Research Article

A New Mixed Element Method for a Class of Time-Fractional Partial Differential Equations

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A kind of new mixed element method for time-fractional partial differential equations is studied. The Caputo-fractional derivative of time direction is approximated by two-step difference method and the spatial direction is discretized by a new mixed element method, whose gradient belongs to the simple $(L^2(\Omega))^2$ space replacing the complex $\mathbf{H}(\text{div}; \Omega)$ space. Some a priori error estimates in L^2 -norm for the scalar unknown u and in $(L^2)^2$ -norm for its gradient σ . Moreover, we also discuss a priori error estimates in H^1 -norm for the scalar unknown u .

1. Introduction

In this paper, we consider the following time-fractional partial differential equation with initial and boundary conditions

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \Delta u &= f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, t) &= 0, \quad (\mathbf{x}, t) \in \partial\Omega \times \bar{J}, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \quad (1)$$

In (1), Ω is a bounded convex polygonal domain in R^d , $d \leq 2$ with Lipschitz continuous boundary $\partial\Omega$, $J = (0, T]$ is the time interval with $0 < T < \infty$. $u_0(\mathbf{x})$ and $f(\mathbf{x}, t)$ are given functions and $\partial^\alpha u(x, t)/\partial t^\alpha$ is Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha}, \quad 0 < \alpha < 1. \quad (2)$$

Fractional partial differential equations (PDEs) mainly include three types: PDEs with space fractional derivative, PDEs with time-fractional derivative, and PDEs with space-time-fractional derivative. So far, more and more people have started to pay attention to looking for the analytical and numerical solutions of fractional PDEs. In [1–14], authors

proposed a lot of finite difference methods for time, space, and space-time-fractional PDEs. Lin and Xu [15] proposed and analyzed the spectral methods for solving time-fractional diffusion equation. In [16, 17], authors presented local discontinuous Galerkin methods for fractional PDEs. Li et al. [18] discussed the detailed error estimate theories of finite element methods for nonlinear space-time-fractional differential equations with subdiffusion and superdiffusion. Jiang and Ma [19] developed high-order finite element methods for one-dimensional time fractional PDE (1). In [20, 21], the finite element methods were analyzed for space fractional PDEs. In [22–24], some time-fractional PDEs were solved by the finite element methods. Zhao and Li [25] presented the fractional difference/finite element approximations for the space-time-fractional telegraph equation.

Based on the summary of the above numerical methods for solving fractional PDEs, we can see that many numerical methods, such as finite difference methods, LDG methods, finite element methods, and spectral methods, have been studied and developed. However mixed finite element methods for solving fractional PDEs have not been proposed in the current literatures.

In recent years, a lot of mixed finite element methods have been proposed by many mathematical scholars. In [26, 27], authors presented a new mixed finite element method based on the linear elliptic equations. Compared to classical mixed

methods, this method has several distinct characteristics: the gradient of the new one belongs to the simple $(L^2(\Omega))^2$ space avoiding $\mathbf{H}(\text{div}; \Omega)$ space, the optimal a priori error estimates in H^1 -norm for the scalar unknown u can be obtained, the number of total degrees of freedom for this method is less than that for classical mixed methods, and the regularity requirements on the solution $\sigma = \nabla u$ are reduced. In view of the method's characteristics, the new mixed method has been developed to solve some integer-order partial differential equations, such as parabolic equation [28–31], Sobolev equation [32], fourth-order parabolic equation [33], and extended Fisher-Kolmogorov equation [34].

In this paper, our aim is to study the new numerical method based on the new mixed finite element method [26, 27] for solving a class of time-fractional PDEs. We derive a new discrete method for time-fractional derivative, formulate a fully discrete mixed finite element scheme, and prove some a priori error estimates in L^2 for the scalar unknown u and in $(L^2)^2$ -norm for its gradient σ . What is more, we derive an a priori error estimate in H^1 -norm for the scalar unknown u .

The layout of the paper is as follows. In Section 2, we introduce a new discrete method for the Caputo time-fractional derivative and give the “proof” of the truncation error's boundedness. In Section 3, we formulate a new mixed scheme for time-fractional PDE (1) and give the detailed proof for the a priori error estimates for two important variables based on fully discrete scheme. In Section 4, we give some remarks and extensions about the new mixed method and fractional PDEs. Throughout this paper, $C > 0$ will denote a generic constant independent of the space-time discretization parameter h and Δt . At the same time, we denote the natural inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ by $(\cdot, \cdot)$ with the corresponding norm $\|\cdot\|$. The other notations and definitions of Sobolev spaces as in [35, 36] are used.

2. Approximation of Time-Fractional Derivative

For the discretization for time-fractional derivative, let $0 = t_0 < t_1 < t_2 < \dots < t_M = T$ be a given partition of the time interval $[0, T]$ with step length $\Delta t = T/M$ and nodes $t_n = n\Delta t$, for some positive integer M . For a smooth function ϕ on $[0, T]$, define $\phi^n = \phi(t_n)$.

Lemma 1. The time-fractional derivative $\partial^\alpha u(x, t)/\partial t^\alpha$ at $t = t_{n+1}$ is approximated by the following: for $0 < \alpha < 1$

$$\begin{aligned} \frac{\partial^\alpha u(x, t_{n+1})}{\partial t^\alpha} &= \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n \left[(n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \right] \\ &\quad \times \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + E_0^{n+1}, \end{aligned} \quad (3)$$

where

$$\begin{aligned} E_0^{n+1} &= \frac{1}{\Gamma(1-\alpha)} \\ &\quad \times \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \left[(\tau - t_{k+1}) \frac{\partial^2 u(x, t_{k+1})}{\partial t^2} \right. \\ &\quad \left. + O((\tau - t_{k+1})^2) + O(\Delta t^2) \right] \\ &\quad \times \frac{d\tau}{(t_{n+1} - \tau)^\alpha}. \end{aligned} \quad (4)$$

Proof. As [37], using Taylor's expansion at time $t = t_{k+1}$, we can arrive at

$$\frac{\partial u(x, t_{k+1})}{\partial t} = \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + O(\Delta t^2). \quad (5)$$

By (5), Taylor's expansion, and some simple calculations of definite integral, we have

$$\begin{aligned} \frac{\partial^\alpha u(x, t_{n+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \frac{\partial u(x, \tau)}{\partial \tau} \frac{d\tau}{(t_{n+1} - \tau)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \left[\frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} \right. \\ &\quad \left. + \frac{\partial u(x, \tau)}{\partial \tau} - \frac{\partial u(x, t_{k+1})}{\partial t} \right. \\ &\quad \left. + O(\Delta t^2) \right] \frac{d\tau}{(t_{n+1} - \tau)^\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} \\ &\quad \times \int_{t_k}^{t_{k+1}} \frac{d\tau}{(t_{n+1} - \tau)^\alpha} + \frac{1}{\Gamma(1-\alpha)} \\ &\quad \times \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \left[(\tau - t_{k+1}) \frac{\partial^2 u(x, t_{k+1})}{\partial t^2} \right. \\ &\quad \left. + O((\tau - t_{k+1})^2) \right. \\ &\quad \left. + O(\Delta t^2) \right] \frac{d\tau}{(t_{n+1} - \tau)^\alpha} \\ &= \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n \left[(n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \right] \\ &\quad \times \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + E_0^{n+1}. \end{aligned} \quad (6)$$

So, the conclusion of Lemma 1 has been arrived at by the above calculations. $\square$

Remark 2. In a number of studies [18, 19, 22], the time-fractional derivative with α ($0 < \alpha < 1$) order is discretized by

$$\frac{\partial^\alpha u(x, t_{n+1})}{\partial t^\alpha} \approx \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n \left[(n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \right] \frac{u^{k+1} - u^k}{\Delta t}. \quad (7)$$

However, the study on the discrete formulation (3) for the Caputo fractional derivative with α ($0 < \alpha < 1$) order is fairly limited.

Lemma 3. The truncation error E_0^{n+1} is bounded by

$$|E_0^{n+1}| \leq \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} (M_\Delta \Delta t + C \Delta t^2), \quad (8)$$

where $M_\Delta = \sup_{t \in [0, T]} \{|\partial^2 u(x, t)/\partial t^2|\}$.

Proof. By the simple calculations, we arrive at

$$\begin{aligned} |E_0^{n+1}| &= \left| \frac{1}{\Gamma(1-\alpha)} \right. \\ &\quad \times \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \left[(\tau - t_{k+1}) \frac{\partial^2 u(x, t_{k+1})}{\partial t^2} \right. \\ &\quad \left. + O((\tau - t_{k+1})^2) \right. \\ &\quad \left. + O(\Delta t^2) \right] \frac{d\tau}{(t_{n+1} - \tau)^\alpha} \Big| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^n \int_{t_k}^{t_{k+1}} |\tau - t_{k+1}| \\ &\quad \times \left| \frac{\partial^2 u(x, t_{k+1})}{\partial t^2} \right| \frac{d\tau}{(t_{n+1} - \tau)^\alpha} \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \\ &\quad \times \sum_{k=0}^n \int_{t_k}^{t_{k+1}} |O((\tau - t_{k+1})^2) \\ &\quad + O(\Delta t^2)| \frac{d\tau}{(t_{n+1} - \tau)^\alpha} \\ &\leq \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} (M_\Delta \Delta t + O(\Delta t^2)). \end{aligned} \quad (9)$$

From (9), we can see easily that the conclusion for Lemma 3 is obtained. $\square$

3. New Mixed Finite Element Method

3.1. Mixed Formulation and Projections. In order to get the mixed scheme, we first split (1) into the following coupled system of two lower-order equations by introducing an auxiliary variable $\sigma = \nabla u$:

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \nabla \cdot \sigma &= f(x, t), \\ \sigma - \nabla u &= 0, \end{aligned} \quad (10)$$

Based on the new mixed method in [26, 27], using Green's formula, the new mixed weak formulation of (10) is to determine $\{u, \sigma\} : [0, T] \mapsto H_0^1 \times (L^2(\Omega))^2$ such that

$$\left(\frac{\partial^\alpha u(x, t)}{\partial t^\alpha}, v \right) + (\sigma, \nabla v) = (f, v), \quad \forall v \in H_0^1, \quad (11a)$$

$$(\sigma, \mathbf{w}) - (\nabla u, \mathbf{w}) = 0, \quad \forall \mathbf{w} \in (L^2(\Omega))^2. \quad (11b)$$

In order to formulate a new mixed finite element scheme, we first define the mixed finite element spaces. As shown in the literatures [26, 27], we choose the mixed space $(V_h, \mathbf{W}_h)$ with finite element pair $P_1 - P_0^2$ as

$$\begin{aligned} V_h &= \{v_h \in C^0(\Omega) \cap H_0^1 \mid v_h \in P_1(K), \forall K \in \mathcal{T}_h\}, \\ \mathbf{W}_h &= \{\mathbf{w}_h = (w_{1h}, w_{2h}) \in (L^2(\Omega))^2 \\ &\quad \mid w_{1h}, w_{2h} \in P_0(K), \forall K \in \mathcal{T}_h\}. \end{aligned} \quad (12)$$

As discussed in [26, 27], we know that $(V_h, \mathbf{W}_h)$ satisfies the so-called discrete Ladyzhenskaya-Babuska-Brezzi condition.

In view of the definition of the above mixed space, the corresponding semidiscrete mixed scheme of (11a) and (11b) is to find $\{u_h, \sigma_h\} : [0, T] \mapsto V_h \times \mathbf{W}_h$ such that

$$\left(\frac{\partial^\alpha u_h(x, t)}{\partial t^\alpha}, v_h \right) + (\sigma_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (13a)$$

$$(\sigma_h, \mathbf{w}_h) - (\nabla u_h, \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h. \quad (13b)$$

Remark 4. (i) If the standard mixed method is considered, the mixed weak formulation for problem (1) is to find $\{u, \sigma\} : [0, T] \mapsto X \times \mathbf{H}$ such that

$$\left(\frac{\partial^\alpha u(x, t)}{\partial t^\alpha}, v \right) - (\nabla \cdot \sigma, v) = (f, v), \quad \forall v \in X, \quad (14a)$$

$$(\sigma, \mathbf{w}) + (u, \nabla \cdot \mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathbf{H}, \quad (14b)$$

where $X = \{w \in L^2(\Omega) \mid w|_{\partial\Omega} = 0\}$, $\mathbf{H} = \mathbf{H}(\text{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$.

(ii) Compared with the classical mixed weak formulation (14a) and (14b), the gradient in our scheme (11a) and (11b) belongs to the simple square integrable $(L^2(\Omega))^2$ space avoiding the use of the complex $\mathbf{H}(\text{div}; \Omega)$ space. Obviously, the regularity requirements on the solution $\sigma = \nabla u$ is reduced.

(iii) So far, we have not seen any related reports on the study of mixed finite element methods for solving Fractional PDEs. Here, we will give some detailed theoretical analysis on a kind of new mixed element method for solving the fractional PDE (1).

In order to analyze the convergence of the method, we first introduce two mixed elliptic projection associated with our equations.

Lemma 5. *There exists a linear operator $\Pi_h : (L^2(\Omega))^2 \rightarrow \mathbf{W}_h$ such that*

$$(\sigma - \Pi_h \sigma, \nabla v_h) = 0, \quad \forall v_h \in V_h, \quad (15)$$

$$\|\sigma - \Pi_h \sigma\|_{L^2(\Omega)} \leq Ch^m \|\sigma\|_m. \quad (16)$$

Lemma 6. *There exists a linear operator $P_h : H_0^1(\Omega) \rightarrow V_h$ such that*

$$(\nabla(u - P_h u), \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \quad (17)$$

$$\|u - P_h u\| + h\|u - P_h u\|_1 \leq Ch^{m+1} \|u\|_{m+1}. \quad (18)$$

From [26–28], we can obtain the proof for Lemmas 5 and 6.

3.2. A Priori Error Estimates for Fully Discrete Scheme. In the following discussion, we will analyze some a priori error estimates for fully discrete schemes based on the case $0 < \alpha < 1$. For the convenience of theoretical analysis, we now denote

$$\begin{aligned} B_{n-k}^\alpha &= (n-k+1)^{1-\alpha} - (n-k)^{1-\alpha}, \\ D_t u^{k+1} &= \frac{3u^{k+1} - 4u^k + u^{k-1}}{2\Delta t}. \end{aligned} \quad (19)$$

By the discrete formula (3) for time-fractional derivative, (11a) and (11b) have the following equivalent formulation:

$$\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t u^{k+1}, v) + (\sigma^{n+1}, \nabla v) \quad (20a)$$

$$= (f^{n+1} + E_0^{n+1}, v), \quad \forall v \in H_0^1,$$

$$(\sigma^{n+1}, \mathbf{w}) - (\nabla u^{n+1}, \mathbf{w}) = 0, \quad \forall \mathbf{w} \in (L^2(\Omega))^2. \quad (20b)$$

Now, we formulate a completely discrete procedure: find $(u_h^{n+1}, \sigma_h^{n+1}) \in V_h \times \mathbf{W}_h$, $(n = 0, 1, \dots, M-1)$ such that

$$\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t u_h^{k+1}, v_h) + (\sigma_h^{n+1}, \nabla v_h) \quad (21a)$$

$$= (f^{n+1}, v_h), \quad \forall v_h \in V_h,$$

$$(\sigma_h^{n+1}, \mathbf{w}_h) - (\nabla u_h^{n+1}, \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h. \quad (21b)$$

For the convenience of the analysis, we now decompose the errors as

$$\begin{aligned} u(t_n) - u_h^n &= (u(t_n) - P_h u^n) + (P_h u^n - u_h^n) \\ &= \eta^n + \zeta^n; \\ \sigma(t_n) - \sigma_h^n &= (\sigma(t_n) - \Pi_h \sigma^n) + (\Pi_h \sigma^n - \sigma_h^n) \\ &= \rho^n + \xi^n. \end{aligned} \quad (22)$$

Subtracting (21a) and (21b) from (20a) and (20b) and using two projections (15) and (17), we get the error equations

$$\begin{aligned} &\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t \zeta^{k+1}, v_h) + (\xi^{n+1}, \nabla v_h) \\ &= -\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t \eta^{k+1}, v_h) \end{aligned} \quad (23a)$$

$$+ (E_0^{n+1}, v_h), \quad \forall v_h \in V_h,$$

$$(\xi^{n+1}, \mathbf{w}_h) - (\nabla \zeta^{n+1}, \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h. \quad (23b)$$

In the following discussion, we will derive the detailed process of proof for the fully discrete a priori error estimates.

Theorem 7. *Supposing that $u_h^0 = P_h u(0)$, $\sigma_h^0 = \Pi_h \sigma(0)$; then the error estimates hold with a parameter $0 < \alpha < 1$:*

$$\|u(t_n) - u_h^n\| \leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1}), \quad (24a)$$

$$\begin{aligned} &\sqrt{\Delta t^\alpha \Gamma(2-\alpha)} \|\sigma^n - \sigma_h^n\| \\ &\leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1} + h^m). \end{aligned} \quad (24b)$$

Proof. Noting that

$$\sum_{k=0}^n B_{n-k}^\alpha (D_t \zeta^{k+1}, v_h) = \sum_{k=0}^n B_k^\alpha (D_t \zeta^{n-k+1}, v_h). \quad (25)$$

Then (23a) may be rewritten as

$$\begin{aligned} &\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_k^\alpha (D_t \zeta^{n-k+1}, v_h) + (\xi^{n+1}, \nabla v_h) \\ &= -\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_k^\alpha (D_t \eta^{n-k+1}, v_h) \\ &\quad + (E_0^{n+1}, v_h), \quad \forall v_h \in V_h. \end{aligned} \quad (26)$$

We add (26) to (23b), take $(\nu_h, \mathbf{w}_h) = (\zeta^{n+1}, \xi^{n+1})$, and multiply by $2\Delta t^\alpha \Gamma(2 - \alpha)$ to arrive at

$$\begin{aligned} & \sum_{k=0}^n B_k^\alpha (3\zeta^{n-k+1} - 4\zeta^{n-k} + \zeta^{n-k-1}, \zeta^{n+1}) \\ & + 2\Delta t^\alpha \Gamma(2 - \alpha) \|\xi^{n+1}\|^2 \\ & = -2\Delta t \sum_{k=0}^n B_k^\alpha (D_t \eta^{n-k+1}, \zeta^{n+1}) \\ & + 2\Delta t^\alpha \Gamma(2 - \alpha) (E_0^{n+1}, \zeta^{n+1}). \end{aligned} \quad (27)$$

Now we consider the first term on the left-hand side of (27). Noting that $B_0^\alpha = 1$ and denoting $B_{n+1}^\alpha = B_{-1}^\alpha = 0$, we have

$$\begin{aligned} & \sum_{k=0}^n B_k^\alpha (3\zeta^{n-k+1} - 4\zeta^{n-k} + \zeta^{n-k-1}, \zeta^{n+1}) \\ & = \left(3 \sum_{k=0}^n B_k^\alpha \zeta^{n-k+1} - 4 \sum_{k=0}^n B_k^\alpha \zeta^{n-k} \right. \\ & \quad \left. + \sum_{k=0}^n B_k^\alpha \zeta^{n-k-1}, \zeta^{n+1} \right) \\ & = \left(3 \sum_{k=-1}^{n-1} B_{k+1}^\alpha \zeta^{n-k} - 4 \sum_{k=0}^n B_k^\alpha \zeta^{n-k} + \sum_{k=1}^{n+1} B_{k-1}^\alpha \zeta^{n-k}, \zeta^{n+1} \right) \\ & = 3 \|\zeta^{n+1}\|^2 \\ & \quad + \left(\sum_{k=0}^n (3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha) \zeta^{n-k}, \zeta^{n+1} \right) \\ & \quad + B_n^\alpha (\zeta^{-1}, \zeta^{n+1}). \end{aligned} \quad (28)$$

Substitute (28) into (27) to arrive at

$$\begin{aligned} & 3 \|\zeta^{n+1}\|^2 + 2\Delta t^\alpha \Gamma(2 - \alpha) \|\xi^{n+1}\|^2 \\ & = -2\Delta t \sum_{k=0}^n B_k^\alpha (D_t \eta^{n-k+1}, \zeta^{n+1}) \\ & \quad + 2\Delta t^\alpha \Gamma(2 - \alpha) (E_0^{n+1}, \zeta^{n+1}) \\ & \quad - \left(\sum_{k=0}^n (3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha) \zeta^{n-k}, \zeta^{n+1} \right) \\ & \quad - B_n^\alpha (\zeta^{-1}, \zeta^{n+1}). \end{aligned} \quad (29)$$

For (29), we use Cauchy-Schwarz inequality to get

$$\begin{aligned} & 3 \|\zeta^{n+1}\|^2 + 2\Delta t^\alpha \Gamma(2 - \alpha) \|\xi^{n+1}\|^2 \\ & \leq \left(\sum_{k=0}^n |3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha| \|\zeta^{n-k}\| + B_n^\alpha \|\zeta^{-1}\| \right. \\ & \quad + \left\| \Delta t \sum_{k=0}^{n-1} B_k^\alpha D_t \eta^{n-k-1} \right\| \\ & \quad \left. + 2\Delta t^\alpha \Gamma(2 - \alpha) \|E_0^{n+1}\| \right) \|\zeta^{n+1}\|. \end{aligned} \quad (30)$$

By (30), we can arrive at

$$\begin{aligned} & 3 \|\zeta^{n+1}\|^2 + 2\Delta t^\alpha \Gamma(2 - \alpha) \|\xi^{n+1}\|^2 \\ & \leq \frac{C(u, \alpha, T)}{B_n^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \|\zeta^{n+1}\|. \end{aligned} \quad (31)$$

Now, we use induction to prove the conclusion (31).

Step 1. Setting $n = 0$ in (30) and noting that $B_0^\alpha > B_1^\alpha$ and $B_{-1}^\alpha = 0$, we can arrive easily at

$$\begin{aligned} & 3 \|\zeta^1\|^2 + 2\Delta t^\alpha \Gamma(2 - \alpha) \|\zeta^1\|^2 \\ & \leq \left(|3B_1^\alpha - 4B_0^\alpha + B_{-1}^\alpha| \|\zeta^0\| + B_0^\alpha \|\zeta^{-1}\| \right. \\ & \quad + \left\| \Delta t \sum_{k=0}^{-1} B_k^\alpha D_t \eta^{-1} \right\| \\ & \quad \left. + 2\Delta t^\alpha \Gamma(2 - \alpha) \|E_0^1\| \right) \|\zeta^1\| \\ & = B_0^\alpha \left(3 \left[1 - \frac{B_1^\alpha}{B_0^\alpha} \right] \|\zeta^0\| + (\|\zeta^0\| + \|\zeta^{-1}\|) \right. \\ & \quad + \frac{1}{B_0^\alpha} \left\| \Delta t \sum_{k=0}^{-1} B_k^\alpha D_t \eta^{-1} \right\| \\ & \quad \left. + \frac{2\Delta t^\alpha \Gamma(2 - \alpha)}{B_0^\alpha} \|E_0^1\| \right) \|\zeta^1\| \\ & \leq C(u, \alpha, T) \frac{1}{B_0^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \|\zeta^1\|. \end{aligned} \quad (32)$$

So, when $n = 0$, (31) holds.

Step 2. Supposing that (31) holds, for $n \leq j$,

$$\begin{aligned} & 3 \|\zeta^{j+1}\|^2 + 2\Delta t^\alpha \Gamma(2 - \alpha) \|\xi^{j+1}\|^2 \\ & \leq \frac{C(u, \alpha, T)}{B_j^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \|\zeta^{j+1}\|. \end{aligned} \quad (33)$$

Now, we consider the case for $n = j + 1$. By (30) and the supposition (33), we have

$$\begin{aligned}
 & 3\|\zeta^{(j+1)+1}\|^2 + 2\Delta t^\alpha \Gamma(2-\alpha) \|\xi^{(j+1)+1}\|^2 \\
 & \leq \left(\sum_{k=0}^{j+1} |3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha| \|\zeta^{j+1-k}\| + B_{j+1}^\alpha \|\zeta^{-1}\| \right. \\
 & \quad \left. + \left\| \Delta t \sum_{k=0}^j B_k^\alpha D_t \eta^{j-k} \right\| \right. \\
 & \quad \left. + 2\Delta t^\alpha \Gamma(2-\alpha) \|E_0^{(j+1)+1}\| \right) \|\zeta^{(j+1)+1}\| \\
 & \leq \left(\frac{C(u, \alpha, T)}{B_{j-k}^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \right. \\
 & \quad \times \sum_{k=0}^{j+1} |3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha| + B_{j+1}^\alpha \|\zeta^{-1}\| \\
 & \quad \left. + \left\| \Delta t \sum_{k=0}^j B_k^\alpha D_t \eta^{j-k} \right\| \right. \\
 & \quad \left. + 2\Delta t^\alpha \Gamma(2-\alpha) \|E_0^{(j+1)+1}\| \right) \|\zeta^{(j+1)+1}\|. \tag{34}
 \end{aligned}$$

Noting that $1/B_{j-k}^\alpha < 1/B_{j+1}^\alpha$ in inequality (34); then we have

$$\begin{aligned}
 & 3\|\zeta^{(j+1)+1}\|^2 + 2\Delta t^\alpha \Gamma(2-\alpha) \|\xi^{(j+1)+1}\|^2 \\
 & \leq \frac{C(u, \alpha, T)}{B_{j+1}^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \\
 & \quad \times \sum_{k=0}^{j+1} |3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha| \|\zeta^{(j+1)+1}\|. \tag{35}
 \end{aligned}$$

In order to obtain the estimate for (35), we have to discuss the boundedness for $\sum_{k=0}^{j+1} |3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha|$. Noting that $B_k^\alpha > B_{k+1}^\alpha$, we have

$$\begin{aligned}
 & \sum_{k=0}^{j+1} |3B_{k+1}^\alpha - 4B_k^\alpha + B_{k-1}^\alpha| \\
 & = \sum_{k=0}^{j+1} |3(B_k^\alpha - B_{k+1}^\alpha) + (B_{k-1}^\alpha - B_k^\alpha)| \\
 & \leq 3 \sum_{k=0}^{j+1} (B_k^\alpha - B_{k+1}^\alpha) + \sum_{k=1}^{j+1} (B_{k-1}^\alpha - B_k^\alpha) + B_0^\alpha \\
 & = 5B_0^\alpha - 3B_{j+2}^\alpha - B_{j+1}^\alpha. \tag{36}
 \end{aligned}$$

Combining (36) with (35), we arrive at

$$\begin{aligned}
 & 3\|\zeta^{(j+1)+1}\|^2 + 2\Delta t^\alpha \Gamma(2-\alpha) \|\xi^{(j+1)+1}\|^2 \\
 & \leq \frac{C(u, \alpha, T)}{B_{j+1}^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \|\zeta^{(j+1)+1}\|. \tag{37}
 \end{aligned}$$

Making use of induction based on (32) and (37), we claim that (31) holds.

Note that the relationship [15] $(n+1)^{-\alpha}/B_n^\alpha \rightarrow 1/(1-\alpha)$ holds; then we have

$$\begin{aligned}
 & \frac{C(u, \alpha, T)}{B_n^\alpha} (\Delta t^{1+\alpha} + \Delta t^{2+\alpha} + h^{m+1}) \\
 & = \frac{C(u, \alpha, T)(n+1)^{-\alpha}}{B_n^\alpha} (n+1)^\alpha \Delta t^\alpha \\
 & \quad \times (\Delta t + \Delta t^2 + \Delta t^{-\alpha} h^{m+1}) \\
 & \leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1}). \tag{38}
 \end{aligned}$$

By a substitution (38) into (31), we get

$$\begin{aligned}
 & \|\zeta^n\| + \sqrt{\Delta t^\alpha \Gamma(2-\alpha)} \|\xi^n\| \\
 & \leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1}). \tag{39}
 \end{aligned}$$

By a combination of (16) and (18) with triangle inequality, we get the error results of theorem. $\square$

Theorem 8. *With the same condition, one has the following a priori error estimates:*

$$\begin{aligned}
 & \|u(t_n) - u_h^n\|_1 + \|\sigma(t_n) - \sigma_h^n\| \\
 & \leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1} + h^m). \tag{40}
 \end{aligned}$$

Proof. By (21a) and (21b), we easily get

$$\begin{aligned}
 & \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t \zeta^{k+1}, \mathbf{w}_h) \\
 & - \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (\nabla D_t \zeta^{k+1}, \mathbf{w}_h) = 0. \tag{41}
 \end{aligned}$$

We take $\mathbf{w}_h = \xi^{n+1}$ in (41) and $v_h = (\Delta t^{1-\alpha}/\Gamma(2-\alpha)) \sum_{k=0}^n B_{n-k}^\alpha \nabla D_t \zeta^{k+1}$ in (41) and use Cauchy-Schwarz inequality and Young's inequality to get

$$\begin{aligned} & \left\| \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha \nabla D_t \zeta^{k+1} \right\|^2 \\ & + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t \xi^{k+1}, \xi^{n+1}) \\ & = \left(-\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha D_t \eta^{k+1} + E_0^{n+1}, \right. \\ & \quad \left. \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha \nabla D_t \zeta^{k+1} \right) \\ & \leq \left(\left\| -\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha D_t \eta^{k+1} \right\| + \|E_0^{n+1}\| \right) \\ & \quad \times \left\| \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha \nabla D_t \zeta^{k+1} \right\| \\ & \leq C(\Delta t^2 + h^{2m+2}) + \frac{1}{2} \left\| \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha \nabla D_t \zeta^{k+1} \right\|^2. \end{aligned} \quad (42)$$

By (42), we have

$$\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^n B_{n-k}^\alpha (D_t \xi^{k+1}, \xi^{n+1}) \leq C(\Delta t^2 + h^{2m+2}). \quad (43)$$

By a similar discussion to Theorem 7, we get

$$\|\xi^n\| \leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1}). \quad (44)$$

Taking $\mathbf{w}_h = \nabla \zeta^n$ in (23b) and using (31), we arrive at

$$\|\nabla \zeta^{n+1}\| \leq \|\xi^{n+1}\| \leq \frac{C(u, \alpha, T) T^\alpha}{1-\alpha} (\Delta t + \Delta t^{-\alpha} h^{m+1}). \quad (45)$$

Combining (44), (45), (16), and (18) with triangle inequality, we complete the proof. $\square$

Remark 9. It is easy to find that a priori error estimate in H^1 -norm for the variable u , which cannot be derived based on the classical mixed scheme (14a) and (14b), is gotten.

4. Some Concluding Remarks and Extensions

As far as I know, the mixed finite element methods for fractional partial differential equations have not been proposed and studied. In this paper, our purpose is to present and analyze a kind of novel mixed finite element method for seeking the numerical solution of time-fractional partial differential equation with α ($0 < \alpha < 1$) order derivative. We discuss two-step difference method in time direction

(the approximations of the time-fractional derivative) and a class of new mixed finite element methods proposed by [26, 27] in spatial direction. We obtain some a priori error estimates in L^2 for the scalar unknown u and in $(L^2)^2$ -norm for its gradient σ . What is more, an a priori error estimate in H^1 -norm for the scalar unknown u is derived, too.

In the near future, we will develop the new mixed finite element method to solve two-dimensional time-fractional Tricomi-type equations, fractional telegraph equation, and so on. At the same time, we will try to find some new approximation method for fractional derivatives and to study some other mixed finite element methods for seeking the numerical solutions of the fractional PDEs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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