=0}^L \xi_j(x) y^j, \quad \text{or} \quad \sum_{j=0}^L \eta_j(y) x^j, \quad \text{or} \quad \sum_{i,j=0}^L c_{ij} x^i y^j, \quad (1)$$

where L is a positive integer and each coefficient c_{ij} is constant.

Theorem 2. Let $f \in C^\infty(\Omega)$, where Ω is stated as in Theorem 1. Then, for any $r \in \mathbb{Z}_+$, there exists a 1-periodic function $F_p \in C^r(\mathbb{R}^2)$ such that $F_p(x, y) = f(x, y) \ ((x, y) \in \Omega)$.

Theorem 3. Let $f \in C^\infty(\Omega)$, where Ω is stated as in Theorem 1. Then, for any $r \in \mathbb{Z}_+$, there exists a function $F^c \in C^r(\mathbb{R}^2)$ with compact support T such that $F^c(x, y) = f(x, y) \ ((x, y) \in \Omega)$.

In Sections 3 and 4, we give constructive proofs of Theorems 1–3. In these three theorems, we assume that $f \in C^\infty(\Omega)$. If $f \in C^q(\Omega)$ (q is a nonnegative integer), by using the similar method of arguments of Theorems 1–3, we also can obtain the corresponding results.

2.2. Applications. Here we show some applications of these theorems.

2.2.1. Approximation by Polynomials. Let F be the smooth extension of f from Ω to T which is stated as in Theorem 1. Then $F \in C^r(T)$ and $F = f$ on Ω . By Δ_N , denote the set of all bivariate polynomials in the form $\sum_{n_1, n_2=-N}^N c_{n_1, n_2} x^{n_1} y^{n_2}$. Then

$$\inf_{P \in \Delta_N} \|f - P\|_{L_p(\Omega)} \leq \inf_{P \in \Delta_N} \|F - P\|_{L_p(T)}, \quad (2)$$

where $\|\cdot\|_{L_p(D)}$ is the norm of the space $L_p(D)$. The right-hand side of formula (2) is the best approximation of the extension F in Δ_N . By (2), we know that the approximation problem of f by polynomials on a domain Ω is reduced to the well-known approximation problem of its smooth extension F by polynomials on the square T [4, 10].

2.2.2. Fourier Analysis

(i) **Approximation by Trigonometric Polynomials.** Let F_p be the smooth periodic extension of f as in Theorem 2. Then $F_p \in C^r(\mathbb{R}^2)$ and $F_p = f$ on Ω . By the well-known results [5, 10], we know that the smooth periodic function F_p can be approximated by bivariate trigonometric polynomials very well. Its approximation error can be estimated by the modulus of continuity of its r time derivatives.

By Δ_N^* , denote the set of all bivariate trigonometric polynomials in the form

$$\sum_{n_1, n_2=-N}^N c_{n_1, n_2}^* e^{2\pi i(n_1 x + n_2 y)}. \quad (3)$$

By Theorem 2, we have

$$\inf_{P^* \in \Delta_N^*} \|f - P^*\|_{L_{p'}(\Omega)} \leq \min_{P^* \in \Delta_N^*} \|F_p - P^*\|_{L_{p'}(T)}. \quad (4)$$

From this and Theorem 2, we see that the approximation problem of f on Ω by trigonometric polynomials is reduced to a well-known approximation problem of smooth periodic functions [5, 7, 10].

(ii) **Fourier Series.** We expand F_p into a Fourier series [9]

$$F_p(x, y) = \sum_{(n_1, n_2) \in \mathbb{Z}^2} \tau_{n_1, n_2} e^{2\pi i(n_1 x + n_2 y)}, \quad (5)$$

where $\tau_{n_1, n_2} = \int_T F_p(x, y) e^{-2\pi i(n_1 x + n_2 y)} dx dy$. By Theorem 2, we obtain that, for $(x, y) \in \Omega$,

$$f(x, y) = \sum_{(n_1, n_2) \in \mathbb{Z}^2} \tau_{n_1, n_2} e^{2\pi i(n_1 x + n_2 y)}. \quad (6)$$

Denote the partial sum

$$s_{n_1, n_2}(x, y) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \tau_{k_1, k_2} e^{2\pi i(k_1 x + k_2 y)}. \quad (7)$$

Then we have

$$\begin{aligned} & \|f(x, y) - s_{n_1, n_2}(x, y)\|_{L_{p'}(\Omega)} \\ & \leq \|F_p(x, y) - s_{n_1, n_2}(x, y)\|_{L_{p'}(T)}. \end{aligned} \quad (8)$$

Since the smooth periodic function F_p can be approximated well by the partial sum of its Fourier series [5, 7, 10], from this inequality, we see that we have constructed a trigonometric polynomial $s_{n_1, n_2}(x, y)$ which can approximate to f on Ω very well.

(iii) **Odd (Even) Periodic Extension.** Let F be the smooth extension of f from Ω to T which is stated in Theorem 1. Define F° on $[-1, 1]^2$ by

$$F^\circ(x, y) = \begin{cases} F(x, y), & (x, y) \in [0, 1]^2, \\ -F(-x, y), & (x, y) \in [-1, 0] \times [0, 1], \\ F(-x, -y), & (x, y) \in [-1, 0]^2, \\ -F(x, -y), & (x, y) \in [0, 1] \times [-1, 0]. \end{cases} \quad (9)$$

Then F^o is an odd function. By Theorem 1, we have $F^o \in C^r([-1, 1]^2)$ and $(\partial^{i+j} F^o / \partial x^i \partial y^j)(x, y) = 0$ on $\partial([-1, 1]^2)$ for $0 \leq i + j \leq r$. Again, doing a 2-periodic extension, we obtain a 2-periodic odd function F_p^o and $F_p^o \in C^r(\mathbb{R}^2)$. By the well-known results [5, 7, 10], F_p^o can be approximated by sine polynomials very well. Moreover, F_p^o can be expanded into the Fourier sine series; that is,

$$F_p^o(x, y) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \alpha_{n_1, n_2} \sin(\pi n_1 x) \sin(\pi n_2 y), \quad (10)$$

where the coefficients $\alpha_{n_1, n_2} = 4 \int_T F_p^o(x, y) \sin(\pi n_1 x) \sin(\pi n_2 y) dx dy$ [9]. Considering the approximation of F_p^o by the partial sum, the Fejer sum, and the Valler-Poussin sum [7, 14] of the Fourier sine series of F_p^o , we will obtain the approximation of the original function f on Ω by sine polynomials.

Define F^e on $[-1, 1]^2$ as follows:

$$F^e(x, y) = \begin{cases} F(x, y), & (x, y) \in [0, 1]^2, \\ F(-x, y), & (x, y) \in [-1, 0] \times [0, 1], \\ F(-x, -y), & (x, y) \in [-1, 0]^2, \\ F(x, -y), & (x, y) \in [0, 1] \times [-1, 0]. \end{cases} \quad (11)$$

Then F^e is an even function on $[-1, 1]^2$. By Theorem 1, $F^e \in C^r([-1, 1]^2)$ and $(\partial^{i+j} F^e / \partial x^i \partial y^j)(x, y) = 0$ on $\partial([-1, 1]^2)$ for $0 \leq i + j \leq r$. Again, doing a 2-periodic extension, we obtain a 2-periodic even function F_p^e and $F_p^e \in C^r(\mathbb{R}^2)$. By the well-known result [5, 10], F_p^e can be approximated by cosine polynomials very well. Moreover, F_p^e can be expanded into the Fourier cosine series. Considering the partial sum, the Fejer sum, and the Valler-Poussin sum [5, 7, 14] of the Fourier cosine series of F_p^e , we will obtain the approximation of the original function f on Ω by cosine polynomials.

2.2.3. Wavelet Analysis

(i) *Periodic Wavelet Series.* Let $F_p \in C^r(\mathbb{R}^2)$ be stated in Theorem 2. Let $\{\psi_\mu\}_1^3$ be a bivariate smooth wavelet [2]. Then, under a mild condition, the families

$$\Psi^{\text{per}} : = \{1\} \cup \left\{ \psi_{\mu, m, n}^{\text{per}}, \mu = 1, 2, 3; m \in \mathbb{Z}_+; n = (n_1, n_2), n_1, n_2 = 0, \dots, 2^m - 1 \right\}, \quad (12)$$

$$\text{where } \psi_{\mu, m, n}^{\text{per}} = \sum_{l \in \mathbb{Z}^2} \psi_{\mu, m, n}(\cdot + l),$$

$$\psi_{\mu, m, n} = 2^m \psi_\mu(2^m \cdot - n)$$

are a periodic wavelet basis. We expand F_p into a periodic wavelet series [2]

$$F_p = d_{0,0} + \sum_{\mu=1}^3 \sum_{m=0}^{\infty} \sum_{n_1, n_2=0}^{2^m-1} d_{\mu, m, n} \psi_{\mu, m, n}^{\text{per}}. \quad (13)$$

From this, we can realize the wavelet approximation of f on Ω , for example, if $r = 2$, its partial sum

$$s_{2^M}(F_p) = d_{0,0} + \sum_{\mu=1}^3 \sum_{m=0}^{M-1} \sum_{n_1, n_2=0}^{2^m-1} d_{\mu, m, n} \psi_{\mu, m, n}^{\text{per}} \quad (14)$$

satisfies $\|F_p - s_{2^M}(F_p)\|_{L^2(T)} = O(2^{-2M})$. From this and $F_p(x, y) = f(x, y)$ ($(x, y) \in \Omega$), we will obtain an estimate of wavelet approximation for a smooth function f on the domain Ω .

(ii) *Wavelet Approximation.* Let F^c be the smooth function with a compact support as in Theorem 3. Let ψ be a univariate Daubechies wavelet and ϕ be the corresponding scaling function [2]. Denoting

$$\begin{aligned} \psi_1(x, y) &= \phi(x) \psi(y), & \psi_2(x, y) &= \psi(x) \phi(y), \\ \psi_3(x, y) &= \psi(x) \psi(y), \end{aligned} \quad (15)$$

then $\{\psi_\mu(x, y)\}_{\mu=1}^3$ is a smooth tensor product wavelet. We expand F^c into the wavelet series

$$F^c(x, y) = \sum_{\mu=1}^3 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^2} c_{\mu, m, n} \psi_{\mu, m, n}(x, y), \quad (16)$$

where $\psi_{\mu, m, n} = 2^m \psi_\mu(2^m \cdot - n)$ and the wavelet coefficients

$$\begin{aligned} c_{\mu, m, n} &= \int_{\mathbb{R}^2} F^c(x, y) \bar{\psi}_{\mu, m, n}(x, y) dx dy \\ &= \int_T F(x, y) \bar{\psi}_{\mu, m, n}(x, y) dx dy. \end{aligned} \quad (17)$$

Since F^c is a smooth function, the wavelet coefficients $c_{\mu, m, n}$ decay fast.

On the other hand, since $F^c(x, y) = 0$, $(x, y) \in \mathbb{R}^2 \setminus T$, a lot of wavelet coefficients vanish. In fact, when $m_0 \in \mathbb{Z}$ and $n_0 \in \mathbb{Z}^2$ satisfy $\text{supp } \psi_{\mu, m_0, n_0} \subset (\mathbb{R}^2 \setminus T)$, we have $c_{\mu, m_0, n_0} = 0$. Besides, by condition (iii) in Theorem 1, we know that F is univariate or bivariate polynomials on $T \setminus \Omega$. By the moment theorem [2], we know that more wavelet coefficients vanish.

For example, let $m^* \in \mathbb{Z}$ and $n^* = (n_1^*, n_2^*) \in \mathbb{Z}^2$ satisfy $\text{supp } \psi_{m^*, n_2^*} \subset [0, \alpha^*]$, where $\alpha^* = \inf\{g(x), x_1 \leq x \leq x_2\}$. Then we have

$$\begin{aligned} c_{1, m^*, n^*} &= 2^{m^*} \left(\int_0^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^1 \right) \bar{\phi}_{m^*, n_1^*}(x) \\ &\quad \times \left(\int_0^{\alpha^*} \bar{\psi}_{m^*, n_2^*}(y) F(x, y) dy \right) dx \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (18)$$

By Lemma 8, we know that

$$F(x, y) = \sum_{j=0}^L \xi_j(x) y^j, \quad (x, y) \in E_1, \quad (19)$$

where $E_1 = \{(x, y) : x_1 \leq x \leq x_2, 0 \leq y \leq g(x)\}$ and $g(x) \geq \alpha^*(x_1 \leq x \leq x_2)$. So

$$I_2 = \int_{x_1}^{x_2} \bar{\phi}_{m^*, n_1^*}(x) \left(\int_0^{\alpha^*} \bar{\psi}_{m^*, n_2^*}(y) \left(\sum_{j=0}^L \xi_j(x) y^j \right) dy \right) dx. \quad (20)$$

If the Daubechies wavelet ψ chosen by us is L time smooth, then, by using the moment theorem and $\text{supp } \psi_{m^*, n_2^*} \subset [0, \alpha^*]$, we have

$$\int_0^{\alpha^*} \bar{\psi}_{m^*, n_2^*}(y) y^j dy = \int_{\mathbb{R}} \bar{\psi}_{m^*, n_2^*}(y) y^j dy = 0, \quad (21)$$

$$(0 \leq j \leq 2r + 1).$$

So $I_2 = 0$. Similarly, since $F(x, y)$ is bivariate polynomials on rectangles H_1 and H_3 (see Lemma 11), we have $I_1 = I_3 = 0$. Furthermore, by (18), we get $c_{1, m^*, n^*} = 0$.

Therefore, the partial sum of the wavelet series (16) can approximate to F^c very well and few wavelet coefficients can reconstruct F^c . Since $F^c = f$ on Ω , the partial sum of the wavelet series (16) can approximate to the original function f on the domain Ω very well.

3. Proofs of the Main Theorems

We first give a partition of the complement $T \setminus \Omega$.

3.1. Partition of the Complement of the Domain Ω in T . Since $\Omega \subset T^o$ and $\partial\Omega$ is a piecewise infinitely many time smooth curve, without loss of generality, we can divide the complement $T \setminus \Omega$ into some rectangles and some trapezoids with a curved side. For convenience of representation, we assume that we can choose four point $(x_\nu, y_\nu) \in \partial\Omega$ ($\nu = 1, 2, 3, 4$) such that $T \setminus \Omega$ can be divided into the four rectangles

$$\begin{aligned} H_1 &= [0, x_1] \times [0, y_1], & H_2 &= [x_2, 1] \times [0, y_2], \\ H_3 &= [x_3, 1] \times [y_3, 1], & H_4 &= [0, x_4] \times [y_4, 1] \end{aligned} \quad (22)$$

and four trapezoids with a curved side

$$\begin{aligned} E_1 &= \{(x, y) : x_1 \leq x \leq x_2, 0 \leq y \leq g(x)\}, \\ E_2 &= \{(x, y) : h(y) \leq x \leq 1, y_2 \leq y \leq y_3\}, \\ E_3 &= \{(x, y) : x_4 \leq x \leq x_3, g^*(x) \leq y \leq 1\}, \\ E_4 &= \{(x, y) : 0 \leq x \leq h^*(y), y_1 \leq y \leq y_4\}, \end{aligned} \quad (23)$$

where $g \in C^\infty([x_1, x_2])$, $h \in C^\infty([y_2, y_3])$, $g^* \in C^\infty([x_4, x_3])$, and $h^* \in C^\infty([y_1, y_4])$ and

$$\begin{aligned} 0 &< g(x) < 1 & (x_1 \leq x \leq x_2), \\ 0 &< h(y) < 1 & (y_2 \leq y \leq y_3), \\ 0 &< g^*(x) < 1 & (x_4 \leq x \leq x_3), \\ 0 &< h^*(y) < 1 & (y_1 \leq y \leq y_4). \end{aligned} \quad (24)$$

FIGURE 1: Partition of the complement of the domain Ω .

From this, we know that T can be expressed into a disjoint union as follows:

$$T = \Omega \cup \left(\bigcup_1^4 E_\nu \right) \cup \left(\bigcup_1^4 H_\nu \right), \quad (25)$$

where each E_ν is a trapezoid with a curved side and each H_ν is a rectangle (see Figure 1).

In Sections 3.2 and 3.3 we will extend f to each E_ν and continue to extend to each H_ν such that the obtained extension F satisfies the conditions of Theorem 1.

3.2. Smooth Extension to Each Trapezoid E_ν with a Curved Side. By (23), the trapezoid E_1 with a curved side $y = g(x)$ ($x_1 \leq x \leq x_2$) is represented as

$$E_1 = \{(x, y) : x_1 \leq x \leq x_2, 0 \leq y \leq g(x)\}. \quad (26)$$

We define two sequences of functions $\{a_{k,1}(x, y)\}_0^\infty$ and $\{b_{k,1}(x, y)\}_0^\infty$ as follows:

$$\begin{aligned} a_{0,1}(x, y) &= \frac{y}{g(x)}, & b_{0,1}(x, y) &= \frac{y - g(x)}{-g(x)}, \\ a_{k,1}(x, y) &= \frac{(y - g(x))^k}{k!} \left(\frac{y}{g(x)} \right)^{k+1}, \\ b_{k,1}(x, y) &= \frac{y^k}{k!} \left(\frac{y - g(x)}{-g(x)} \right)^{k+1}, & k &= 1, 2, \dots \end{aligned} \quad (27)$$

By (27), we deduce that for $x_1 \leq x \leq x_2$,

$$\begin{aligned} \frac{\partial^l a_{k,1}}{\partial y^l}(x, g(x)) &= 0 \quad (0 \leq l \leq k-1), \\ \frac{\partial^k a_{k,1}}{\partial y^k}(x, g(x)) &= 1; \quad \frac{(\partial^l a_{k,1})}{\partial y^l}(x, 0) = 0 \quad (0 \leq l \leq k), \\ \frac{\partial^l b_{k,1}}{\partial y^l}(x, g(x)) &= 0 \quad (0 \leq l \leq k); \\ \frac{\partial^l b_{k,1}}{\partial y^l}(x, 0) &= 0 \quad (0 \leq l \leq k-1), \quad \frac{(\partial^k b_{k,1})}{\partial y^k}(x, 0) = 1. \end{aligned} \quad (28)$$

On E_1 , we define a sequence of functions $\{S_1^{(k)}(x, y)\}_0^\infty$ by induction.

Let

$$\begin{aligned} S_1^{(0)}(x, y) &= f(x, g(x)) a_{0,1}(x, y) \\ (x_1 \leq x \leq x_2, 0 \leq y \leq g(x)). \end{aligned} \quad (29)$$

Then, by (27),

$$\begin{aligned} S_1^{(0)}(x, 0) &= 0, \quad S_1^{(0)}(x, g(x)) = f(x, g(x)), \\ (x_1 \leq x \leq x_2). \end{aligned} \quad (30)$$

Let

$$\begin{aligned} S_1^{(1)}(x, y) &= S_1^{(0)}(x, y) \\ &+ a_{1,1}(x, y) \left(\frac{\partial f}{\partial y}(x, g(x)) - \frac{\partial S_1^{(0)}}{\partial y}(x, g(x)) \right) \\ &- b_{1,1}(x, y) \frac{\partial S_1^{(0)}}{\partial y}(x, 0) \\ &(x_1 \leq x \leq x_2, 0 \leq y \leq g(x)). \end{aligned} \quad (31)$$

Then, by (27)–(30), we obtain that, for $x_1 \leq x \leq x_2$,

$$\begin{aligned} S_1^{(1)}(x, g(x)) &= f(x, g(x)), \\ \frac{\partial S_1^{(1)}}{\partial y}(x, g(x)) &= \frac{\partial f}{\partial y}(x, g(x)), \\ S_1^{(1)}(x, 0) &= 0, \quad \frac{\partial S_1^{(1)}}{\partial y}(x, 0) = 0. \end{aligned} \quad (32)$$

In general, let

$$\begin{aligned} S_1^{(k)}(x, y) &= S_1^{(k-1)}(x, y) + a_{k,1}(x, y) \\ &\times \left(\frac{\partial^k f}{\partial y^k}(x, g(x)) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, g(x)) \right) \\ &- b_{k,1}(x, y) \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, 0) \\ &(x_1 \leq x \leq x_2, 0 \leq y \leq g(x)). \end{aligned} \quad (33)$$

Lemma 4. For any $k \in \mathbb{Z}_+$, one has $S_1^{(k)} \in C^\infty(E_1)$ and

$$S_1^{(k)}(x, y) = \sum_{j=0}^{2k+1} \zeta_{j,1}(x) y^j, \quad (x, y) \in E_1. \quad (34)$$

Proof. Since $f \in C^\infty(\Omega)$ and $g \in C^\infty([x_1, x_2])$, and $g(x) > 0$ ($x_1 \leq x \leq x_2$), by the above construction, we know that $S_1^{(k)} \in C^\infty(E_1)$ for any $k = 0, 1, \dots$

For $k = 0$, since

$$S_1^{(0)}(x, y) = f(x, g(x)) a_{0,1}(x, y) = \frac{f(x, g(x))}{g(x)} y, \quad (35)$$

(34) holds. We assume that (34) holds for $k = l - 1$; that is,

$$S_1^{(l-1)}(x, y) = \sum_{j=0}^{2l-1} \zeta_{j,1}^{(l-1)}(x) y^j. \quad (36)$$

This implies that

$$\begin{aligned} \frac{\partial^l S_1^{(l-1)}}{\partial y^l}(x, g(x)) &= \sum_{j=l}^{2l-1} \frac{j!}{(j-l)!} \zeta_{j,1}^{(l-1)}(x) (g(x))^{j-l}, \\ \frac{\partial^l S_1^{(l-1)}}{\partial y^l}(x, 0) &= l! \zeta_{l,1}^{(l-1)}(x). \end{aligned} \quad (37)$$

Again, notice that $a_{l,1}(x, y)$ and $b_{l,1}(x, y)$ are polynomials of y whose degrees are both $2l + 1$. From this and (33), it follows that (34) holds for $k = l$. By induction, (34) holds for all k . Lemma 4 is proved. $\square$

Below we compute derivatives $(\partial^l S_1^{(k)} / \partial y^l)(x, y)$ ($0 \leq l \leq k$) on the curved side $\Gamma_1 = \{(x, g(x)) : x_1 \leq x \leq x_2\}$ and the bottom side $\Delta_1 = \{(x, 0) : x_1 \leq x \leq x_2\}$ of E_1 .

Lemma 5. Let $S_1^{(k)}(x, y)$ be stated as above. For any $k \in \mathbb{Z}_+$, one has

$$\begin{aligned} \frac{\partial^l S_1^{(k)}}{\partial y^l}(x, g(x)) &= \frac{\partial^l f}{\partial y^l}(x, g(x)), \\ \frac{\partial^l S_1^{(k)}}{\partial y^l}(x, 0) &= 0 \quad (x_1 \leq x \leq x_2, 0 \leq l \leq k). \end{aligned} \quad (38)$$

Proof. By (30), We have known that, for $k = 0$, (38) holds.

Now we assume that (38) holds for $k - 1$.

For $x_1 \leq x \leq x_2$, by (33), we have

$$\begin{aligned} \frac{\partial^l S_1^{(k)}}{\partial y^l}(x, g(x)) &= \frac{\partial^l S_1^{(k-1)}}{\partial y^l}(x, g(x)) + \frac{\partial^l a_{k,1}}{\partial y^l}(x, g(x)) \\ &\quad \times \left(\frac{\partial^k f}{\partial y^k}(x, g(x)) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, g(x)) \right) \\ &\quad - \frac{\partial^l b_{k,1}}{\partial y^l}(x, g(x)) \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, 0), \\ &\quad (0 \leq l \leq k). \end{aligned} \quad (39)$$

For $l = 0, 1, \dots, k-1$, by the assumption of induction, we have

$$\frac{\partial^l S_1^{(k-1)}}{\partial y^l}(x, g(x)) = \frac{\partial^l f}{\partial y^l}(x, g(x)). \quad (40)$$

By (28), we have $(\partial^l a_{k,1}/\partial y^l)(x, g(x)) = 0$, $(\partial^l b_{k,1}/\partial y^l)(x, g(x)) = 0$. So we get

$$\frac{\partial^l S_1^{(k)}}{\partial y^l}(x, g(x)) = \frac{\partial^l f}{\partial y^l}(x, g(x)). \quad (41)$$

For $l = k$, note that $(\partial^k a_{k,1}/\partial y^k)(x, g(x)) = 1$ and $(\partial^k b_{k,1}/\partial y^k)(x, g(x)) = 0$. By (39), we get

$$\begin{aligned} \frac{\partial^k S_1^{(k)}}{\partial y^k}(x, g(x)) &= \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, g(x)) \\ &\quad + \left(\frac{\partial^k f}{\partial y^k}(x, g(x)) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, g(x)) \right) \\ &= \frac{\partial^k f}{\partial y^k}(x, g(x)). \end{aligned} \quad (42)$$

The first formula of (38) holds for k .

By (33), we have

$$\begin{aligned} \frac{\partial^l S_1^{(k)}}{\partial y^l}(x, 0) &= \frac{\partial^l S_1^{(k-1)}}{\partial y^l}(x, 0) + \frac{\partial^l a_{k,1}}{\partial y^l}(x, 0) \\ &\quad \times \left(\frac{\partial^k f}{\partial y^k}(x, g(x)) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, g(x)) \right) \\ &\quad - \frac{\partial^l b_{k,1}}{\partial y^l}(x, 0) \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, 0), \\ &\quad (x_1 \leq x \leq x_2, \quad 0 \leq l \leq k). \end{aligned} \quad (43)$$

For $l = 0, \dots, k-1$, by the assumption of induction and (28), we have $(\partial^l S_1^{(k-1)}/\partial y^l)(x, 0) = 0$ and $(\partial^l a_{k,1}/\partial y^l)(x, 0) = (\partial^l b_{k,1}/\partial y^l)(x, 0) = 0$. So

$$\frac{\partial^l S_1^{(k)}}{\partial y^l}(x, 0) = 0. \quad (44)$$

For $l = k$, since $(\partial^k a_{k,1}/\partial y^k)(x, 0) = 0$, $(\partial^k b_{k,1}/\partial y^k)(x, 0) = 1$, by (43), we have

$$\frac{\partial^k S_1^{(k)}}{\partial y^k}(x, 0) = \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, 0) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, 0) = 0. \quad (45)$$

The second formula of (38) holds. By induction, (38) holds for all k . From this, we get Lemma 5. $\square$

Now we compute the mixed derivatives of $S_1^{(k)}(x, y)$ on the curved side Γ_1 and bottom side Δ_1 of E_1 .

Lemma 6. Let Γ_1 and Δ_1 be the curved side and the bottom side of E_1 , respectively. Then, for $k \in \mathbb{Z}_+$,

$$\begin{aligned} \text{(i)} \quad &(\partial^{i+j} S_1^{(k)} / \partial x^i \partial y^j)(x, y) = (\partial^{i+j} f / \partial x^i \partial y^j)(x, y) \\ &((x, y) \in \Gamma_1), \\ \text{(ii)} \quad &(\partial^{i+j} S_1^{(k)} / \partial x^i \partial y^j)(x, y) = 0 \quad ((x, y) \in \Delta_1), \end{aligned}$$

where $0 \leq i + j \leq k$.

Proof. Let $x_1 \leq x \leq x_2$. Then we have

$$\begin{aligned} \frac{d}{dx} \left(\frac{\partial^{l-1} f}{\partial y^{l-1}}(x, g(x)) \right) &= \frac{\partial^l f}{\partial x \partial y^{l-1}}(x, g(x)) \\ &\quad + \frac{\partial^l f}{\partial y^l}(x, g(x)) g'(x), \quad (l \geq 1). \end{aligned} \quad (46)$$

By the Newton-Leibniz formula, we have

$$\begin{aligned} \frac{\partial^{l-1} f}{\partial y^{l-1}}(x, g(x)) &= \frac{\partial^{l-1} f}{\partial y^{l-1}}(x_1, g(x_1)) \\ &\quad + \int_{x_1}^x \left(\frac{\partial^l f}{\partial x \partial y^{l-1}}(t, g(t)) + \frac{\partial^l f}{\partial y^l}(t, g(t)) g'(t) \right) dt. \end{aligned} \quad (47)$$

Similarly, replacing f by $S_1^{(k)}$ in this formula, we have

$$\begin{aligned} & \frac{\partial^{l-1} S_1^{(k)}}{\partial y^{l-1}}(x, g(x)) \\ &= \frac{\partial^{l-1} S_1^{(k)}}{\partial y^{l-1}}(x_1, g(x_1)) \\ &+ \int_{x_1}^x \left(\frac{\partial^l S_1^{(k)}}{\partial x \partial y^{l-1}}(t, g(t)) + \frac{\partial^l S_1^{(k)}}{\partial y^l}(t, g(t)) g'(t) \right) dt, \\ & \quad (l \geq 1). \end{aligned} \quad (48)$$

From this and Lemma 5, it follows that, for any $x_1 \leq x \leq x_2$, we have

$$\begin{aligned} & \int_{x_1}^x \frac{\partial^l S_1^{(k)}}{\partial x \partial y^{l-1}}(t, g(t)) dt \\ &= \int_{x_1}^x \frac{\partial^l f}{\partial x \partial y^{l-1}}(t, g(t)) dt \quad (1 \leq l \leq k). \end{aligned} \quad (49)$$

Finding derivatives on the both sides of this formula, we get

$$\frac{\partial^l S_1^{(k)}}{\partial x \partial y^{l-1}}(x, g(x)) = \frac{\partial^l f}{\partial x \partial y^{l-1}}(x, g(x)) \quad (1 \leq l \leq k). \quad (50)$$

Now we start from the equality

$$\begin{aligned} \frac{d}{dx} \left(\frac{\partial^{l-1} f}{\partial x \partial y^{l-2}}(x, g(x)) \right) &= \frac{\partial^l f}{\partial x^2 \partial y^{l-2}}(x, g(x)) \\ &+ \frac{\partial^l f}{\partial x \partial y^{l-1}}(x, g(x)) g'(x), \\ & \quad (l \geq 2). \end{aligned} \quad (51)$$

Similar to the argument from (46) to (50), we get

$$\frac{\partial^l S_1^{(k)}}{\partial x^2 \partial y^{l-2}}(x, g(x)) = \frac{\partial^l f}{\partial x^2 \partial y^{l-2}}(x, g(x)) \quad (2 \leq l \leq k). \quad (52)$$

Continuing this procedure, we deduce that (i) holds for $0 < i + j \leq k$. Letting $l = 0$ in Lemma 5, we have $S_1^{(k)}(x, g(x)) = f(x, g(x))$; that is, (i) holds for $i = j = 0$. So we get (i).

By Lemma 5, $(\partial^l S_1^{(k)} / \partial y^j)(x, 0) = 0$ ($0 \leq j \leq k$). From this and $S_1^{(k)} \in C^\infty(E_1)$, we have

$$\frac{\partial^{i+j} S_1^{(k)}}{\partial x^i \partial y^j}(x, 0) = 0 \quad (0 \leq i + j \leq k), \quad (53)$$

so (ii) holds. Lemma 6 is proved. $\square$

From this, we get the following.

Lemma 7. For any positive integer r , denote $l_r = r(r+1)(r+2)(r+3)$. Let

$$F(x, y) = \begin{cases} S_1^{(l_r)}(x, y), & (x, y) \in E_1, \\ f(x, y), & (x, y) \in \Omega. \end{cases} \quad (54)$$

Then (i) $F \in C^l(\Omega \cup E_1)$ and $F(x, y) = f(x, y)$ ($(x, y) \in \Omega$); (ii) $(\partial^{i+j} F / \partial x^i \partial y^j)(x, y) = 0$ ($(x, y) \in (E_1 \cap \partial T)$, $0 \leq i + j \leq l_r$).

Proof. By the assumption $f \in C^\infty(\Omega)$, Lemma 4: $S_1^{(k)} \in C^\infty(E_1)$, and Lemma 6(i):

$$\begin{aligned} \frac{\partial^{i+j} S_1^{(k)}}{\partial x^i \partial y^j}(x, y) &= \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, y) \\ & \quad ((x, y) \in \Gamma_1, 0 \leq i + j \leq k), \end{aligned} \quad (55)$$

where $\Gamma_1 = \Omega \cap E_1$, we get (i). By Lemma 6(ii) and $E_1 \cap \partial T = \Delta_1$, we get (ii). Lemma 7 is proved. $\square$

For $v = 2, 3, 4$, by using a similar method, we define $S_v^{(k)}(x, y)$ on the each trapezoid E_v with a curve side. The representations of $S_v^{(k)}(x, y)$ are stated in Section 4.1.

Lemma 8. For any $v = 1, 2, 3, 4$, let

$$F(x, y) = \begin{cases} S_v^{(l_r)}(x, y), & (x, y) \in E_v, \\ f(x, y), & (x, y) \in \Omega, \end{cases} \quad (56)$$

where $l_r = r(r+1)(r+2)(r+3)$. Then, for $v = 1, 2, 3, 4$, one has the following:

- (i) $F \in C^l(\Omega \cup E_v)$;
- (ii) $(\partial^{i+j} F / \partial x^i \partial y^j)(x, y) = 0$, $(x, y) \in (E_v \cap \partial T)$ for $0 \leq i + j \leq l_r$;
- (iii) $F(x, y)$ can be expressed in the form:

$$\begin{aligned} F(x, y) &= \sum_{j=0}^{2l_r+1} \zeta_{j,1}(x) y^j, \quad (x, y) \in E_1, \\ F(x, y) &= \sum_{j=0}^{2l_r+1} \zeta_{j,2}(y) x^j, \quad (x, y) \in E_2, \\ F(x, y) &= \sum_{j=0}^{2l_r+1} \zeta_{j,3}(x) y^j, \quad (x, y) \in E_3, \\ F(x, y) &= \sum_{j=0}^{2l_r+1} \zeta_{j,4}(y) x^j, \quad (x, y) \in E_4. \end{aligned} \quad (57)$$

Proof. By Lemma 7, we have

$$\begin{aligned} F &\in C^{l_r}(\Omega \cup E_1), \quad \frac{\partial^{i+j} F}{\partial x^i \partial y^j}(x, y) = 0, \\ & \quad ((x, y) \in (E_1 \cap \partial T), 0 \leq i + j \leq l_r). \end{aligned} \quad (58)$$

Similar to the argument of Lemma 7, for $v = 2, 3, 4$, we have

$$F \in C^{l_r}(\Omega \cup E_v), \quad \frac{\partial^{i+j} F}{\partial x^i \partial y^j}(x, y) = 0, \quad (59)$$

$$((x, y) \in (E_v \cap \partial T), 0 \leq i + j \leq l_r).$$

From this, we get (i) and (ii).

The proof of (iii) is similar to the argument of Lemma 4(iii). Lemma 8 is proved. $\square$

3.3. Smooth Extension to Each Rectangle H_v . We have completed the smooth extension of f to each trapezoid E_v with a curved side. In this subsection we complete the smooth extension of the obtained function F to each rectangle H_v . First we consider the smooth extension of F to H_1 . We divide this procedure in two steps.

Step 1. In Lemma 8, we know that $F(x, y) = S_4^{l_r}(x, y)$ on E_4 . Now we construct the smooth extension of $S_4^{(l_r)}(x, y)$ from E_4 to H_1 , where $S_4^{(l_r)}(x, y)$ is stated in Section 4.2 and $l_r = r(r+1)(r+2)(r+3)$.

Let

$$\alpha_{k,11}(y) = \frac{(y - y_1)^k}{k!} \left(\frac{y}{y_1} \right)^{k+1}, \quad (60)$$

$$\beta_{k,11}(y) = \frac{y^k}{k!} \left(\frac{y - y_1}{-y_1} \right)^{k+1}, \quad k = 0, 1, \dots,$$

and let

$$M_1^{(0)}(x, y) = S_4^{(l_r)}(x, y_1) \alpha_{0,11}(y),$$

$$M_1^{(k)}(x, y) = M_1^{(k-1)}(x, y) + \alpha_{k,11}(y)$$

$$\times \left(\frac{\partial^k S_4^{(l_r)}}{\partial y^k}(x, y_1) - \frac{\partial^k M_1^{(k-1)}}{\partial y^k}(x, y_1) \right) \quad (61)$$

$$- \beta_{k,11}(y) \frac{\partial^k M_1^{(k-1)}}{\partial y^k}(x, 0),$$

$$k = 1, 2, \dots, \tau_r \quad ((x, y) \in H_1),$$

where $\tau_r = r(r+2)$.

Lemma 9. Let $\{J_{1,i}\}_1^4$ be four sides of the rectangle H_1 :

$$J_{1,1} = \{(x, y_1), 0 \leq x \leq x_1\},$$

$$J_{1,2} = \{(0, y), 0 \leq y \leq y_1\},$$

$$J_{1,3} = \{(x, 0), 0 \leq x \leq x_1\},$$

$$J_{1,4} = \{(x_1, y), 0 \leq y \leq y_1\}. \quad (62)$$

Then one has the following

$$(i) \quad M_1^{(\tau_r)}(x, y) = \sum_{i,j=0}^{2l_r+1} d_{i,j}^{(1)} x^i y^j, \text{ where } d_{i,j}^{(1)} \text{ is a constant;}$$

$$(ii) \quad (\partial^{i+j} M_1^{(\tau_r)} / \partial x^i \partial y^j)(x, y) = (\partial^{i+j} S_4^{(l_r)} / \partial x^i \partial y^j)(x, y) \quad ((x, y) \in J_{1,1});$$

$$(iii) \quad (\partial^{i+j} M_1^{(\tau_r)} / \partial x^i \partial y^j)(x, y) = 0 \quad ((x, y) \in J_{1,2});$$

$$(iv) \quad (\partial^{i+j} M_1^{(\tau_r)} / \partial x^i \partial y^j)(x, y) = 0 \quad ((x, y) \in J_{1,3}),$$

where $0 \leq i + j \leq \tau_r$.

Proof. By Lemma 8(iii), we have

$$S_4^{(l_r)}(x, y) = \sum_{j=0}^{2l_r+1} \zeta_{j,4}(y) x^j. \quad (63)$$

So $(\partial^k S_4^{(l_r)} / \partial y^k)(x, y_1)$ is a polynomial of degree $2l_r + 1$ with respect to x . Since $\alpha_{\tau_r,11}(y)$ and $\beta_{\tau_r,11}(y)$ are both polynomials of degree $2\tau_r + 1$, (i) follows from (61).

Similar to the argument of Lemma 6, we get (ii) and (iv).

Since $(0, y_1) \in (E_4 \cap \partial T)$, by Lemma 7, we have

$$\frac{\partial^{i+j} S_4^{(l_r)}}{\partial x^i \partial y^j}(0, y_1) = \frac{\partial^{i+j} F}{\partial x^i \partial y^j}(0, y_1) = 0, \quad (0 \leq i + j \leq l_r). \quad (64)$$

By the definition of $M_1^{(0)}$ and (64), we have

$$\frac{\partial^{i+j} M_1^{(0)}}{\partial x^i \partial y^j}(0, y) = \frac{\partial^i S_4^{(l_r)}}{\partial x^i}(0, y_1) \frac{d^j \alpha_{0,11}}{dy^j}(y) = 0, \quad (65)$$

$$(0 \leq i + j \leq l_r, y \in \mathbb{R}).$$

We assume that

$$\frac{\partial^{i+j} M_1^{(k-1)}}{\partial x^i \partial y^j}(0, y) = 0, \quad \left(0 \leq i + j \leq l_r - \frac{1}{2}k(k-1) \right). \quad (66)$$

By (61), we get

$$\frac{\partial^{i+j} M_1^{(k)}}{\partial x^i \partial y^j}(0, y) = \frac{\partial^{i+j} M_1^{(k-1)}}{\partial x^i \partial y^j}(0, y) + \frac{d^j \alpha_{k,11}}{dy^j}(y)$$

$$\times \left(\frac{\partial^{k+i} S_4^{(l_r)}}{\partial x^i \partial y^k}(0, y_1) - \frac{\partial^{k+i} M_1^{(k-1)}}{\partial x^i \partial y^k}(0, y_1) \right)$$

$$- \frac{d^j \beta_{k,11}}{dy^j}(y) \frac{\partial^{k+i} M_1^{(k-1)}}{\partial x^i \partial y^k}(0, 0). \quad (67)$$

for $0 \leq i + j \leq l_r - (1/2)k(k+1)$, we have $0 \leq i + j \leq l_r - (1/2)k(k-1)$ and $0 \leq i + k \leq l_r - (1/2)k(k-1)$. Again, by the assumption of induction, we get

$$\frac{\partial^{i+j} M_1^{(k-1)}}{\partial x^i \partial y^j}(0, y) = 0, \quad \frac{\partial^{k+i} M_1^{(k-1)}}{\partial x^i \partial y^k}(0, y_1) = 0. \quad (68)$$

By (64), we have $(\partial^{k+i} S_4^{(l_r)} / \partial x^i \partial y^k)(0, y_1) = 0$. From this and (67), we get

$$\frac{\partial^{i+j} M_1^{(k)}}{\partial x^i \partial y^j}(0, y) = 0, \quad (69)$$

$$\left(0 \leq y \leq y_1, \quad 0 \leq i+j \leq l_r - \frac{k}{2}(k+1)\right).$$

Taking $k = \tau_r$, we have

$$\frac{\partial^{i+j} M_1^{(\tau_r)}}{\partial x^i \partial y^j}(0, y) = 0, \quad (70)$$

$$\left(0 \leq y \leq y_1, \quad 0 \leq i+j \leq l_r - \frac{\tau_r}{2}(\tau_r + 1)\right).$$

Since $l_r - (\tau_r/2)(\tau_r + 1) = (l_r/2) \geq \tau_r$, we get (iii). Lemma 9 is proved. $\square$

Step 2. In Lemma 8, we know that $F(x, y) = S_1^{(l_r)}(x, y)$ on E_1 . We consider the difference $S_1^{(l_r)}(x, y) - M_1^{(\tau_r)}(x, y)$. Obviously, it is infinitely many time differentiable on E_1 since $M_1^{(\tau_r)}(x, y)$ is a polynomial. Now we construct its smooth extension from E_1 to the rectangle H_1 as follows. Let

$$\alpha_{k,14}(x) = \frac{(x - x_1)^k}{k!} \left(\frac{x}{x_1}\right)^{k+1}, \quad (71)$$

$$\beta_{k,14}(x) = \frac{x^k}{k!} \left(\frac{x - x_1}{-x_1}\right)^{k+1}, \quad k = 0, 1, \dots,$$

and let

$$N_1^{(0)}(x, y) = (S_1^{(l_r)}(x_1, y) - M_1^{(\tau_r)}(x_1, y)) \alpha_{0,14}(x),$$

$$N_1^{(k)}(x, y) = N_1^{(k-1)}(x, y) + \alpha_{k,14}(x)$$

$$\times \left(\frac{\partial^k (S_1^{(l_r)} - M_1^{(\tau_r)})}{\partial x^k}(x_1, y) \right. \\ \left. - \frac{\partial^k N_1^{(k-1)}}{\partial x^k}(x_1, y) \right) \quad (72)$$

$$- \beta_{k,14}(x) \frac{\partial^k N_1^{(k-1)}}{\partial x^k}(0, y),$$

$$k = 0, 1, \dots, r, \quad ((x, y) \in H_1).$$

From this, we obtain the following.

Lemma 10. $N_1^{(r)}(x, y)$ possesses the following properties:

- (i) $(\partial^{i+j} N_1^{(r)} / \partial x^i \partial y^j)(x, y) = (\partial^{i+j} S_1^{(l_r)} / \partial x^i \partial y^j)(x, y) - (\partial^{i+j} M_1^{(\tau_r)} / \partial x^i \partial y^j)(x, y)$ on $J_{1,4}$;
- (ii) $(\partial^{i+j} N_1^{(r)} / \partial x^i \partial y^j)(x, y) = 0$ on $J_{1,2}$;
- (iii) $(\partial^{i+j} N_1^{(r)} / \partial x^i \partial y^j)(x, y) = 0$ on $J_{1,1}$;

(iv) $(\partial^{i+j} N_1^{(r)} / \partial x^i \partial y^j)(x, y) = 0$ on $J_{1,3}$, where $0 \leq i+j \leq r$ and $\{J_{1,v}\}_1^4$ are stated in (62);

(v) $N_1^{(r)}(x, y) = \sum_{i,j=0}^{2l_r+1} \tau_{i,j}^{(1)} x^i y^j$, where $\tau_{i,j}^{(1)}$ is a constant.

Proof. The arguments similar to Lemma 9(ii) and (iv) give the conclusions (i) and (ii) of this theorem. Now we prove (iii) and (iv).

By Lemma 6(i) and Lemma 9(ii), as well as $l_r \geq \tau_r$, we get that, for $0 \leq i+j \leq \tau_r$,

$$\frac{\partial^{i+j} S_1^{(l_r)}}{\partial x^i \partial y^j}(x_1, y_1) = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_1, y_1) = \frac{\partial^{i+j} S_4^{(l_r)}}{\partial x^i \partial y^j}(x_1, y_1) \quad (73)$$

$$= \frac{\partial^{i+j} M_1^{(\tau_r)}}{\partial x^i \partial y^j}(x_1, y_1).$$

So we have

$$\frac{\partial^{i+j} N_1^{(0)}}{\partial x^i \partial y^j}(x, y_1) = \frac{\partial^j (S_1^{(l_r)} - M_1^{(\tau_r)})}{\partial y^j}(x_1, y_1) \frac{d^i \alpha_{0,14}}{dx^i}(x) = 0, \quad (74)$$

$$(0 \leq x \leq x_1, \quad 0 \leq i+j \leq \tau_r).$$

Now we assume that

$$\frac{\partial^{i+j} N_1^{(k-1)}}{\partial x^i \partial y^j}(x, y_1) = 0, \quad (75)$$

$$\left(0 \leq x \leq x_1, \quad 0 \leq i+j \leq \tau_r - \frac{k}{2}(k-1)\right).$$

By (72) and (73),

$$\frac{\partial^{i+j} N_1^{(k)}}{\partial x^i \partial y^j}(x, y_1) = \frac{\partial^{i+j} N_1^{(k-1)}}{\partial x^i \partial y^j}(x, y_1) + \frac{d^i \alpha_{k,14}}{dx^i}(x)$$

$$\times \left(\frac{\partial^{k+j} (S_1^{(l_r)} - M_1^{(\tau_r)})}{\partial x^k \partial y^j}(x_1, y_1) \right. \\ \left. - \frac{\partial^{k+j} N_1^{(k-1)}}{\partial x^k \partial y^j}(x_1, y_1) \right) \\ - \frac{d^i \beta_{k,14}}{dx^i}(x) \frac{\partial^{k+j} N_1^{(k-1)}}{\partial x^k \partial y^j}(0, y_1) = 0, \quad (76)$$

$$\left(0 \leq x \leq x_1, \quad 0 \leq i+j \leq \tau_r - \frac{k(k+1)}{2}\right).$$

By induction, we get

$$\frac{\partial^{i+j} N_1^{(k)}}{\partial x^i \partial y^j}(x, y_1) = 0, \quad (77)$$

$$\left(0 \leq x \leq x_1, \quad 0 \leq i+j \leq \tau_r - \frac{k(k+1)}{2}\right).$$

From this and $\tau_r - (1/2)r(r+1) \geq r$, we get (iii). By Lemma 6(ii) and Lemma 9(iii), we get that

$$\begin{aligned} \frac{\partial^{i+j} S_1^{(l_r)}}{\partial x^i \partial y^j} (x, 0) &= 0, \quad (0 \leq i+j \leq l_r), \\ \frac{\partial^{i+j} M_1^{(\tau_r)}}{\partial x^i \partial y^j} (x, 0) &= 0, \quad (0 \leq i+j \leq \tau_r). \end{aligned} \quad (78)$$

From this and (72), by using an argument similar to the proof of (iii), we get (iv).

By Lemma 8(iii) and Lemma 9(i), we deduce that $(S_1^{(l_r)} - M_1^{(\tau_r)})(x_1, y)$ is a polynomial of degree $2l_r + 1$ with respect to y . From this and (72), we get (v). Lemma 10 is proved. $\square$

By Lemmas 9 and 10, we obtain that for $0 \leq i+j \leq r$,

$$\frac{\partial^{i+j} (M_1^{(\tau_r)} + N_1^{(r)})}{\partial x^i \partial y^j} (x, y) = \frac{\partial^{i+j} S_4^{(l_r)}}{\partial x^i \partial y^j} (x, y) \quad \text{on } J_{1,1}, \quad (79)$$

$$\frac{\partial^{i+j} (M_1^{(\tau_r)} + N_1^{(r)})}{\partial x^i \partial y^j} (x, y) = 0 \quad \text{on } J_{1,2} \cup J_{1,3}, \quad (80)$$

$$\frac{\partial^{i+j} (M_1^{(\tau_r)} + N_1^{(r)})}{\partial x^i \partial y^j} (x, y) = \frac{\partial^{i+j} S_1^{(l_r)}}{\partial x^i \partial y^j} (x, y) \quad \text{on } J_{1,4}. \quad (81)$$

Lemma 11. Let

$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in \Omega, \\ S_1^{(l_r)}(x, y), & (x, y) \in E_1, \\ S_4^{(l_r)}(x, y), & (x, y) \in E_4, \\ M_1^{(\tau_r)}(x, y) + N_1^{(r)}(x, y), & (x, y) \in H_1. \end{cases} \quad (82)$$

Then one has

- (i) $F \in C^r(\Omega \cup E_1 \cup E_4 \cup H_1)$,
- (ii) $(\partial^{i+j} F / \partial x^i \partial y^j)(x, y) = 0$, $(x, y) \in ((E_1 \cup E_4 \cup H_1) \cap \partial T)$ for $0 \leq i+j \leq r$;
- (iii) $F(x, y) = \sum_{i,j=0}^{2l_r+1} c_{ij}^{(1)} x^i y^j$ ($(x, y) \in H_1$), where each $c_{ij}^{(1)}$ is constant.

Proof. By Lemma 7, we have $F \in C^r(\Omega \cup E_1 \cup E_4)$. Since $S_1^{(l_r)} \in C^r(E_1)$,

$$M_1^{(\tau_r)} + N_1^{(r)} \in C^r(H_1), \quad E_1 \cap H_1 = J_{1,4}, \quad (83)$$

by (81), we deduce that $F \in C^r(E_1 \cup H_1)$. Since $S_4^{(l_r)} \in C^r(E_4)$,

$$M_1^{(\tau_r)} + N_1^{(r)} \in C^r(H_1), \quad E_4 \cap H_1 = J_{1,1}, \quad (84)$$

by (79), we deduce that $F \in C^r(H_1 \cup E_4)$. So we get (i).

By Lemma 8(ii),

$$\frac{\partial^{i+j} F}{\partial x^i \partial y^j} (x, y) = 0, \quad (x, y) \in ((E_1 \cup E_4) \cap \partial T). \quad (85)$$

Since $H_1 \cap \partial T = J_{1,2} \cup J_{1,3}$, by (80), we deduce that

$$\frac{\partial^{i+j} F}{\partial x^i \partial y^j} (x, y) = 0, \quad (x, y) \in (H_1 \cap \partial T). \quad (86)$$

So we get (ii).

From Lemma 9(i), Lemma 10(v), and $F(x, y) = M_1^{(\tau_r)}(x, y) + N_1^{(r)}(x, y)$ ($(x, y) \in H_1$), we get (iii). Lemma 11 is proved. $\square$

For $\nu = 2, 3, 4$, by using a similar method, we define $F(x, y) = M_\nu^{(\tau_r)}(x, y) + N_\nu^{(r)}(x, y)$ ($(x, y) \in H_\nu$), where representations of $M_\nu^{(\tau_r)}(x, y)$ and $N_\nu^{(r)}(x, y)$ see Section 4.2.

3.4. The Proofs of the Theorems

Proof of Theorem 1. Let

$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in \Omega, \\ S_\nu^{(l_r)}(x, y), & (x, y) \in E_\nu \quad (\nu = 1, 2, 3, 4), \\ M_\nu^{(\tau_r)}(x, y) + N_\nu^{(r)}(x, y), & (x, y) \in H_\nu \quad (\nu = 1, 2, 3, 4). \end{cases} \quad (87)$$

By (25), F has been defined on the unit square T . The argument similar to Lemma 11(i)-(ii) shows that

$$\begin{aligned} F &\in C^r(\Omega \cup E_1 \cup E_4 \cup H_1); \\ F &\in C^r(\Omega \cup E_1 \cup E_2 \cup H_2); \\ F &\in C^r(\Omega \cup E_2 \cup E_3 \cup H_3); \\ F &\in C^r(\Omega \cup E_3 \cup E_4 \cup H_4); \end{aligned} \quad (88)$$

and for $0 \leq i+j \leq r$,

$$\frac{\partial^{i+j} F}{\partial x^i \partial y^j} (x, y) = 0, \quad (x, y) \in ((E_1 \cup E_4 \cup H_1) \cap \partial T);$$

$$\frac{\partial^{i+j} F}{\partial x^i \partial y^j} (x, y) = 0, \quad (x, y) \in ((E_1 \cup E_2 \cup H_2) \cap \partial T);$$

$$\frac{\partial^{i+j} F}{\partial x^i \partial y^j} (x, y) = 0, \quad (x, y) \in ((E_2 \cup E_3 \cup H_3) \cap \partial T);$$

$$\frac{\partial^{i+j} F}{\partial x^i \partial y^j} (x, y) = 0, \quad (x, y) \in ((E_3 \cup E_4 \cup H_4) \cap \partial T). \quad (89)$$

From this and $\Omega \cap \partial T = \emptyset$, by (25), we have $F \in C^r(T)$ and $(\partial^{i+j} F / \partial x^i \partial y^j)(x, y) = 0$ ($(x, y) \in \partial T$, $0 \leq i+j \leq r$). So we get (i) and (ii).

Similar to the argument of Lemma 11(iii), we get

$$F(x, y) = \sum_{i,j=0}^{2l_r+1} c_{ij}^{(\nu)} x^i y^j, \quad (x, y) \in H_\nu \quad (\nu = 1, 2, 3, 4), \quad (90)$$

where each $c_{ij}^{(\nu)}$ is a constant. From this and Lemma 8(iii), we know that, on $T \setminus \Omega$, $F(x, y)$ can be expressed locally in the form

$$\sum_{j=0}^{2l_r+1} \xi_j(x) y^j \quad \text{or} \quad \sum_{j=0}^{2l_r+1} \eta_j(x) x^j \quad \text{or} \quad \sum_{i,j=0}^{2l_r+1} c_{ij} x^i y^j; \quad (91)$$

(iii) holds. We have completed the proof of Theorem 1. $\square$

The representation of F satisfying the conditions of Theorem 1 is given in Section 4.

Proof of Theorem 2. Let F be the smooth extension of f from Ω to T which is stated as in Theorem 1. Define F_p by

$$F_p(x+k, y+l) = F(x, y) \quad ((x, y) \in T; k, l \in \mathbb{Z}). \quad (92)$$

Then F_p is a 1-periodic function of $\mathbb{R}^2$. By Theorem 1, we know that $F_p \in C^r(T)$ and

$$\frac{\partial^{i+j} F_p}{\partial x^i \partial y^j}(x, y) = 0 \quad ((x, y) \in \partial T; 0 \leq i+j \leq r). \quad (93)$$

Let $T_{n_1, n_2} = [n_1, n_1+1] \times [n_2, n_2+1]$ ($n_1, n_2 \in \mathbb{Z}$). Since F_p is 1-periodic function, we have $F_p \in C^r(T_{n_1, n_2})$ and for any $n_1, n_2 \in \mathbb{Z}$,

$$\frac{\partial^{i+j} F_p}{\partial x^i \partial y^j}(x, y) = 0 \quad ((x, y) \in \partial T_{n_1, n_2}; 0 \leq i+j \leq r). \quad (94)$$

Noticing that $\mathbb{R}^2 = \bigcup_{n_1, n_2 \in \mathbb{Z}} T_{n_1, n_2}$, we have $F_p \in C^r(\mathbb{R}^2)$. By (92) and Theorem 1(i), we get

$$F_p(x, y) = F(x, y) = f(x, y) \quad ((x, y) \in \Omega). \quad (95)$$

Theorem 2 is proved. $\square$

Proof of Theorem 3. Let F be the smooth extension of f from Ω to T which is stated as in Theorem 1. Define F^c by

$$F^c(x, y) = \begin{cases} F(x, y), & (x, y) \in T, \\ 0, & (x, y) \in \mathbb{R}^2 \setminus T. \end{cases} \quad (96)$$

From Theorem 1(ii), we have

$$\frac{\partial^{i+j} F^c}{\partial x^i \partial y^j}(x, y) = 0 \quad ((x, y) \in \partial T; 0 \leq i+j \leq r). \quad (97)$$

From this and (96), we get $F^c(x, y) \in C^r(\mathbb{R}^2)$. By (96) and Theorem 1(i), we get

$$F^c(x, y) = F(x, y) = f(x, y) \quad ((x, y) \in \Omega). \quad (98)$$

Theorem 3 is proved. $\square$

4. Representation of the Extension F Satisfying Theorem 1

Let f and Ω be stated as in Theorem 1 and let Ω be divided as in Section 3.1. The representation of F satisfying conditions of Theorem 1 is as follows:

$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in \Omega, \\ S_v^{(l_r)}(x, y), & (x, y) \in E_v \quad (v = 1, 2, 3, 4), \\ M_v^{(r_r)}(x, y) + N_v^{(r)}(x, y), & (x, y) \in H_v \quad (v = 1, 2, 3, 4), \end{cases} \quad (99)$$

where

$$T = \Omega \cup \left(\bigcup_1^4 E_v \right) \cup \left(\bigcup_1^4 H_v \right) \quad (100)$$

and the rectangles $\{H_v\}_1^4$ and the trapezoids $\{E_v\}_1^4$ with a curved side are stated in (22) and (23) and $l_r = r(r+1)(r+2)(r+3)$ and $r_r = r(r+2)$.

Below we write out the representations of $\{S_v^{(k)}(x, y)\}_1^4$, $\{M_v^{(k)}(x, y)\}_1^4$, and $\{N_v^{(k)}(x, y)\}_1^4$.

4.1. The Construction of Each $S_v^{(k)}(x, y)$. (i) Denote

$$a_{k,1}(x, y) = \frac{(y-g(x))^k}{k!} \left(\frac{y}{g(x)} \right)^{k+1}, \quad (101)$$

$$b_{k,1}(x, y) = \frac{y^k}{k!} \left(\frac{y-g(x)}{-g(x)} \right)^{k+1}, \quad k = 0, 1, \dots$$

Define $S_1^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} S_1^{(0)}(x, y) &= f(x, g(x)) a_{0,1}(x, y), \\ S_1^{(k)}(x, y) &= S_1^{(k-1)}(x, y) + a_{k,1}(x, y) \\ &\quad \times \left(\frac{\partial^k f}{\partial y^k}(x, g(x)) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, g(x)) \right) \\ &\quad - b_{k,1}(x, y) \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, 0), \\ &\quad k = 1, 2, \dots, \quad ((x, y) \in E_1). \end{aligned} \quad (102)$$

(ii) Denote

$$\begin{aligned} a_{k,2}(x, y) &= \frac{(x-h(y))^k}{k!} \left(\frac{1-x}{1-h(y)} \right)^{k+1}, \\ b_{k,2}(x, y) &= \frac{(x-1)^k}{k!} \left(\frac{h(y)-x}{h(y)-1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (103)$$

Define $S_2^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} S_2^{(0)}(x, y) &= f(h(y), y) a_{0,2}(x, y), \\ S_2^{(k)}(x, y) &= S_2^{(k-1)}(x, y) + a_{k,2}(x, y) \\ &\quad \times \left(\frac{\partial^k f}{\partial x^k}(h(y), y) - \frac{\partial^k S_2^{(k-1)}}{\partial x^k}(h(y), y) \right) \\ &\quad - b_{k,2}(x, y) \frac{\partial^k S_2^{(k-1)}}{\partial x^k}(1, y), \\ &\quad k = 1, 2, \dots, \quad ((x, y) \in E_2). \end{aligned} \quad (104)$$

(iii) Denote

$$\begin{aligned} a_{k,3}(x, y) &= \frac{(y - g^*(x))^k}{k!} \left(\frac{1 - y}{1 - g^*(x)} \right)^{k+1}, \\ b_{k,3}(x, y) &= \frac{(y - 1)^k}{k!} \left(\frac{g^*(x) - y}{g^*(x) - 1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (105)$$

Define $S_3^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} S_3^{(0)}(x, y) &= f(x, g^*(x)) a_{0,3}(x, y), \\ S_3^{(k)}(x, y) &= S_3^{(k-1)}(x, y) + a_{k,3}(x, y) \\ &\quad \times \left(\frac{\partial^k f}{\partial y^k}(x, g^*(x)) - \frac{\partial^k S_3^{(k-1)}}{\partial y^k}(x, g^*(x)) \right) \\ &\quad - b_{k,3}(x, y) \frac{\partial^k S_3^{(k-1)}}{\partial y^k}(x, 1), \\ &\quad k = 1, 2, \dots, \quad ((x, y) \in E_3). \end{aligned} \quad (106)$$

(iv) Denote

$$\begin{aligned} a_{k,4}(x, y) &= \frac{(x - h^*(y))^k}{k!} \left(\frac{x}{h^*(y)} \right)^{k+1}, \\ b_{k,4}(x, y) &= \frac{x^k}{k!} \left(\frac{x - h^*(y)}{-h^*(y)} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (107)$$

Define $S_4^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} S_4^{(0)}(x, y) &= f(h^*(y), y) a_{0,4}(x, y), \\ S_4^{(k)}(x, y) &= S_4^{(k-1)}(x, y) + a_{k,4}(x, y) \\ &\quad \times \left(\frac{\partial^k f}{\partial x^k}(h^*(y), y) - \frac{\partial^k S_4^{(k-1)}}{\partial x^k}(h^*(y), y) \right) \\ &\quad - b_{k,4}(x, y) \frac{\partial^k S_4^{(k-1)}}{\partial x^k}(0, y), \\ &\quad k = 1, 2, \dots, \quad ((x, y) \in E_4). \end{aligned} \quad (108)$$

4.2. The Constructions of Each $M_v^{(k)}(x, y)$ and $N_v^{(k)}(x, y)$. (i) Denote

$$\begin{aligned} \alpha_{k,11}(y) &= \frac{(y - y_1)^k}{k!} \left(\frac{y}{y_1} \right)^{k+1}, \\ \beta_{k,11}(y) &= \frac{y^k}{k!} \left(\frac{y - y_1}{-y_1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (109)$$

Define $M_1^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} M_1^{(0)}(x, y) &= S_4^{(l_r)}(x, y_1) \alpha_{0,11}(y), \\ M_1^{(k)}(x, y) &= M_1^{(k-1)}(x, y) + \alpha_{k,11}(y) \\ &\quad \times \left(\frac{\partial^k S_4^{(l_r)}}{\partial y^k}(x, y_1) - \frac{\partial^k M_1^{(k-1)}}{\partial y^k}(x, y_1) \right) \\ &\quad - \beta_{k,11}(y) \frac{\partial^k M_1^{(k-1)}}{\partial y^k}(x, 0), \\ &\quad k = 1, 2, \dots, \quad ((x, y) \in H_1). \end{aligned} \quad (110)$$

Denote

$$\begin{aligned} \alpha_{k,14}(x) &= \frac{(x - x_1)^k}{k!} \left(\frac{x}{x_1} \right)^{k+1}, \\ \beta_{k,14}(x) &= \frac{x^k}{k!} \left(\frac{x - x_1}{-x_1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (111)$$

Define $N_1^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} N_1^{(0)}(x, y) &= (S_1^{(l_r)}(x_1, y) - M_1^{(2r)}(x_1, y)) \alpha_{0,14}(x) \\ N_1^{(k)}(x, y) &= N_1^{(k-1)}(x, y) + \alpha_{k,14}(x) \\ &\quad \times \left(\frac{\partial^k (S_1^{(4r)} - M_1^{(2r)})}{\partial x^k}(x_1, y) \right. \\ &\quad \left. - \frac{\partial^k N_1^{(k-1)}}{\partial x^k}(x_1, y) \right) \\ &\quad - \beta_{k,14}(x) \frac{\partial^k N_1^{(k-1)}}{\partial x^k}(0, y), \\ &\quad k = 1, 2, \dots, \quad ((x, y) \in H_1). \end{aligned} \quad (112)$$

(ii) Denote

$$\begin{aligned} \alpha_{k,21}(x) &= \frac{(x - x_2)^k}{k!} \left(\frac{1 - x}{1 - x_2} \right)^{k+1}, \\ \beta_{k,21}(x) &= \frac{(x - 1)^k}{k!} \left(\frac{x_2 - x}{x_2 - 1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (113)$$

Define $M_2^{(\tau_r)}(x, y)$ by induction as follows:

$$\begin{aligned} M_2^{(0)}(x, y) &= S_1^{(l_r)}(x_2, y) \alpha_{0,21}(x), \\ M_2^{(k)}(x, y) &= M_2^{(k-1)}(x, y) + \alpha_{k,21}(x) \\ &\quad \times \left(\frac{\partial^k S_1^{(l_r)}}{\partial x^k}(x_2, y) \right. \\ &\quad \left. - \frac{\partial^k M_2^{(k-1)}}{\partial x^k}(x_2, y) \right) \\ &\quad - \beta_{k,21}(x) \frac{\partial^k M_2^{(k-1)}}{\partial x^k}(1, y), \\ k &= 1, 2, \dots \quad ((x, y) \in H_2). \end{aligned} \quad (114)$$

Denote

$$\begin{aligned} \alpha_{k,22}(y) &= \frac{(y - y_2)^k}{k!} \left(\frac{y}{y_2} \right)^{k+1}, \\ \beta_{k,22}(y) &= \frac{y^k}{k!} \left(\frac{y - y_2}{-y_2} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (115)$$

Define $N_2^{(r)}(x, y)$ by induction as follows:

$$\begin{aligned} N_2^{(0)}(x, y) &= (S_2^{(l_r)} - M_2^{(\tau_r)})(x, y_2) \alpha_{0,22}(y), \\ N_2^{(k)}(x, y) &= N_2^{(k-1)}(x, y) + \alpha_{k,22}(y) \\ &\quad \times \left(\frac{\partial^k (S_2^{(l_r)} - M_2^{(\tau_r)})}{\partial y^k}(x, y_2) \right. \\ &\quad \left. - \frac{\partial^k N_2^{(k-1)}}{\partial y^k}(x, y_2) \right) \\ &\quad - \beta_{k,22}(y) \frac{\partial^k N_2^{(k-1)}}{\partial y^k}(x, 0), \\ k &= 1, 2, \dots \quad ((x, y) \in H_2). \end{aligned} \quad (116)$$

(iii) Denote

$$\begin{aligned} \alpha_{k,31}(y) &= \frac{(y - y_3)^k}{k!} \left(\frac{1 - y}{1 - y_3} \right)^{k+1}, \\ \beta_{k,31}(y) &= \frac{(y - 1)^k}{k!} \left(\frac{y_3 - y}{y_3 - 1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (117)$$

Define $M_3^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} M_3^{(0)}(x, y) &= S_2^{(l_r)}(x, y_3) \alpha_{0,31}(y), \\ M_3^{(k)}(x, y) &= M_3^{(k-1)}(x, y) + \alpha_{k,31}(y) \\ &\quad \times \left(\frac{\partial^k S_2^{(l_r)}}{\partial y^k}(x, y_3) - \frac{\partial^k M_3^{(k-1)}}{\partial y^k}(x, y_3) \right) \\ &\quad - \beta_{k,31}(y) \frac{\partial^k M_3^{(k-1)}}{\partial y^k}(x, 1), \\ k &= 1, 2, \dots \quad ((x, y) \in H_3). \end{aligned} \quad (118)$$

Denote

$$\begin{aligned} \alpha_{k,32}(x) &= \frac{(x - x_3)^k}{k!} \left(\frac{1 - x}{1 - x_3} \right)^{k+1}, \\ \beta_{k,32}(x) &= \frac{(x - 1)^k}{k!} \left(\frac{x_3 - x}{x_3 - 1} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (119)$$

Define $N_3^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} N_3^{(0)}(x, y) &= (S_3^{(l_r)} - M_3^{(\tau_r)})(x_3, y) \alpha_{0,32}(x), \\ N_3^{(k)}(x, y) &= N_3^{(k-1)}(x, y) + \alpha_{k,32}(x) \\ &\quad \times \left(\frac{\partial^k (S_3^{(l_r)} - M_3^{(\tau_r)})}{\partial y^k}(x_3, y) \right. \\ &\quad \left. - \frac{\partial^k N_3^{(k-1)}}{\partial x^k}(x_3, y) \right) \\ &\quad - \beta_{k,32}(x) \frac{\partial^k N_3^{(k-1)}}{\partial x^k}(1, y), \\ k &= 1, 2, \dots \quad ((x, y) \in H_3). \end{aligned} \quad (120)$$

(iv) Denote

$$\begin{aligned} \alpha_{k,41}(x) &= \frac{(x - x_4)^k}{k!} \left(\frac{x}{x_4} \right)^{k+1}, \\ \beta_{k,41}(x) &= \frac{(x - x_4)^k}{k!} \left(\frac{x}{-x_4} \right)^{k+1}, \quad k = 0, 1, \dots \end{aligned} \quad (121)$$

Define $M_4^{(k)}(x, y)$ by induction as follows:

$$\begin{aligned} M_4^{(0)}(x, y) &= S_3^{(l_r)}(x_4, y) \alpha_{0,41}(x), \\ M_4^{(k)}(x, y) &= M_4^{(k-1)}(x, y) + \alpha_{k,41}(x) \\ &\quad \times \left(\frac{\partial^k S_3^{(l_r)}}{\partial x^k}(x_4, y) - \frac{\partial^k M_4^{(k-1)}}{\partial x^k}(x_4, y) \right) \\ &\quad - \beta_{k,41}(x, y) \frac{\partial^k M_4^{(k-1)}}{\partial x^k}(0, y), \\ k &= 1, 2, \dots \quad ((x, y) \in H_4). \end{aligned} \quad (122)$$

Denote

$$\alpha_{k,42}(y) = \frac{(y-y_4)^k}{k!} \left(\frac{1-y}{1-y_4} \right)^{k+1},$$

$$\beta_{k,42}(y) = \frac{(y-1)^k}{k!} \left(\frac{y_4-y}{y_4-1} \right)^{k+1}, \quad k = 0, 1, \dots$$
(123)

Define $N_4^{(k)}(x, y)$ by induction as follows:

$$N_4^{(0)}(x, y) = (S_4^{(l_r)} - M_4^{(\tau_r)})(x, y_4) \alpha_{0,42}(y),$$

$$N_4^{(k)}(x, y) = N_4^{(k-1)}(x, y) + \alpha_{k,42}(y) \times \left(\frac{\partial^k (S_4^{(l_r)} - M_4^{(\tau_r)})}{\partial y^k}(x, y_4) - \frac{\partial^k N_4^{(k-1)}}{\partial y^k}(x, y_4) \right) - \beta_{k,42}(x, y) \frac{\partial^k N_4^{(k-1)}}{\partial x^k}(x, 1),$$

$$k = 1, 2, \dots \quad ((x, y) \in H_4).$$
(124)

5. Corollaries

By using the extension method given in Section 3, we discuss the two important special cases.

5.1. Smooth Extensions of Functions on a Kind of Domains. Let Ω be a trapezoid with two curved sides:

$$\Omega = \{(x, y) : x_1 \leq x \leq x_2, \eta(x) \leq y \leq \xi(x)\}, \quad (125)$$

where $L_1 < \eta(x) < \xi(x) < L_2$ ($x_1 \leq x \leq x_2$), $\eta, \xi \in C^m([x_1, x_2])$. Denote the rectangle $D = [x_1, x_2] \times [L_1, L_2]$. Then $D = G_1 \cup \Omega \cup G_2$, where G_1 and G_2 are both trapezoids with a curved side:

$$G_1 = \{(x, y) : x_1 \leq x \leq x_2, L_1 \leq y \leq \eta(x)\},$$

$$G_2 = \{(x, y) : x_1 \leq x \leq x_2, \xi(x) \leq y \leq L_2\}.$$
(126)

Suppose that $f \in C^q(\Omega)$ (q is a nonnegative integer). We will smoothly extend f from Ω to the trapezoids G_1 and G_2 with a curved side, respectively, as in Section 3.2, such that the extension function F is smooth on the rectangle D . Moreover, we will give a precise formula. It shows that the index of smoothness of F depends on not only smoothness of f but also smoothness of η, ξ .

Denote $a_{0,1}(x, y) = (y - L_1)/(\eta(x) - L_1)$ and

$$a_{k,1}(x, y) = \frac{(y - \eta(x))^k}{k!} \left(\frac{y - L_1}{\eta(x) - L_1} \right)^{k+1},$$

$$k = 1, 2, \dots \quad (x_1 \leq x \leq x_2, y \in \mathbb{R}).$$
(127)

We define $\{S_1^{(k)}(x, y)\}$ on G_1 as follows. Let

$$S_1^{(0)}(x, y) = f(x, \eta(x)) a_{1,0}(x, y) \quad ((x, y) \in G_1), \quad (128)$$

and let k_0 be the maximal integer satisfying $1 + 2 + \dots + k_0 \leq q$. For $k = 1, 2, \dots, k_0$, we define

$$S_1^{(k)}(x, y) = S_1^{(k-1)}(x, y) + a_{1,k}(x, y) \times \left(\frac{\partial^k f}{\partial y^k}(x, \eta(x)) - \frac{\partial^k S_1^{(k-1)}}{\partial y^k}(x, \eta(x)) \right).$$
(129)

Then $S_1^{(k)} \in C^{\lambda_k}(G_1)$, where $\lambda_k = \min\{q - 1 - 2 - \dots - k, m\}$.

Denote $a_{0,2}(x, y) = (L_2 - y)/(L_2 - \xi(x))$ and

$$a_{k,2}(x, y) = \frac{(y - \xi(x))^k}{k!} \left(\frac{L_2 - y}{L_2 - \xi(x)} \right)^{k+1},$$

$$k = 1, 2, \dots \quad (x_1 \leq x \leq x_2, y \in \mathbb{R}).$$
(130)

We define $\{S_2^{(k)}(x, y)\}$ on G_2 as follows. Let

$$S_2^{(0)}(x, y) = f(x, \xi(x)) a_{0,2}(x, y) \quad ((x, y) \in G_2). \quad (131)$$

For $k = 1, 2, \dots, k_0$, define

$$S_2^{(k)}(x, y) = S_2^{(k-1)}(x, y) + a_{k,2}(x, y) \times \left(\frac{\partial^k f}{\partial y^k}(x, \xi(x)) - \frac{\partial^k S_2^{(k-1)}}{\partial y^k}(x, \xi(x)) \right),$$

$$((x, y) \in G_2).$$
(132)

Then $S_2^{(k)} \in C^{\lambda_k}(G_2)$, where λ_k is stated as above.

An argument similar to Lemmas 5 and 6 shows that, for $0 \leq k \leq k_0$ and $0 \leq i + j \leq \min\{k, \lambda_k\}$,

$$\frac{\partial^{i+j} S_1^{(k)}}{\partial x^i \partial y^j}(x, \eta(x)) = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, \eta(x)),$$

$$\frac{\partial^{i+j} S_2^{(k)}}{\partial x^i \partial y^j}(x, \xi(x)) = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, \xi(x)),$$

$$(x_1 \leq x \leq x_2).$$
(133)

A direct calculation shows that the number

$$\tau(q, m) = \min \left\{ \left\lceil \sqrt{2q + \frac{9}{4}} - \frac{3}{2} \right\rceil, m \right\} \quad (134)$$

is the maximal value of integers k satisfying $k \leq \lambda_k$, where $[\cdot]$ expresses the integral part. So $\tau(q, m) \leq \lambda_{\tau(q, m)}$.

By (133), we get that, for $0 \leq i + j \leq \tau(q, m)$,

$$\frac{\partial^{i+j} S_1^{(\tau(q, m))}}{\partial x^i \partial y^j}(x, \eta(x)) = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, \eta(x)),$$

$$\frac{\partial^{i+j} S_2^{(\tau(q, m))}}{\partial x^i \partial y^j}(x, \xi(x)) = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x, \xi(x)),$$

$$(x_1 \leq x \leq x_2).$$
(135)

Note that

$$S_1^{(\tau(q,m))} \in C^{\lambda_{\tau(q,m)}}(G_1), \quad S_2^{(\tau(q,m))} \in C^{\lambda(q,m)}(G_2), \quad (136)$$

$$\tau(q, m) \leq \lambda_{\tau(q,m)} \leq q,$$

and the assumption $f \in C^q(\Omega)$. Now we define a function on D by

$$F_{q,m}(x, y) = \begin{cases} f(x, y), & (x, y) \in \Omega, \\ S_1^{(\tau(q,m))}(x, y), & (x, y) \in G_1, \\ S_2^{(\tau(q,m))}(x, y), & (x, y) \in G_2. \end{cases} \quad (137)$$

From this and (135), we have $F_{q,m} \in C^{\tau(q,m)}(D)$. This implies the following theorem.

Theorem 12. Let the domain Ω and the rectangle D be stated as above. If $f \in C^q(\Omega)$, then the function $F_{q,m}(x, y)$, defined in (137), is a smooth extension of f from Ω to D and $F_{q,m} \in C^{\tau(q,m)}(D)$, where $\tau(q, m)$ is stated in (134).

Especially, for $q = 0$ and $m \geq 0$, we have $\tau(q, m) = 0$, and so $F_{0,m} \in C(D)$; for $q = 2$ and $m \geq 1$, we have $\tau(q, m) = 1$, and so $F_{2,1} \in C^1(D)$; for $q = 5$ and $m \geq 2$, we have $\tau(q, m) = 2$, and so $F_{5,2} \in C^2(D)$.

5.2. Smooth Extensions of Univariate Functions on Closed Intervals. Let $f \in C^q([x_1, x_2])$ and $[x_1, x_2] \subset (0, 1)$. In order to extend smoothly f from $[x_1, x_2]$ to $[0, x_1]$, we construct two polynomials

$$a_0^{(k)}(x) = \frac{(x - x_1)^k}{k!} \left(\frac{x}{x_1} \right)^{k+1}, \quad (138)$$

$$b_0^{(k)}(x) = \frac{x^k}{k!} \left(\frac{x_1 - x}{x_1} \right)^{k+1}, \quad k = 0, 1, \dots$$

Define $S_0^{(0)}(x) = f(x_1)(x/x_1)$ and for $k = 1, \dots, q$,

$$S_0^{(k)}(x) = S_0^{(k-1)}(x) - a_0^{(k)}(x) \left(f^{(k)}(x_1) - (S_0^{(k-1)})^{(k)}(x_1) \right) - b_0^{(k)}(x) (S_0^{(k-1)})^{(k)}(0) \quad (0 \leq x \leq x_1). \quad (139)$$

Then $S_0^{(q)}(x)$ is a polynomial of degree $\leq 2q + 1$.

Similar to the proof of Lemma 5, we get

$$(S_0^{(q)})^{(k)}(0) = 0, \quad (S_0^{(q)})^{(k)}(x_1) = f^{(k)}(x_1), \quad (140)$$

$$k = 0, 1, \dots, q.$$

It is also easy to check directly them.

Again extend smoothly f from $[x_1, x_2]$ to $[x_2, 1]$, we construct two polynomials

$$a_1^{(k)}(x) = \frac{(x - x_2)^k}{k!} \left(\frac{1 - x}{1 - x_2} \right)^{k+1}, \quad (141)$$

$$b_1^{(k)}(x) = \frac{(x - 1)^k}{k!} \left(\frac{x_2 - x}{x_2 - 1} \right)^{k+1}, \quad k = 0, 1, \dots$$

Define $S_1^{(0)}(x) = f(x_2)((1 - x)/(1 - x_2))$ and for $k = 1, \dots, q$,

$$S_1^{(k)}(x) = S_1^{(k-1)}(x) - a_1^{(k)}(x) \left(f^{(k)}(x_2) - (S_1^{(k-1)})^{(k)}(x_2) \right) - b_1^{(k)}(x) (S_1^{(k-1)})^{(k)}(1) \quad (x_2 \leq x \leq 1). \quad (142)$$

Then $S_1^{(q)}(x)$ is a polynomial of degree $\leq 2q + 1$. Similar to the proof of Lemma 5, we get

$$(S_1^{(q)})^{(k)}(x_2) = f^{(k)}(x_2), \quad (143)$$

$$(S_1^{(q)})^{(k)}(1) = 0 \quad (k = 0, 1, \dots, q).$$

Therefore, we obtain the smooth extension F from $[x_1, x_2]$ to $[0, 1]$ by

$$F(x) = \begin{cases} f(x), & x \in [x_1, x_2], \\ S_0^{(q)}(x), & x \in [0, x_1], \\ S_1^{(q)}(x), & x \in [x_2, 1], \end{cases} \quad (144)$$

where $S_0^{(q)}(x)$ and $S_1^{(q)}(x)$ are polynomials of degree $2q + 1$ defined as above, and $F \in C^q([0, 1])$ and $F^{(l)}(0) = F^{(l)}(1) = 0$ ($l = 0, 1, \dots, q$). From this, we get the following.

Theorem 13. Let $f \in C^q([x_1, x_2])$ and $[x_1, x_2] \subset (0, 1)$. Then there exists a function $F \in C^q([0, 1])$ satisfying $F(x) = f(x)$ ($x_1 \leq x \leq x_2$) and $F^{(l)}(0) = F^{(l)}(1) = 0$ ($l = 0, 1, \dots, q$).

Let $f \in C^q([x_1, x_2])$ and $[x_1, x_2] \subset (0, 1)$, and let F be the smooth extension of f from $[x_1, x_2]$ to $[0, 1]$ which is stated as in Theorem 12. Let F_p be the 1-periodic extension satisfying $F_p(x + n) = F(x)$ ($0 \leq x \leq 1$, $n \in \mathbb{Z}$). Then $F_p \in C^q(\mathbb{R})$ and $F_p(x) = f(x)$ ($x \in [x_1, x_2]$). We expand $F(x)$ into the Fourier series which converges fast. From this, we get trigonometric approximation of $f \in C^q([x_1, x_2])$. We also may do odd extension or even extension of F from $[0, 1]$ to $[-1, 1]$, and then doing periodic extension, we get the odd periodic extension $F_p^o \in C^q(\mathbb{R})$ or the even periodic extension $F_p^e \in C^q(\mathbb{R})$. We expand F_p^o or F_p^e into the sine series and the cosine series, respectively. From this, we get the sine polynomial approximation and the cosine polynomial approximation of f on $[x_1, x_2]$. For $F \in C^q(x)$ ($x \in [0, 1]$), we pad zero in the outside of $[0, 1]$ and then the obtained function $F^c \in C^q(\mathbb{R})$. We expand F^c into a wavelet series which converges fast. By the moment theorem, a lot of wavelet coefficients are equal to zero. From this, we get wavelet approximation of $f \in C^q([x_1, x_2])$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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