Research Article

Spaces of Ideal Convergent Sequences

M. Mursaleen¹ and Sunil K. Sharma²

¹ Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

² Department of Mathematics, Model Institute of Engineering & Technology, Kot Bhalwal, J&K 181122, India

Correspondence should be addressed to M. Mursaleen; mursaleenm@gmail.com

Received 20 August 2013; Accepted 26 November 2013; Published 28 January 2014

Academic Editors: F. Başar and J. Xu

Copyright © 2014 M. Mursaleen and S. K. Sharma. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In the present paper, we introduce some sequence spaces using ideal convergence and Musielak-Orlicz function $\mathcal{M} = (M_k)$. We also examine some topological properties of the resulting sequence spaces.

1. Introduction and Preliminaries

The notion of ideal convergence was first introduced by Kostyrko et al. [1] as a generalization of statistical convergence which was further studied in topological spaces by Das et al. see [2]. More applications of ideals can be seen in [2, 3]. We continue in this direction and introduce I -convergence of generalized sequences with respect to Musielak-Orlicz function.

A family $\mathcal{I} \subset 2^X$ of subsets of a nonempty set X is said to be an ideal in X if

- (1) $\emptyset \in \mathcal{I}$,
- (2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
- (3) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$,

while an admissible ideal $\mathcal{I}$ of X further satisfies $\{x\} \in \mathcal{I}$ for each $x \in X$; see [1]. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be I -convergent to $x \in X$. If for each $\epsilon > 0$, the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to $\mathcal{I}$; see [1]. For more details about ideal convergent sequence spaces, see [4–10] and references therein.

Mursaleen and Noman [11] introduced the notion of λ -convergent and λ -bounded sequences as follows.

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity; that is,

$$0 < \lambda_0 < \lambda_1 < \cdots, \quad \lambda_k \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (1)$$

The sequence $x = (x_k) \in w$ is λ -convergent to the number L , called the λ -limit of x , if $\Lambda_m(x) \rightarrow L$, as $m \rightarrow \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) x_k. \quad (2)$$

The sequence $x = (x_k) \in w$ is λ -bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [11] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^m (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0. \quad (3)$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) (x_k - a) \right| = 0, \quad (4)$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ -convergent to a .

Let X be a linear metric space. A function $p : X \rightarrow \mathbb{R}$ is called paranorm if

- (1) $p(x) \geq 0$, for all $x \in X$,
- (2) $p(-x) = p(x)$, for all $x \in X$,
- (3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,

- (4) if (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, then $p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$.

A paranorm p for which $p(x) = 0$ implies that $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [12, Theorem 10.4.2, P-183]). For more details about sequence spaces, see [13–15] and references therein.

An Orlicz function M is a function which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$. Then,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\} \quad (5)$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \quad (6)$$

It is shown in [16] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). The Δ_2 -condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see; [17, 18]. A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup \{ |v| u - (M_k) : u \geq 0 \}, \quad k = 1, 2, \dots, \quad (7)$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$\begin{aligned} t_{\mathcal{M}} &= \{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \}, \\ h_{\mathcal{M}} &= \{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \}, \end{aligned} \quad (8)$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}. \quad (9)$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}, \quad (10)$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{M}}(kx)) : k > 0 \right\}. \quad (11)$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and let $p = (p_k)$ be a bounded sequence of positive real numbers. We define the following sequence spaces:

$$\begin{aligned} c^I(\mathcal{M}, \Lambda, p) &= \left\{ x = (x_k) \in w : I - \lim_k M_k \left(\frac{|\Lambda_k(x) - L|}{\rho} \right)^{p_k} = 0, \right. \\ &\quad \left. \text{for some } L \text{ and } \rho > 0 \right\}, \end{aligned}$$

$$\begin{aligned} c_0^I(\mathcal{M}, \Lambda, p) &= \left\{ x = (x_k) \in w : I - \lim_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} = 0, \right. \\ &\quad \left. \text{for some } \rho > 0 \right\}, \end{aligned}$$

$$\begin{aligned} l_{\infty}(\mathcal{M}, \Lambda, p) &= \left\{ x = (x_k) \in w : \sup_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} < \infty, \right. \\ &\quad \left. \text{for some } \rho > 0 \right\}. \end{aligned} \quad (12)$$

We can write

$$\begin{aligned} m^I(\mathcal{M}, \Lambda, p) &= c^I(\mathcal{M}, \Lambda, p) \cap l_{\infty}(\mathcal{M}, \Lambda, p), \\ m_0^I(\mathcal{M}, \Lambda, p) &= c_0^I(\mathcal{M}, \Lambda, p) \cap l_{\infty}(\mathcal{M}, \Lambda, p). \end{aligned} \quad (13)$$

If we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} c^I(\mathcal{M}, \Lambda) &= \left\{ x = (x_k) \in w : I - \lim_k M_k \left(\frac{|\Lambda_k(x) - L|}{\rho} \right) = 0, \right. \\ &\quad \left. \text{for some } L \text{ and } \rho > 0 \right\}, \end{aligned}$$

$$\begin{aligned} c_0^I(\mathcal{M}, \Lambda) &= \left\{ x = (x_k) \in w : I - \lim_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right) = 0, \right. \\ &\quad \left. \text{for some } \rho > 0 \right\}, \end{aligned}$$

$$\begin{aligned} l_{\infty}(\mathcal{M}, \Lambda) &= \left\{ x = (x_k) \in w : \sup_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right) < \infty, \right. \\ &\quad \left. \text{for some } \rho > 0 \right\}. \end{aligned} \quad (14)$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \leq D \{ |a_k|^{p_k} + |b_k|^{p_k} \}, \quad (15)$$

for all k , and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some ideal convergent sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

2. Main Results

Theorem 1. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then, the spaces $c^I(\mathcal{M}, \Lambda, p)$, $c_0^I(\mathcal{M}, \Lambda, p)$, $m^I(\mathcal{M}, \Lambda, p)$, and $m_0^I(\mathcal{M}, \Lambda, p)$ are linear.

Proof. Let $x, y \in c^I(\mathcal{M}, \Lambda, p)$ and let α, β be scalars. Then, there exist positive numbers ρ_1 and ρ_2 such that

$$\begin{aligned} I - \lim_k M_k \left(\frac{|\Lambda_k(x) - L_1|}{\rho_1} \right)^{p_k} &= 0, \quad \text{for some } L_1 \in \mathbb{C}, \\ I - \lim_k M_k \left(\frac{|\Lambda_k(y) - L_2|}{\rho_2} \right)^{p_k} &= 0, \quad \text{for some } L_2 \in \mathbb{C}. \end{aligned} \quad (16)$$

For a given $\epsilon > 0$, we have

$$\begin{aligned} D_1 &= \left\{ k \in \mathbb{N} : M_k \left(\frac{|\Lambda_k(x) - L_1|}{\rho_1} \right)^{p_k} \leq \epsilon \right\}, \\ D_2 &= \left\{ k \in \mathbb{N} : M_k \left(\frac{|\Lambda_k(y) - L_2|}{\rho_2} \right)^{p_k} \leq \epsilon \right\}. \end{aligned} \quad (17)$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function, so by using inequality (15), we have

$$\begin{aligned} &\lim_k M_k \left(\frac{|\Lambda_k((\alpha x + \beta y) - (\alpha L_1 + \beta L_2))|}{\rho_3} \right)^{p_k} \\ &\leq \lim_k M_k \left(\frac{|\alpha| |\Lambda_k(x) - L_1|}{\rho_3} + \frac{|\beta| |\Lambda_k(y) - L_2|}{\rho_3} \right)^{p_k} \\ &\leq \lim_k M_k \left(\frac{|\Lambda_k(x) - L_1|}{\rho_1} \right)^{p_k} \\ &\quad + \lim_k M_k \left(\frac{|\Lambda_k(y) - L_2|}{\rho_2} \right)^{p_k}. \end{aligned} \quad (18)$$

Now, by (17), we have

$$\begin{aligned} &\left\{ k \in \mathbb{N} : \lim_k M_k \left(\frac{|\Lambda_k((\alpha x + \beta y) - (\alpha L_1 + \beta L_2))|}{\rho_3} \right)^{p_k} \right. \\ &\quad \left. > \epsilon \right\} \subset D_1 \cup D_2. \end{aligned} \quad (19)$$

Therefore, $\alpha x + \beta y \in c^I(\mathcal{M}, \Lambda, p)$. Hence $c^I(\mathcal{M}, \Lambda, p)$ is a linear space. Similarly, we can prove that $c_0^I(\mathcal{M}, \Lambda, p)$, $m^I(\mathcal{M}, \Lambda, p)$, and $m_0^I(\mathcal{M}, \Lambda, p)$ are linear spaces. $\square$

Theorem 2. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then,

$$c_0^I(\mathcal{M}, \Lambda, p) \subset c^I(\mathcal{M}, \Lambda, p) \subset l_\infty(\mathcal{M}, \Lambda, p). \quad (20)$$

Proof. Let $x \in c^I(\mathcal{M}, \Lambda, p)$. Then, there exist $L \in \mathbb{C}$ and $\rho > 0$ such that

$$I - \lim_k M_k \left(\frac{|\Lambda_k(x) - L|}{\rho} \right)^{p_k} = 0. \quad (21)$$

We have

$$M_k \left(\frac{|\Lambda_k(x)|}{2\rho} \right)^{p_k} \leq \frac{1}{2} M_k \left(\frac{|\Lambda_k(x) - L|}{\rho} \right)^{p_k} + M_k \left(\frac{|L|}{2\rho} \right)^{p_k}. \quad (22)$$

Taking supremum over k on both sides, we get $x \in l_\infty(\mathcal{M}, \Lambda, p)$. The inclusion $c_0^I(\mathcal{M}, \Lambda, p) \subset c^I(\mathcal{M}, \Lambda, p)$ is obvious. Thus,

$$c_0^I(\mathcal{M}, \Lambda, p) \subset c^I(\mathcal{M}, \Lambda, p) \subset l_\infty(\mathcal{M}, \Lambda, p). \quad (23)$$

This completes the proof of the theorem. $\square$

Theorem 3. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then, $l_\infty(\mathcal{M}, \Lambda, p)$ is a paranormed space with paranorm defined by

$$g(x) = \inf \left\{ \rho > 0 : \sup_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} \leq 1 \right\}. \quad (24)$$

Proof. It is clear that $g(x) = g(-x)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Let us take $x, y \in l_\infty(\mathcal{M}, \Lambda, p)$. Let

$$\begin{aligned} B(x) &= \left\{ \rho > 0 : \sup_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} \leq 1 \right\}, \\ B(y) &= \left\{ \rho > 0 : \sup_k M_k \left(\frac{|\Lambda_k(y)|}{\rho} \right)^{p_k} \leq 1 \right\}. \end{aligned} \quad (25)$$

Let $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned} & \sup_k M_k \left(\frac{|\Lambda_k(x+y)|}{\rho} \right) \\ & \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_k M_k \left(\frac{|\Lambda_k(x)|}{\rho_1} \right) \\ & \quad + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_k M_k \left(\frac{|\Lambda_k(y)|}{\rho_2} \right). \end{aligned} \quad (26)$$

Thus, $\sup_k M_k(|\Lambda(x+y)|/(\rho_1 + \rho_2))^{p_k} \leq 1$ and

$$\begin{aligned} g(x+y) & \leq \inf \{(\rho_1 + \rho_2) > 0 : \rho_1 \in B(x), \rho_2 \in B(y)\} \\ & \leq \inf \{\rho_1 > 0 : \rho_1 \in B(x)\} \\ & \quad + \inf \{\rho_2 > 0 : \rho_2 \in B(y)\} \\ & = g(x) + g(y). \end{aligned} \quad (27)$$

Let $\sigma^s \rightarrow \sigma$, where $\sigma, \sigma^s \in \mathbb{C}$ and let $g(x^s - x) \rightarrow 0$ as $s \rightarrow \infty$. We have to show that $g(\sigma^s x^s - \sigma x) \rightarrow 0$ as $s \rightarrow \infty$. Let

$$\begin{aligned} B(x^s) & = \left\{ \rho_s > 0 : \sup_k M_k \left(\frac{|\Lambda_k(x^s)|}{\rho_s} \right)^{p_k} \leq 1 \right\}, \\ B(x^s - x) & = \left\{ \rho'_s > 0 : \sup_k M_k \left(\frac{|\Lambda_k(x^s - x)|}{\rho'_s} \right)^{p_k} \leq 1 \right\}. \end{aligned} \quad (28)$$

If $\rho_s \in B(x^s)$ and $\rho'_s \in B(x^s - x)$, then we observe that

$$\begin{aligned} & M_k \left(\frac{|\Lambda_k(\sigma^s x^s - \sigma x)|}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} \right) \\ & \leq M_k \left(\frac{|\Lambda_k(\sigma^s x^s - \sigma x^s)|}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} + \frac{|\Lambda_k(\sigma x^s - \sigma x)|}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} \right) \\ & \leq \frac{|\sigma^s - \sigma| \rho_s}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} M_k \left(\frac{|\Lambda_k(x^s)|}{\rho_s} \right) \\ & \quad + \frac{|\sigma| \rho'_s}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} M_k \left(\frac{|\Lambda_k(x^s - x)|}{\rho'_s} \right). \end{aligned} \quad (29)$$

From the above inequality, it follows that

$$M_k \left(\frac{|\Lambda_k(\sigma^s x^s - \sigma x)|}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} \right)^{p_k} \leq 1 \quad (30)$$

and, consequently,

$$\begin{aligned} g(\sigma^s x^s - \sigma x) & \leq \inf \left\{ (\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|) > 0 : \right. \\ & \quad \left. \rho_s \in B(x^s), \rho'_s \in B(x^s - x) \right\} \\ & \leq (|\sigma^s - \sigma|) > 0 \inf \{\rho > 0 : \rho \in B(x^s)\} \\ & \quad + (|\sigma|) > 0 \inf \left\{ (\rho'_s)^{p_n/H} : \rho'_s \in B(x^s - x) \right\} \\ & \rightarrow 0 \quad \text{as } s \rightarrow \infty. \end{aligned} \quad (31)$$

This completes the proof. $\square$

Theorem 4. Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ be Musielak-Orlicz functions that satisfy the Δ_2 -condition. Then,

- (i) $Z(\mathcal{M}'', \Lambda, p) \subseteq Z(\mathcal{M}' \circ \mathcal{M}'', \Lambda, p)$,
- (ii) $Z(\mathcal{M}', \Lambda, p) \cap Z(\mathcal{M}'', \Lambda, p) \subseteq Z(\mathcal{M}' + \mathcal{M}'', \Lambda, p)$ for $Z = c^I, c_0^I, m^I, m_0^I$.

Proof. (i) Let $x \in c_0^I(\mathcal{M}'', \Lambda, p)$. Then, there exists $\rho > 0$ such that

$$I - \lim_k \mathcal{M}''_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} = 0. \quad (32)$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M'_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_k = M'_k(|\Lambda_k(x)|/\rho)^{p_k}$ and consider

$$\lim_{\substack{0 \leq y_k \leq \delta \\ k \in \mathbb{N}}} M'_k(y_k) = \lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} M'_k(y_k) + \lim_{\substack{y_k > \delta \\ k \in \mathbb{N}}} M'_k(y_k). \quad (33)$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2 -condition, we have

$$\lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} M'_k(y_k) \leq M'_k(2) \lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} (y_k). \quad (34)$$

For $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}. \quad (35)$$

Since $\mathcal{M}' = (M'_k)$ is nondecreasing and convex, it follows that

$$M'_k(y_k) < M'_k \left(1 + \frac{y_k}{\delta} \right) < \frac{1}{2} M'_k(2) + \frac{1}{2} \frac{M'_k(2y_k)}{\delta}. \quad (36)$$

Since $\mathcal{M}' = (M'_k)$ satisfies Δ_2 -condition, we have

$$M'_k(y_k) < \frac{1}{2} K \frac{y_k}{\delta} M'_k(2) + \frac{1}{2} K \frac{y_k}{\delta} M'_k(2) = K \frac{y_k}{\delta} M'_k(2). \quad (37)$$

Hence,

$$\lim_{\substack{y_k > \delta \\ k \in \mathbb{N}}} M'_k(y_k) \leq \max \left(1, K \delta^{-1} M'_k(2) \right) \lim_{\substack{y_k \leq \delta \\ k \in \mathbb{N}}} (y_k). \quad (38)$$

From (32), (34), and (38), we have $x = (x_k) \in c_0^I(\mathcal{M}' \circ \mathcal{M}'', \Lambda, p)$. Thus, $c_0^I(\mathcal{M}'', \Lambda, p) \subseteq c_0^I(\mathcal{M}' \circ \mathcal{M}'', \Lambda, p)$. Similarly, we can prove the other cases.

(ii) Let $x \in c_0^I(\mathcal{M}', \Lambda, p) \cap c_0^I(\mathcal{M}'', \Lambda, p)$. Then, there exists $\rho > 0$ such that

$$I - \lim_k M'_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} = 0, \quad (39)$$

$$I - \lim_k M''_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} = 0.$$

The rest of the proof follows from the following equality:

$$\begin{aligned} \lim_{k \in \mathbb{N}} (M'_k + M''_k) \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} \\ = \lim_{k \in \mathbb{N}} M'_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} + \lim_{k \in \mathbb{N}} M''_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k}. \end{aligned} \quad (40)$$

□

Corollary 5. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function which satisfies Δ_2 -condition. Then, $Z(p, \Lambda) \subseteq Z(\mathcal{M}, \Lambda, p)$ holds for $Z = c^I, c_0^I, m^I$, and m_0^I .

Proof. The proof follows from Theorem 3 by putting $M'_k(x) = x$ and $M''_k(x) = M_k(x) \forall x \in [0, \infty)$. □

Theorem 6. The spaces $c_0^I(\mathcal{M}, \Lambda, p)$ and $m_0^I(\mathcal{M}, \Lambda, p)$ are solid.

Proof. We will prove for the space $c_0^I(\mathcal{M}, \Lambda, p\Lambda)$. Let $x \in c_0^I(\mathcal{M}, \Lambda, p)$. Then, there exists $\rho > 0$ such that

$$I - \lim_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} = 0. \quad (41)$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1 \forall k \in \mathbb{N}$. Then, the result follows from the following inequality:

$$\lim_k M_k \left(\frac{|\Lambda_k(\alpha x)|}{\rho} \right)^{p_k} \leq \lim_k M_k \left(\frac{|\Lambda_k(x)|}{\rho} \right)^{p_k} \quad (42)$$

and this completes the proof. Similarly, we can prove for the space $m_0^I(\mathcal{M}, \Lambda, p)$. □

Corollary 7. The spaces $c_0^I(\mathcal{M}, \Lambda, p)$ and $m_0^I(\mathcal{M}, \Lambda, p)$ are monotone.

Proof. It is easy to prove, so we omit the details. □

Theorem 8. The spaces $c^I(\mathcal{M}, \Lambda, p)$ and $c_0^I(\mathcal{M}, \Lambda, p)$ are sequence algebra.

Proof. Let $x, y \in c_0^I(\mathcal{M}, \Lambda, p)$. Then,

$$I - \lim_k M_k \left(\frac{|\Lambda_k(x)|}{\rho_1} \right)^{p_k} = 0, \quad \text{for some } \rho_1 > 0, \quad (43)$$

$$I - \lim_k M_k \left(\frac{|\Lambda_k(y)|}{\rho_2} \right)^{p_k} = 0, \quad \text{for some } \rho_2 > 0.$$

Let $\rho = \rho_1 + \rho_2$. Then, we can show that

$$I - \lim_k M_k \left(\frac{|\Lambda_k(x \cdot y)|}{\rho} \right)^{p_k} = 0. \quad (44)$$

Thus, $(x \cdot y) \in c_0^I(\mathcal{M}, \Lambda, p)$. Hence, $c_0^I(\mathcal{M}, \Lambda, p)$ is a sequence algebra. Similarly, we can prove that $c^I(\mathcal{M}, \Lambda, p)$ is a sequence algebra. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] P. Kostyrko, T. Šalát, and W. Wilczyński, "I-convergence," *Real Analysis Exchange*, vol. 26, no. 2, pp. 669–685, 2000.
- [2] P. Das, P. Kostyrko, W. Wilczyński, and P. Malik, "I and I*-convergence of double sequences," *Mathematica Slovaca*, vol. 58, no. 5, pp. 605–620, 2008.
- [3] P. Das and P. Malik, "On the statistical and I variation of double sequences," *Real Analysis Exchange*, vol. 33, no. 2, pp. 351–363, 2008.
- [4] V. Kumar, "On I and I*-convergence of double sequences," *Mathematical Communications*, vol. 12, no. 2, pp. 171–181, 2007.
- [5] M. Mursaleen and A. Alotaibi, "On I-convergence in random 2-normed spaces," *Mathematica Slovaca*, vol. 61, no. 6, pp. 933–940, 2011.
- [6] M. Mursaleen, S. A. Mohiuddine, and O. H. H. Edely, "On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 603–611, 2010.
- [7] M. Mursaleen and S. A. Mohiuddine, "On ideal convergence of double sequences in probabilistic normed spaces," *Mathematical Reports*, vol. 12(62), no. 4, pp. 359–371, 2010.
- [8] M. Mursaleen and S. A. Mohiuddine, "On ideal convergence in probabilistic normed spaces," *Mathematica Slovaca*, vol. 62, no. 1, pp. 49–62, 2012.
- [9] A. Şahiner, M. Gürdal, S. Saltan, and H. Gunawan, "Ideal convergence in 2-normed spaces," *Taiwanese Journal of Mathematics*, vol. 11, no. 5, pp. 1477–1484, 2007.
- [10] B. C. Tripathy and B. Hazarika, "Some I-convergent sequence spaces defined by Orlicz functions," *Acta Mathematicae Applicatae Sinica*, vol. 27, no. 1, pp. 149–154, 2011.
- [11] M. Mursaleen and A. K. Noman, "On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ ," *Filomat*, vol. 25, no. 2, pp. 33–51, 2011.
- [12] A. Wilansky, *Summability through Functional Analysis*, vol. 85 of *North-Holland Mathematics Studies*, North-Holland Publishing, Amsterdam, The Netherlands, 1984.

- [13] K. Raj and S. K. Sharma, "Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function," *Acta Universitatis Sapientiae*, vol. 3, no. 1, pp. 97–109, 2011.
- [14] K. Raj and S. K. Sharma, "Some generalized difference double sequence spaces defined by a sequence of Orlicz-functions," *Cubo*, vol. 14, no. 3, pp. 167–190, 2012.
- [15] K. Raj and S. K. Sharma, "Some multiplier sequence spaces defined by a Musielak-Orlicz function in n -normed spaces," *New Zealand Journal of Mathematics*, vol. 42, pp. 45–56, 2012.
- [16] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.
- [17] L. Maligranda, *Orlicz Spaces and Interpolation*, vol. 5 of *Seminars in Mathematics*, Polish Academy of Science, 1989.
- [18] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, 1983.