

Research Article

Scheduling Jobs with Variable Job Processing Times on Unrelated Parallel Machines

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Received 21 March 2014; Accepted 30 April 2014; Published 26 May 2014

Academic Editor: Chin-Chia Wu

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m unrelated parallel machines scheduling problems with variable job processing times are considered, where the processing time of a job is a function of its position in a sequence, its starting time, and its resource allocation. The objective is to determine the optimal resource allocation and the optimal schedule to minimize a total cost function that depends on the total completion (waiting) time, the total machine load, the total absolute differences in completion (waiting) times on all machines, and total resource cost. If the number of machines is a given constant number, we propose a polynomial time algorithm to solve the problem.

1. Introduction

In classical scheduling theory, it is assumed that the job processing times are fixed and constant values. In practice, however, we often encounter settings in which job processing times may be subject to change due to the deterioration effects and/or learning effects and/or controllable processing times. Extensive surveys of different scheduling models and problems involving deteriorating jobs can be found in Alidaee and Womer [1], Cheng et al. [2], and Gawiejnowicz [3]. A recent survey of the scheduling problems with learning effects could be found in Biskup [4]. A recent survey of the scheduling problems with controllable processing times was given by Shabtay and Steiner [5]. More recent papers that have considered scheduling problems with deterioration effects and/or learning effects and/or controllable processing times include Bai et al. [6, 7], Cheng et al. [8], Gorczyca and Janiak [9], Hsu and Yang [10], Huang and Wang [11], Lee et al. [12], Leyvand et al. [13], Nian and Mao [14], Shabtay and Steiner [15], Wang et al. [16], J. B. Wang and M. Z. Wang [17], J. B. Wang and J. J. Wang [18], Wang et al. [19–21], Wei et al. [22], Wu and Lee [23], Wu and Liu [24], Yang and Kuo [25], S. J. Yang and D. L. Yang [26], Yang et al. [27], Yin et al.

[28–31], and Zhao and Tang [32]. Mosheiov and Sidney [33] considered the single machine model:

$$p_j = a_j r^{b_j}, \quad (1)$$

where a_j is the original (normal) processing time of job J_j , p_j is the actual processing time of job J_j , r is the position of job J_j when scheduled on the machine, and $b_j \leq 0$ is the job-dependent learning index of job J_j . Yang and Kuo [25] considered single machine scheduling with the deterioration and learning effect in which the job processing times are

$$p_j = a_j r^b + \alpha t, \quad (2)$$

where t is the starting time of job J_j , $b \leq 0$ is a learning index, and $\alpha \geq 0$ is a common deterioration rate for all the jobs. Shabtay and Steiner [15] considered single machine scheduling with the resource allocation model in which the job processing times are

$$p_j = a_j - \beta_j u_j, \quad (3)$$

where u_j is the amount of a nonrenewable resource allocated to job J_j , with $0 \leq u_j \leq \bar{u}_j < a_j/\beta_j$, where \bar{u}_j denote

the maximum amount of resource allocated to job J_j and β_j is the positive compression rate of job J_j .

Wang et al. [19] considered single machine scheduling problem where the job processing time is a function of position in a sequence, starting time, and resource allocation to this job on a single machine; that is, the model

$$p_j = a_j r^b + \alpha t - \beta_j u_j, \tag{4}$$

where $0 \leq u_j \leq \bar{u}_j < a_j n^b / \beta_j$. For minimizing a cost function containing makespan, total completion (waiting) time, total absolute differences in completion (waiting) times, and total resource cost, they proved that the problem can be solved in polynomial time.

Wang et al. [19] considered single machine scheduling problem with effect of deterioration, learning, and resource allocation simultaneously. The example of the phenomena of deterioration, learning effect, and resource allocation occurring simultaneously can be found in Wang et al. [19]. However, the parallel machine scheduling is interesting and closer to real problems in practice (Cheng et al. [8], Hsu and Yang [10], Huang and Wang [11], Yang et al. [27], and Yin et al. [30]). This paper extends the single machine scheduling results of Wang et al. [19], by considering unrelated parallel machine scheduling problems that include the one given in Wang et al. [19] as a special case. The remainder of this paper is organized as follows. In Section 2 we formulate the model. In Sections 3 and 4, we show that the problem can be solved in polynomial time for two different cost functions. The last section contains some conclusions.

2. Problem Formulation

The model is described as follows. There are n independent jobs $\{J_1, J_2, \dots, J_n\}$ to be processed on m unrelated parallel machines $\{M_1, M_2, \dots, M_m\}$. Each of them is available at time 0. The machine can handle one job at a time, and preemption is not allowed. Let n_i denote the number of jobs assigned to M_i ($i = 1, 2, \dots, m$) and $P(n, m) = (n_1, n_2, \dots, n_m)$ denote a job-allocation vector, where $(n_1 + n_2 + \dots + n_m = n)$. We assume, as in most practical situations, that $m < n$ and m are a given constant. Each job can be processed on any one of the m unrelated parallel machines. Associated with each job J_j ($j = 1, 2, \dots, n$) on machine M_i , there is a normal processing time a_{ij} . Let p_{ij} denote the actual processing time for job J_j on machine M_i . In this paper, we consider a general unrelated parallel machines model stemming from Yang and Kuo [25], Shabtay and Steiner [15], and Wang et al. [19]; that is,

$$p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij}, \tag{5}$$

where r is the position of job J_j when scheduled on machine M_i , t is the starting time of job J_j on machine M_i , $b_{ij} \leq 0$ is a job-dependent learning index, $\alpha \geq 0$ is a common deterioration rate for all the jobs, $\theta_{ij} \geq 0$ is the positive compression rate of job J_j on machine M_i , and u_{ij} is the amount of resource that can be allocated to job J_j on machine M_i , with $0 \leq u_{ij} \leq \bar{u}_{ij} < a_{ij} n^{b_{ij}} / \theta_{ij}$, where \bar{u}_{ij} is the upper bound on

the amount of resource that can be allocated to job J_j on machine M_i .

Let $J_{i[j]}$ denote the j th job on machine M_i and $C_{i[j]}(W_{i[j]})$ denote the completion (waiting) time of job $J_{i[j]}$; $p_{i[j]}$, $a_{i[j]}$, $b_{i[j]}$, $\theta_{i[j]}$, $u_{i[j]}$, and $\bar{u}_{i[j]}$ are defined similarly, where $W_{i[j]} = C_{i[j]} - p_{i[j]}$. Let $L_i = \max\{C_{ij} \mid j = 1, 2, \dots, n_i\}$, $TC_i = \sum_{j=1}^{n_i} C_{ij}$ ($TW_i = \sum_{j=1}^{n_i} W_{ij}$), and $TADC_i = \sum_{j=1}^{n_i} \sum_{l=j}^{n_i} |C_{ij} - C_{il}|$ ($TADW_i = \sum_{j=1}^{n_i} \sum_{l=j}^{n_i} |W_{ij} - W_{il}|$) be the load, the total completion (waiting) times, and the total absolute differences in completion (waiting) times of machine M_i . Then, the total machine load, the total completion (waiting) time, and the total absolute deviation of job completion (waiting) time on all machines are $\sum_{i=1}^m L_i$, $\sum_{i=1}^m TC_i$ ($\sum_{i=1}^m TW_i$), and $\sum_{i=1}^m TADC_i$ ($\sum_{i=1}^m TADW_i$), respectively. The objective is to determine the optimal resource allocations and the optimal schedule on the machines so that the corresponding value of the following cost functions is optimal:

$$Z_1(\pi, u) = \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}, \tag{6}$$

$$Z_2(\pi, u) = \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}, \tag{7}$$

where weights $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$, and $\alpha_4 \geq 0$ are given constants and G_{ij} is the per time unit cost associated with the resource allocation. Using the three-field notation introduced by Graham et al. [34], the corresponding scheduling problem is denoted by $Rm \mid p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$ and $Rm \mid p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$.

3. Problem $Rm \mid p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$

If the number of jobs on machine M_i is known in advance, from Wang et al. [19], we have

$$C_{i[l]} = \sum_{l=1}^j (1 + \alpha)^{j-l} (a_{i[l]} t^{b_{i[l]}} - \beta_{i[l]} u_{i[l]}),$$

$$p_{i[j]} = a_{i[j]} j^{b_{i[j]}} - \beta_{i[j]} u_{i[j]} + \alpha \sum_{l=1}^{j-1} (1 + \alpha)^{j-1-l} (a_{i[l]} t^{b_{i[l]}} - \beta_{i[l]} u_{i[l]}). \tag{8}$$

From Kanet [35], we have $TADC_i = \sum_{j=1}^{n_i} (j-1)(n_i - j + 1)P_{i[j]}$; in addition, $C_{i[j]} = \sum_{l=1}^j P_{i[l]}$, $L_i = \sum_{j=1}^{n_i} P_{i[j]}$, and $TC_i = \sum_{j=1}^{n_i} C_{i[j]} = \sum_{j=1}^{n_i} (n_i - j + 1)P_{i[j]}$; hence, we have

$$\begin{aligned}
 & Z_1(\pi, u) \\
 &= \alpha_1 \sum_{i=1}^m \sum_{j=1}^{n_i} P_{i[j]} + \alpha_2 \sum_{i=1}^m \sum_{j=1}^{n_i} (n_i - j + 1) P_{i[j]} \\
 &+ \alpha_3 \sum_{i=1}^m \sum_{j=1}^{n_i} (j-1)(n_i - j + 1) P_{i[j]} \\
 &+ \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\
 &= \sum_{i=1}^m \sum_{j=1}^{n_i} [\alpha_1 + \alpha_2 (n_i + 1 - j) + \alpha_3 (j-1)(n_i - j + 1)] P_{i[j]} \\
 &+ \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\
 &= \sum_{i=1}^m \sum_{j=1}^{n_i} \omega_{ij} P_{i[j]} + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\
 &= \sum_{i=1}^m \sum_{j=1}^{n_i} \omega_{ij} \left(a_{i[j]} j^{b_{i[j]}} - \beta_{i[j]} u_{i[j]} \right. \\
 &\quad \left. + \alpha \sum_{l=1}^{j-1} (1 + \alpha)^{j-1-l} (a_{i[l]} l^{b_{i[l]}} - \beta_{i[l]} u_{i[l]}) \right) \\
 &+ \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\
 &= \sum_{i=1}^m \left[\omega_{i1} (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]}) \right. \\
 &\quad + \omega_{i2} (a_{i[2]} 2^{b_{i[2]}} - \beta_{i[2]} u_{i[2]}) \\
 &\quad \left. + \alpha (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]}) \right) \\
 &\quad + \omega_{i3} (a_{i[3]} 3^{b_{i[3]}} - \beta_{i[3]} u_{i[3]}) \\
 &\quad \left. + \alpha (a_{i[2]} 2^{b_{i[2]}} - \beta_{i[2]} u_{i[2]} + (1 + \alpha) \right. \\
 &\quad \left. \times (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]})) \right) \\
 &\quad + \omega_{i4} (a_{i[4]} 4^{b_{i[4]}} - \beta_{i[4]} u_{i[4]}) \\
 &\quad \left. + \alpha (a_{i[3]} 3^{b_{i[3]}} - \beta_{i[3]} u_{i[3]} + (1 + \alpha) \right. \\
 &\quad \left. \times (a_{i[2]} 2^{b_{i[2]}} - \beta_{i[2]} u_{i[2]}) \right. \\
 &\quad \left. + (1 + \alpha)^2 (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]}) \right) + \dots
 \end{aligned}$$

$$\begin{aligned}
 & + \omega_{i, n_i-1} \left(a_{i[n_i-1]} (n_i - 1)^{b_{i[n_i-1]}} - \beta_{i[n_i-1]} u_{i[n_i-1]} \right. \\
 &\quad \left. + \alpha (a_{i[n_i-2]} (n_i - 2)^{b_{i[n_i-2]}} \right. \\
 &\quad \left. - \beta_{i[n_i-2]} u_{i[n_i-2]} + (1 + \alpha) \right. \\
 &\quad \left. \times (a_{i[n_i-3]} (n_i - 3)^{b_{i[n_i-3]}} \right. \\
 &\quad \left. - \beta_{i[n_i-3]} u_{i[n_i-3]}) + \dots \right. \\
 &\quad \left. + (1 + \alpha)^{n_i-4} (a_{i[2]} 2^{b_{i[2]}} - \beta_{i[2]} u_{i[2]}) \right. \\
 &\quad \left. + (1 + \alpha)^{n_i-3} (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]}) \right) \\
 &+ \omega_{i, n_i} \left(a_{i[n_i]} n_i^{b_{i[n_i]}} - \beta_{i[n_i]} u_{i[n_i]} \right. \\
 &\quad \left. + \alpha (a_{i[n_i-1]} (n_i - 1)^{b_{i[n_i-1]}} - \beta_{i[n_i-1]} u_{i[n_i-1]} \right. \\
 &\quad \left. + (1 + \alpha) (a_{i[n_i-2]} (n_i - 2)^{b_{i[n_i-2]}} \right. \\
 &\quad \left. - \beta_{i[n_i-2]} u_{i[n_i-2]}) + \dots \right. \\
 &\quad \left. + (1 + \alpha)^{n_i-3} (a_{i[2]} 2^{b_{i[2]}} - \beta_{i[2]} u_{i[2]}) \right. \\
 &\quad \left. + (1 + \alpha)^{n_i-2} (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]}) \right) \\
 &+ \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\
 &= \sum_{i=1}^m \left[(\omega_{i1} + \alpha \omega_{i2} + \alpha (1 + \alpha) \omega_{i3} + \dots + \alpha (1 + \alpha)^{n_i-2} \omega_{i, n_i}) \right. \\
 &\quad \times (a_{i[1]} 1^{b_{i[1]}} - \beta_{i[1]} u_{i[1]}) \\
 &\quad + (\omega_{i2} + \alpha \omega_{i3} + \alpha (1 + \alpha) \omega_{i4} + \dots \\
 &\quad \left. + \alpha (1 + \alpha)^{n_i-3} \omega_{i, n_i}) (a_{i[2]} 2^{b_{i[2]}} - \beta_{i[2]} u_{i[2]}) \right. \\
 &\quad \left. + (\omega_{i3} + \alpha \omega_{i4} + \alpha (1 + \alpha) \omega_{i5} + \dots \right. \\
 &\quad \left. + \alpha (1 + \alpha)^{n_i-4} \omega_{i, n_i}) (a_{i[3]} 3^{b_{i[3]}} - \beta_{i[3]} u_{i[3]}) \right. \\
 &\quad \left. + \dots + (\omega_{i, n_i-1} + \alpha \omega_{i, n_i}) (a_{i[n_i-1]} - \beta_{i[n_i-1]} u_{i[n_i-1]}) \right. \\
 &\quad \left. + \omega_{i, n_i} (a_{i[n_i]} n_i^{b_{i[n_i]}} - \beta_{i[n_i]} u_{i[n_i]}) \right] \\
 &+ \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\
 &= \sum_{i=1}^m \sum_{j=1}^{n_i} \Omega_{ij} a_{i[j]} j^{b_{i[j]}} + \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij}) u_{i[j]},
 \end{aligned}$$

where $\omega_{ij} = \alpha_1 + \alpha_2(n_i + 1 - j) + \alpha_3(j - 1)(n_i - j + 1)$ and

$$\begin{aligned} \Omega_{i1} &= \omega_{i1} + \alpha\omega_{i2} + \alpha(1 + \alpha)\omega_{i3} + \dots + \alpha(1 + \alpha)^{n_i-2}\omega_{in_i}, \\ \Omega_{i2} &= \omega_{i2} + \alpha\omega_{i3} + \alpha(1 + \alpha)\omega_{i4} + \dots + \alpha(1 + \alpha)^{n_i-3}\omega_{in_i}, \\ \Omega_{i3} &= \omega_{i3} + \alpha\omega_{i4} + \alpha(1 + \alpha)\omega_{i5} + \dots + \alpha(1 + \alpha)^{n_i-4}\omega_{in_i}, \\ &\vdots \\ \Omega_{i,n_i-1} &= \omega_{i,n_i-1} + \alpha\omega_{in_i}, \\ \Omega_{in_i} &= \omega_{in_i}. \end{aligned} \tag{10}$$

From (9), for any given sequence on all machines, the optimal resource allocation on all machines can be obtained by the following.

Lemma 1. For a given sequence, the optimal resource allocation of the problem $Rm \mid p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ can be determined by

$$u_{i[j]}^* = \begin{cases} \bar{u}_{i[j]}, & \text{if } \alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij} < 0, \\ u_{i[j]} \in [0, \bar{u}_{i[j]}], & \text{if } \alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij} = 0, \\ 0, & \text{if } \alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij} > 0, \end{cases} \tag{11}$$

where $u_{i[j]}^*, i = 1, 2, \dots, m; j = 1, 2, \dots, n_i$, represents the optimal resource allocation of the job in position j on machine M_i .

Proof. For the problem $Rm \mid p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$, taking the derivative by $u_{i[j]}$ to (9), we have $df(\pi, u)/u_{i[j]} = \alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij}$ for $i = 1, 2, \dots, m; j = 1, 2, \dots, n_i$. Hence, if $\alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij} > 0$, we should not allocate any resource to job $J_{i[j]}$; if $\alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij} < 0$, we will allocate the maximal feasible amount of resource to job $J_{i[j]}$; and if $\alpha_4 G_{i[j]} - \beta_{i[j]} \Omega_{ij} = 0$, any feasible resource allocation can be optimal. \square

We define $x_{ijr} = 1$ if job J_j is scheduled in position r on machine M_i , and $x_{ijr} = 0$ otherwise. If the number of jobs on machine M_i is known in advance, then we formulate the $Rm \mid p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ problem as the following assignment problem:

$$\text{Min } Z = \sum_{i=1}^m \sum_{r=1}^{n_i} \sum_{j=1}^{n_i} \lambda_{ijr} x_{ijr} \tag{12}$$

subject to

$$\sum_{i=1}^m \sum_{r=1}^{n_i} x_{ijr} = 1, \quad j = 1, 2, \dots, n, \tag{13}$$

$$\sum_{j=1}^n x_{ijr} = 1, \quad i = 1, 2, \dots, m; r = 1, 2, \dots, n_i, \tag{14}$$

$$x_{ijr} = 0 \text{ or } 1, \quad i = 1, 2, \dots, m; \tag{15}$$

$$r = 1, 2, \dots, n_i; \quad j = 1, 2, \dots, n,$$

where

$$\begin{aligned} \lambda_{ijr} &= \begin{cases} \Omega_{ir} a_{ij} r^{b_{ij}}, & \text{if } \alpha_4 G_{ij} - \beta_{ij} \Omega_{ir} \geq 0, \\ \Omega_{ir} a_{ij} r^{b_{ij}} + (\alpha_4 G_{ij} - \beta_{ij} \Omega_{ir}) \bar{u}_{ij}, & \text{if } \alpha_4 G_{ij} - \beta_{ij} \Omega_{ir} < 0. \end{cases} \end{aligned} \tag{16}$$

Constraint (13) makes sure that each job is scheduled exactly once. Constraint (14) makes sure that each position on each machine is taken by one job.

Next, the question is how many $P(n, m) = (n_1, n_2, \dots, n_m)$ vectors exist. Note that n_i may be $0, 1, 2, \dots, n$ for $i = 1, 2, \dots, m$. So, if we get the numbers of jobs on the first $m - 1$ machines, the number of jobs processed on the last machine is then determined uniquely due to $n_1 + n_2 + \dots + n_m = n$. Therefore, the upper bound of the number of $P(n, m)$ vectors is $(n + 1)^{m-1}$. Based on the above analysis, we have the following result.

Theorem 2. The problem $Rm \mid p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ can be solved in $O(n^{m+2})$ time; that is, the problem is polynomially solvable because m is a constant.

Base on the above analysis, we can determine the optimal solution for the problem $Rm \mid p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} \mid \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ via the following algorithm.

Algorithm 3. Consider the following.

Step 1. For all the possible vectors (n_1, n_2, \dots, n_m) , solve the assignment problems ((12)–(16)). Then, obtain the optimal schedule and the corresponding total cost $\alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ for each possible vector (n_1, n_2, \dots, n_m) .

Step 2. The optimal solution for the problem is the one with the minimum value of the total cost $\alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$.

Step 3. Calculate the optimal resources allocation by using (11).

The following example illustrates the working of Algorithm 3 for the problem $Rm | p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$.

Example 4. Data. $n = 4, m = 2, \alpha = 0.05$, and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, and the other corresponding parameters are shown in Table 1.

Solution. When $n_1 = 0, n_2 = 4, \Omega_{21} = 5.9931, \Omega_{22} = 7.6125, \Omega_{23} = 7.2500, \Omega_{24} = 5$, then the optimal schedule on machine M_2 is $[J_4, J_1, J_2, J_3]$ and $Z_1 = 155.2178$.

When $n_1 = 1, n_2 = 3, \Omega_{11} = 2, \Omega_{21} = 4.4600, \Omega_{22} = 5.2000, \Omega_{23} = 4$, the optimal schedule on machine M_1 is $[J_3]$ and on machine M_2 is $[J_4, J_1, J_2]$, and $Z_1 = 102.7967$.

When $n_1 = 2, n_2 = 2, \Omega_{11} = 3.1500, \Omega_{12} = 3, \Omega_{21} = 3.1500, \Omega_{22} = 3$, the optimal schedule on machine M_1 is $[J_2, J_3]$ and on machine M_2 is $[J_4, J_1]$, and $Z_1 = 99.9928$.

When $n_1 = 3, n_2 = 1, \Omega_{11} = 4.4600, \Omega_{12} = 5.2000, \Omega_{13} = 4, \Omega_{21} = 2$, the optimal schedule on machine M_1 is $[J_2, J_3, J_4]$ and on machine M_2 is $[J_1]$, and $Z_1 = 115.8243$.

When $n_1 = 4, n_2 = 0, \Omega_{11} = 5.9931, \Omega_{12} = 7.6125, \Omega_{13} = 7.2500, \Omega_{14} = 5$, the optimal schedule on machine M_1 is $[J_4, J_3, J_2, J_1]$, and $Z_1 = 168.1833$.

Hence, the optimal schedule on machine M_1 is $[J_2, J_3]$ and on machine M_2 is $[J_4, J_1], u_{1[1]}^* = 4, u_{1[2]}^* = 3, u_{2[1]}^* = 2, u_{2[2]}^* = 3$ and optimal cost is $Z_1 = 99.9928$.

4. Problem $Rm | p_{ij} = a_{ij}r^{b_{ij}} + \alpha t -$

$$\theta_{ij}u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$$

Similar to the analysis of the problem $Rm | p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ presented in the previous section, if we substitute $W_{i[j]} = \sum_{l=1}^{j-1} p_{i[l]}, TW_i = \sum_{j=1}^{n_i} W_{i[j]}$, and $TADW_i = \sum_{j=1}^{n_i} j(n_i - j)p_{i[j]}$ (Bagchi [36]) into (7), we have

$$\begin{aligned} Z_2(\pi, u) &= \alpha_1 \sum_{i=1}^m \sum_{j=1}^{n_i} p_{i[j]} + \alpha_2 \sum_{i=1}^m \sum_{j=1}^{n_i} (n_i - j) p_{i[j]} \\ &\quad + \alpha_3 \sum_{i=1}^m \sum_{j=1}^{n_i} j(n_i - j) p_{i[j]} + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{i[j]} u_{i[j]} \\ &= \sum_{i=1}^m \sum_{j=1}^{n_i} \Psi_{ij} a_{i[j]} j^{b_{i[j]}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^{n_i} (\alpha_4 G_{i[j]} - \beta_{i[j]} \Psi_{ij}) u_{i[j]}, \end{aligned} \tag{17}$$

where $v_{ij} = \alpha_1 + \alpha_2(n_i - j) + \alpha_3 j(n_i - j)$,

$$\Psi_{i1} = v_{i1} + \alpha v_{i2} + \alpha(1 + \alpha) v_{i3} + \dots + \alpha(1 + \alpha)^{n_i-2} v_{i n_i},$$

TABLE 1: The date of Example 4.

J_j	J_1	J_2	J_3	J_4
a_{1j}	16	20	18	14
a_{2j}	17	19	20	12
b_{1j}	-0.35	-0.21	-0.25	-0.31
b_{2j}	-0.23	-0.31	-0.27	-0.25
θ_{1j}	2	3	4	5
θ_{2j}	4	5	2	3
\bar{u}_{1j}	3	4	3	1
\bar{u}_{2j}	3	2	5	2
G_{1j}	6	4	2	3
G_{2j}	3	7	5	4

$$\begin{aligned} \Psi_{i2} &= v_{i2} + \alpha v_{i3} + \alpha(1 + \alpha) v_{i4} + \dots + \alpha(1 + \alpha)^{n_i-3} v_{i n_i}, \\ \Psi_{i3} &= v_{i3} + \alpha v_{i4} + \alpha(1 + \alpha) v_{i5} + \dots + \alpha(1 + \alpha)^{n_i-4} v_{i n_i}, \\ &\vdots \\ \Psi_{i, n_i-1} &= v_{i, n_i-1} + \alpha v_{i n_i}, \\ \Psi_{i n_i} &= v_{i n_i}. \end{aligned} \tag{18}$$

Applying a similar analysis in the previous section, we have the following results.

Lemma 5. For a given sequence, the optimal resource allocation of the problem $Rm | p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$ can be determined by

$$u_{i[j]}^* = \begin{cases} \bar{u}_{i[j]}, & \text{if } \alpha_4 G_{i[j]} - \beta_{i[j]} \Psi_{ij} < 0, \\ u_{i[j]} \in [0, \bar{u}_{i[j]}], & \text{if } \alpha_4 G_{i[j]} - \beta_{i[j]} \Psi_{ij} = 0, \\ 0, & \text{if } \alpha_4 G_{i[j]} - \beta_{i[j]} \Psi_{ij} > 0, \end{cases} \tag{19}$$

where $u_{i[j]}^*, i = 1, 2, \dots, m; j = 1, 2, \dots, n_i$ represents the optimal resource allocation of the job in position j on machine M_i .

Lemma 6. For the problem $Rm | p_{ij} = a_{ij}r^{b_{ij}} + \alpha t - \theta_{ij}u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 TW_i + \alpha_3 TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij}u_{ij}$, if the vector $P(n, m) = (n_1, n_2, \dots, n_m)$ is given, the optimal sequence can be determined by solving the following assignment problem:

$$\text{Min } Z = \sum_{i=1}^m \sum_{r=1}^{n_i} \sum_{j=1}^n \theta_{ijr} x_{ijr} \tag{20}$$

subject to (13), (14) and (15),

where

$$\theta_{ijr} = \begin{cases} \Psi_{ir} a_{ij} r^{b_{ij}}, & \text{if } \delta_4 G_{ij} - \beta_{ij} \Psi_{ir} \geq 0, \\ \Omega_{ir} a_{ij} r^{b_{ij}} + (\alpha_4 G_{ij} - \beta_{ij} \Psi_{ir}) m_{ij}, & \text{if } \alpha_4 G_{ij} - \beta_{ij} \Psi_{ir} < 0. \end{cases} \quad (21)$$

Theorem 7. The problem $Rm | p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 TW_i + \alpha_3 TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$ can be solved in $O(n^{m+2})$ time; that is, the problem is polynomially solvable because m is a constant.

The optimal solution for the problem $Rm | p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 TW_i + \alpha_3 TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$ can be obtained by the following algorithm.

Algorithm 8. Consider the following.

Step 1. For all the possible vectors (n_1, n_2, \dots, n_m) , solve the assignment problems ((20), (13)–(15), (21)). Then, obtain the optimal schedule and the corresponding total cost $\alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$ for each possible vector (n_1, n_2, \dots, n_m) .

Step 2. The optimal solution for the problem is the one with the minimum value of the total cost $\alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$.

Step 3. Calculate the optimal resources allocation by using (19).

The following example illustrates the working of Algorithm 8 for the problem $Rm | p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$.

Example 9. The same data in Example 4 is used.

Solution. When $n_1 = 0, n_2 = 4, \Psi_{21} = 7.6676, \Psi_{22} = 7.3025, \Psi_{23} = 5.0500, \Psi_{24} = 1$, then the optimal schedule on machine M_2 is $[J_4, J_1, J_2, J_3]$ and $Z_2 = 126.7347$.

When $n_1 = 1, n_2 = 3, \Psi_{11} = 1, \Psi_{21} = 5.2525, \Psi_{22} = 4.0500, \Psi_{23} = 1$, the optimal schedule on machine M_1 is $[J_3]$ and on machine M_2 is $[J_4, J_1, J_2]$, and $Z_2 = 84.13484$.

When $n_1 = 2, n_2 = 2, \Psi_{11} = 3.0500, \Psi_{12} = 1, \Psi_{21} = 3.0500, \Psi_{22} = 1$, the optimal schedule on machine M_1 is $[J_3, J_4]$ and on machine M_2 is $[J_1, J_2]$, and $Z_2 = 73.1692$.

When $n_1 = 3, n_2 = 1, \Psi_{11} = 5.2525, \Psi_{12} = 4.0500, \Psi_{13} = 1, \Psi_{21} = 1$, the optimal schedule on machine M_1 is $[J_3, J_4, J_2]$ and on machine M_2 is $[J_1]$, and $Z_2 = 95.8810$.

When $n_1 = 4, n_2 = 0, \Psi_{11} = 7.6676, \Psi_{12} = 7.3025, \Psi_{13} = 5.0500, \Psi_{14} = 1$, the optimal schedule on machine M_1 is $[J_4, J_3, J_2, J_1]$, and $Z_2 = 144.1351$.

Hence, the optimal schedule on machine M_1 is $[J_3, J_4]$ and on machine M_2 is $[J_1, J_2]$, $u_{1[1]}^* = 3, u_{1[2]}^* = 1, u_{2[1]}^* = 3, u_{2[2]}^* = 0$ and optimal cost is $Z_2 = 73.1692$.

5. Conclusions

In this paper, we have studied the problem of scheduling n jobs on m unrelated parallel machines with simultaneous consideration of learning effect, deteriorating jobs, and controllable processing times. We provide an $O(n^{m+2})$ time algorithm for the problems $Rm | p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TC_i + \alpha_3 \sum_{i=1}^m TADC_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$ and $Rm | p_{ij} = a_{ij} r^{b_{ij}} + \alpha t - \theta_{ij} u_{ij} | \alpha_1 \sum_{i=1}^m L_i + \alpha_2 \sum_{i=1}^m TW_i + \alpha_3 \sum_{i=1}^m TADW_i + \alpha_4 \sum_{i=1}^m \sum_{j=1}^{n_i} G_{ij} u_{ij}$, respectively. The algorithms can also be easily applied to the cases $b_{ij} > 0$ (aging effect) and $\alpha < 0$. Future research may focus on similar problems with more general processing time model and extend the problems to flow shop machine settings.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful for three anonymous referees for their helpful comments on earlier version of the paper. This research was supported by the National Natural Science Foundation of China (Grant nos. 11001181 and 71271039), the New Century Excellent Talents in University (NCET-13-0082), the Changjiang Scholars and Innovative Research Team in University (IRT1214), and the Fundamental Research Funds for the Central Universities (DUT14YQ211).

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