Research Article

Eventually Periodic Solutions of a Max-Type Difference Equation

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We study the following max-type difference equation $x_n = \max\{A_n/x_{n-r}, x_{n-k}\}$, $n = 1, 2, \dots$, where $\{A_n\}_{n=1}^{+\infty}$ is a periodic sequence with period p and $k, r \in \{1, 2, \dots\}$ with $\gcd(k, r) = 1$ and $k \neq r$, and the initial conditions $x_{1-d}, x_{2-d}, \dots, x_0$ are real numbers with $d = \max\{r, k\}$. We show that if $p = 1$ (or $p \geq 2$ and k is odd), then every well-defined solution of this equation is eventually periodic with period k , which generalizes the results of (Elsayed and Stević (2009), Iričanin and Elsayed (2010), Qin et al. (2012), and Xiao and Shi (2013)) to the general case. Besides, we construct an example with $p \geq 2$ and k being even which has a well-defined solution that is not eventually periodic.

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [1]). In recent years, the discrete case involving difference equations with maximum has been receiving increasing attention, for some results in this area; see, for example, [2–4].

In this paper, we consider the following max-type equation:

$$x_n = \max \left\{ \frac{A_n}{x_{n-r}}, x_{n-k} \right\}, \quad n = 1, 2, \dots, \quad (1)$$

where $\{A_n\}_{n=1}^{+\infty}$ is a periodic sequence with period p and $k, r \in \{1, 2, \dots\}$ with $\gcd(k, r) = 1$ and $k \neq r$, and the initial conditions $x_{1-d}, x_{2-d}, \dots, x_0$ are real numbers with $d = \max\{r, k\}$.

In [5], Iričanin and Elsayed showed that every well-defined solution of (1) is eventually periodic with period 4 when $k = 4$, $r = 1$, and $p = 1$. Elsayed and Stević [6] showed that every well-defined solution of (1) is eventually periodic with period 3 when $k = 3$, $r = 1$, and $p = 1$. In [7], Xiao and Shi showed that if $k = 2$, $r = 1$, and $p = 1$, then every well-defined solution of (1) is eventually periodic with period

2. Qin et al. [8] showed that every well-defined solution of (1) is eventually periodic with period k when $r = 1$ and $p = 1$.

In this paper, we will generalize the results of [5–8] to the general case.

2. Main Results and Example

In this section, we are ready to state and prove the main results.

Theorem 1. Let $\{A_n\}_{n=1}^{+\infty}$ be a periodic sequence with period p , and $k, r \in \{1, 2, \dots\}$ with $\gcd(k, r) = 1$ and $k \neq r$.

- (1) If $p \geq 2$ and k is odd, then every well-defined solution of (1) is eventually periodic with period k .
- (2) If $p = 1$, then every well-defined solution of (1) is eventually periodic with period k .

Proof. Let $\{x_n\}_{n=1-d}^{+\infty}$ be a well-defined solution of (1). It follows from (1) that, for any $n \geq 0$ and any $i \geq 0$,

$$x_{(n+1)k+i} = \max \left\{ \frac{A_{(n+1)k+i}}{x_{(n+1)k+i-r}}, x_{nk+i} \right\} \geq x_{nk+i}. \quad (2)$$

Then, for every $i \geq 0$, $\{x_{nk+i}\}_{n=0}^{+\infty}$ is increasing, and $x_{nk+i} < 0$ for all $n \geq 0$ or there exists $N_i > 0$ such that $x_{nk+i} > 0$ for all $n \geq N_i$.

We claim that $\{x_{nk}\}_{n=0}^{+\infty}$ is a constant sequence eventually. Indeed, if $\{x_{nk}\}_{n=0}^{+\infty}$ is not constant sequence eventually, then there exist $kr < n_1 < n_2 < \dots$ such that $x_{n_i k} > x_{(n_i-1)k}$ and $A_{n_i k}$ is a constant sequence for all $i \geq 1$ since $\{A_n\}_{n=1}^{+\infty}$ is a periodic sequence. Thus we have

$$\begin{aligned} x_{n_{i+1}k} &= \max \left\{ \frac{A_{n_{i+1}k}}{x_{n_{i+1}k-r}}, x_{(n_{i+1}-1)k} \right\} \\ &= \frac{A_{n_{i+1}k}}{x_{n_{i+1}k-r}} > x_{(n_{i+1}-1)k} \\ &\geq x_{n_i k} = \max \left\{ \frac{A_{n_i k}}{x_{n_i k-r}}, x_{(n_i-1)k} \right\} \\ &= \frac{A_{n_i k}}{x_{n_i k-r}}. \end{aligned} \quad (3)$$

From this we obtain that, for all $i \geq 1$,

$$\frac{A_{n_i k} (x_{n_{i+1}k-r} - x_{n_i k-r})}{x_{n_{i+1}k-r} x_{n_i k-r}} < 0. \quad (4)$$

It follows that, for all $i \geq 1$,

$$x_{n_i k} x_{n_i k-r} = A_{n_i k} < 0, \quad x_{n_{i+1}k-r} > x_{n_i k-r}. \quad (5)$$

Therefore we have $x_{nk} x_{nk-r} < 0$ eventually. By induction, we can show that $x_{nk-ir} x_{nk-(i+1)r} < 0$ eventually for all $1 \leq i \leq k-1$ and every $\{x_{nk-ir}\}_{n=r}^{+\infty}$ ($1 \leq i \leq k-1$) is not constant sequence eventually.

If $p \geq 2$ and k is odd, then we have $x_{nk} x_{nk-kr} \prod_{i=1}^{k-1} x_{nk-ir} x_{nk-(i+1)r} < 0$ eventually. This is a contradiction.

If $p = 1$, then we write $A_n = A$ for all $n \geq 1$ and choose $m_0 \geq m_1 \geq \dots \geq m_{2k}$ such that $x_{m_j k-jr} < x_{(m_j+1)k-jr}$ and $x_{(m_j+1)k-(j+1)r} = x_{(m_{j+1}+1)k-(j+1)r}$ for any $j \in \{0, 1, \dots, 2k-1\}$. Thus

$$\begin{aligned} x_{(m_0+1)k} &= \max \left\{ \frac{A}{x_{(m_0+1)k-r}}, x_{m_0 k} \right\} \\ &= \frac{A}{x_{(m_0+1)k-r}} \\ &= \frac{A}{x_{(m_1+1)k-r}} \\ &= \frac{A}{\max \{A/x_{(m_1+1)k-2r}, x_{m_1 k-r}\}} \end{aligned}$$

$$\begin{aligned} &= x_{(m_1+1)k-2r} \\ &= x_{(m_2+1)k-2r} \\ &\vdots \\ &= x_{(m_{2k}+1)k-2kr} \\ &\leq x_{m_0 k}, \end{aligned} \quad (6)$$

which is a contradiction. This completes the proof of the claim.

By the above claim we may choose an $N > 0$ such that $x_{nk} = x_{Nk}$ for all $n \geq N$. Since A_n is a periodic sequence, we can choose an $N_1 > N$ such that $A_{N_1 k+r}/x_{Nk} = \max\{A_{nk+r}/x_{Nk} : n \geq N\}$. Then, for all $n \geq N_1$,

$$x_{nk+r} = \max \left\{ \frac{A_{nk+r}}{x_{Nk}}, x_{(n-1)k+r} \right\}. \quad (7)$$

Thus $x_{nk+r} = x_{(n-1)k+r}$ for all $n \geq N_1 + 1$ (otherwise, if $x_{mk+r} > x_{(m-1)k+r}$ for some $m \geq N_1 + 1$, then we have $x_{mk+r} = A_{mk+r}/x_{Nk} > x_{(m-1)k+r} \geq x_{N_1 k+r} \geq A_{N_1 k+r}/x_{Nk}$, which is a contradiction). By induction, we can show that x_{nk+jr} is a constant sequence eventually for every $1 \leq j \leq k$. Note $\{jr : 1 \leq j \leq k\}_{\text{mod } k} = \{0, 1, 2, \dots, k-1\}$ since $\gcd(k, r) = 1$. Then x_{nk+i} is a constant sequence eventually for every $i \in \{0, 1, \dots, k-1\}$, which implies that $\{x_n\}_{n=1-d}^{+\infty}$ is eventually periodic with period k . $\square$

From the proof of Theorem 1 we obtain the following corollary.

Corollary 2. Let $k, r \in \{1, 2, \dots\}$ with $k \neq r$. If $\{A_n\}_{n=1}^{+\infty}$ is a periodic sequence, then every positive (or negative) solution of (1) is eventually periodic with period k .

Theorem 3. Let $\{A_n\}_{n=1}^{+\infty}$ be a periodic sequence with period $p \geq 2$, and $k, r \in \{1, 2, \dots\}$ with $\gcd(k, r) = 1$ and $k \neq r$. If $A_s \geq 0$ for some $s \in \{1, 2, \dots, p\}$, then every well-defined solution of (1) is eventually periodic with period k .

Proof. Let $\{x_n\}_{n=1-d}^{+\infty}$ be a well-defined solution of (1). Using arguments similar to the ones developed in the proof of Theorem 1, we know that, for every $i \geq 0$, $\{x_{nk+i}\}_{n=0}^{+\infty}$ is increasing, and $x_{nk+i} < 0$ for all $n \geq 0$ or there exists $N_i > 0$ such that $x_{nk+i} > 0$ for all $n \geq N_i$.

We may assume without loss of generality that $A_1 \geq 0$. We claim that $\{x_{nk}\}_{n=0}^{+\infty}$ is a constant sequence eventually. Indeed, if $\{x_{nk}\}_{n=0}^{+\infty}$ is not constant sequence eventually, then there exist $kr < n_1 < n_2 < \dots$ such that $x_{n_i k} > x_{(n_i-1)k}$ with $A_{n_i k}$ being a constant sequence for all $i \geq 1$. Thus we have

$$\begin{aligned} x_{n_{i+1}k} &= \max \left\{ \frac{A_{n_{i+1}k}}{x_{n_{i+1}k-r}}, x_{(n_{i+1}-1)k} \right\} \\ &= \frac{A_{n_{i+1}k}}{x_{n_{i+1}k-r}} > x_{(n_{i+1}-1)k} \end{aligned}$$

$$\begin{aligned}
&\geq x_{n_k} = \max \left\{ \frac{A_{n_k}}{x_{n_k-r}}, x_{(n-1)k} \right\} \\
&= \frac{A_{n_k}}{x_{n_k-r}}, \\
x_{npk} &= \max \left\{ \frac{A_{npk}}{x_{npk-r}}, x_{(np-1)k} \right\} \geq \frac{A_{npk}}{x_{npk-r}}.
\end{aligned} \tag{8}$$

From this we obtain that, for all $i \geq 1$,

$$\begin{aligned}
x_{n_k} x_{n_k-r} &= A_{n_k}, \\
x_{npk} x_{npk-r} &\geq A_1 \geq 0.
\end{aligned} \tag{9}$$

Thus $A_{n_k} \geq 0$ and

$$\frac{A_{n_k} (x_{n_{i+1}k-r} - x_{n_k k-r})}{x_{n_{i+1}k-r} x_{n_k k-r}} < 0. \tag{10}$$

It follows that, for all $i \geq 1$,

$$x_{n_{i+1}k-r} < x_{n_k k-r}. \tag{11}$$

This is a contradiction. $\square$

Using arguments similar to the ones developed in the proof of Theorem 1, we can show that x_{nk+jr} is a constant sequence eventually for every $1 \leq j \leq k$. Note $\{jr : 1 \leq j \leq k\}_{\text{mod } k} = \{0, 1, 2, \dots, k-1\}$ since $\gcd(k, r) = 1$. Then x_{nk+i} is a constant sequence eventually for every $i \in \{0, 1, \dots, k-1\}$, which implies that $\{x_n\}_{n=1}^{+\infty}$ is eventually periodic with period k .

Now we construct an example with $p \geq 2$ and k being even which has a well-defined solution that is not eventually periodic.

Example 4. Consider the max-type equation

$$x_n = \max \left\{ \frac{A_n}{x_{n-r}}, x_{n-k} \right\}, \quad n = 1, 2, \dots, \tag{12}$$

where $k, r \in \{1, 2, \dots\}$ and k is even with $\gcd(k, r) = 1$ and $k \neq r$ and A_n is a periodic sequence with $A_{2i} = A < A_{2i-1} = B < 0$ for all $i \geq 1$. Choose the initial conditions $x_{-i} = B$ for odd $i \in \{0, 1, \dots, d\}$ and $x_{-i} = 1$ for even $i \in \{0, 1, \dots, d\}$ with $d = \max\{r, k\}$; we can obtain a solution $\{x_n\}_{n=-1}^{\infty}$ of (12) such that

$$\begin{aligned}
&(1) \text{ If } r < k, \text{ then} \\
x_{nk+i} &= \begin{cases} \left\{ \frac{B}{A} \right\}^n B, & \text{if } i \in \{1, 3, \dots, r\}, \\ \left\{ \frac{B}{A} \right\}^{n+1} B, & \text{if } i \in \{r+2, \dots, k-1\}, \\ \left\{ \frac{A}{B} \right\}^{n+1}, & \text{if } i \in \{2, \dots, k\}. \end{cases} \tag{13}
\end{aligned}$$

It is easy to verify that $\lim_{n \rightarrow \infty} x_{2n} = \infty$ and $\lim_{n \rightarrow \infty} x_{2n-1} = 0$.

(2) If $r > k$, then

$$x_{n2r+i} = \begin{cases} \left\{ \frac{B}{A} \right\}^n B, & \text{if } i \in \{1, 3, \dots, r\}, \\ \left\{ \frac{B}{A} \right\}^{n+1} B, & \text{if } i \in \{r+2, \dots, 2r-1\}, \\ \left\{ \frac{A}{B} \right\}^{n+1}, & \text{if } i \in \{2, \dots, 2r\}. \end{cases} \tag{14}$$

It is easy to verify that $\lim_{n \rightarrow \infty} x_{2n} = \infty$ and $\lim_{i \rightarrow \infty} x_{2n-1} = 0$.

Remark 5. Consider the max-type equation

$$x_n = \max \left\{ \frac{A_n}{x_{n-sr}}, x_{n-sk} \right\}, \quad n = 1, 2, \dots, \tag{15}$$

where $\{A_n\}_{n=1}^{+\infty}$ is a periodic sequence with period ps and $s, k, r \in \{1, 2, \dots\}$ with $\gcd(k, r) = 1$ and $k \neq r$, and $p = 1$ (or $p \geq 2$), and the initial conditions $x_{1-d}, x_{2-d}, \dots, x_0$ are real numbers with $d = \max\{sr, sk\}$. Write $y_n^i = x_{ns+i}$ for every $1 \leq i \leq s$ and $n = 0, 1, 2, \dots$. Then (12) reduces to the equation

$$y_n^i = \max \left\{ \frac{A_{ns+i}}{y_{n-r}^i}, y_{n-k}^i \right\}, \quad 1 \leq i \leq s, \quad n = 0, 1, 2, \dots \tag{16}$$

- (1) If $p = 1$ (or $p \geq 2$ and k is odd), then it follows from Theorem 1 that, for every $1 \leq i \leq s$, every well-defined solution of equation $y_n^i = \max\{A_{ns+i}/y_{n-r}^i, y_{n-k}^i\}$ is eventually periodic with period k . Thus every well-defined solution of (15) is eventually periodic with period sk .
- (2) If $p \geq 2$ and, for every $1 \leq i \leq s$, there exists some j_i such that $A_{j_i s+i} \geq 0$, then it follows from Theorem 3 that for every $1 \leq i \leq s$, every well-defined solution of equation $y_n^i = \max\{A_{ns+i}/y_{n-r}^i, y_{n-k}^i\}$ is eventually periodic with period k . Thus every well-defined solution of (15) is eventually periodic with period sk .
- (3) If $p \geq 2$ and k is even, then it follows from Example 4 that, for every $1 \leq i \leq s$, we can construct an equation $y_n^i = \max\{A_{ns+i}/y_{n-r}^i, y_{n-k}^i\}$ such that it has a well-defined solution which is not eventually periodic. Thus we can construct an equation

$$x_n = \max \left\{ \frac{A_n}{x_{n-sr}}, x_{n-sk} \right\}, \quad n = 1, 2, \dots \tag{17}$$

such that it has a well-defined solution which is not eventually periodic.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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