Research Article

The Non-Relativistic Limit for the e-MHD Equations

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We investigate the non-relativistic limit for the e-MHD equations in a three-dimension unit periodic torus. With the prepared initial data, our result shows that the small parameter problems have unique solutions existing in the finite time interval where the corresponding limit problems (incompressible Euler equations) have smooth solutions. Moreover, the formal limit is rigorously justified.

1. Introduction

Let us consider the following so-called electron-magnetohydrodynamics (e-MHD) system [1, 2]:

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u + E + \gamma u \times B &= 0, \\ \partial_t B + \frac{1}{\gamma} \nabla \times E &= 0, \\ u &= -\frac{1}{\gamma} \nabla \times B, \quad \operatorname{div} B = 0,\end{aligned}\quad (1)$$

for $(x, t) \in \Omega \times [0, T]$, where $\Omega = (\mathbb{R}/2\pi)^3$ is the 3-dimensional torus. The unknowns are $u \in \mathbb{R}^3$, $E \in \mathbb{R}^3$, and $B \in \mathbb{R}^3$. In the above equations, the non-dimensionless parameter σ is proportional to $1/c$, where $c = (\epsilon_0 \nu_0)^{-1/2}$ is the speed of light, with ϵ_0 and ν_0 being the vacuum permittivity and permeability. Let us notice that, as $\gamma \rightarrow 0$ in (1), we get $\nabla \times E = 0 (\Rightarrow E = \nabla p^I)$, $\nabla \times B = 0$, $\operatorname{div} B = 0 (\Rightarrow B = 0)$, and the incompressible Euler equations of ideal fluid:

$$\partial_t u^I + (u^I \cdot \nabla) u^I + \nabla p^I = 0, \quad \operatorname{div} u^I = 0. \quad (2)$$

The goal of this paper is to justify the above formal derivation of the incompressible Euler equations for periodic initial-value problems (IVPs) with an emphasis on three space dimensions. Indeed, it is well known that the phenomenon

of non-relativistic is important in many physical situation involving various non-equilibrium processes. For instance, important examples occur in inviscid radiation hydrodynamics [3], quantum mechanics [4], Klein-Gordon-Maxwell system [5], and so on.

The rigorous derivations of the e-MHD equation (1) from Vlasov-Maxwell system equations by a scaling limit and from Euler-Maxwell system by a quasineutral regime are obtained, respectively, in [2] and in [6]. In [7], the incompressible Euler equation (2) of ideal fluid from the e-MHD system (1) via a non-relativistic limit was gotten only in formal derivation. The aim of this work is to give a rigorous justification of the asymptotic limit using the energy method.

Let us recall that the non-relativistic limit $\gamma \rightarrow 0$ or $c \rightarrow \infty$ has been investigated in a few fields. For instance, the limit $\gamma \rightarrow 0$ has been performed in Vlasov-Poisson system in [8], in isentropic relativistic Euler equations [9], in a model system for multiple space dimensions radiation hydrodynamics in [10], and in Euler-Maxwell equations [11, 12].

Now we recall some results on the Moser-type calculus inequalities in Sobolev spaces and the local existence of smooth solutions for symmetrizable hyperbolic equations for later use in this paper.

Lemma 1 (Moser-type calculus inequalities; see [13, 14]). *Let $s \geq 1$ be an integer. Suppose $u \in H^s(\mathcal{T}^3)$, $\nabla u \in L^\infty(\mathcal{T}^3)$, and*

$v \in H^{s-1}(\mathcal{T}^3) \cap L^\infty(\mathcal{T}^3)$. Then for all multi-indexes $|\alpha| \leq s$, one has $(\partial_x^\alpha(uv) - u\partial_x^\alpha v) \in L^2(\mathcal{T}^3)$ and

$$\begin{aligned} & \|\partial_x^\alpha(uv) - u\partial_x^\alpha v\| \\ & \leq C_s \left(\|\nabla u\|_{0,\infty} \|D^{|\alpha|-1}v\| + \|D^{|\alpha|}u\| \|v\|_{0,\infty} \right), \end{aligned} \quad (3)$$

where

$$\|D^h u\| = \sum_{|\alpha|=h} \|\partial_x^\alpha u\|, \quad \forall h \in \mathbb{N}. \quad (4)$$

Moreover, if $s \geq 3$, then the embedding $H^{s-1}(\mathcal{T}^3) \hookrightarrow L^\infty(\mathcal{T}^3)$ is continuous and one has

$$\begin{aligned} & \|uv\|_{s-1} \leq C_s \|u\|_{s-1} \|v\|_{s-1}, \\ & \|\partial_x^\alpha(uv) - u\partial_x^\alpha v\| \leq C_s \|u\|_s \|v\|_{s-1}. \end{aligned} \quad (5)$$

Recalling the classical result on the existence of sufficiently regular solutions of the incompressible Euler equations (2), we have the following regularity result about (u^I, p^I) .

Lemma 2 (see [15, 16]). *Let u_0^I satisfy $u_0^I \in C^\infty$ and $\operatorname{div} u_0^I = 0$. Then there exist $0 < T_* < \infty$, the maximal existence time, and a unique smooth solution (u^I, p^I) of the incompressible Euler equations (2) on $[0, T_*)$ with initial datum u_0^I satisfying, for any $T_0 < T_*$,*

$$(u^I, p^I) \in C^\infty(\Omega \times [0, T_0)). \quad (6)$$

This paper is organized as follows. In Section 2 we give some analytical backgrounds and the main result. The proof of the main result is given in Section 3.

2. Main Result

First we will state the existence of smooth local solutions for system (1) for the smooth initial data given by

$$u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \quad (7)$$

Proposition 3 (see [2, 17]). *Assume that (u_0, B_0) belongs to $C^\infty(\Omega)$ and satisfies*

$$-\nabla \times B_0 = \gamma u_0, \quad \operatorname{div} B_0 = 0. \quad (8)$$

Then there exist $0 < T^\gamma < +\infty$, the maximal existence time, which depends only on the initial data, and a unique smooth solution $(u, B, E) \in C^\infty(\Omega \times [0, T^\gamma))$ of the incompressible e-MHD equations (1) defined on $[0, T^\gamma)$.

For the convergence of the e-MHD system (1), our main result is stated as follows.

Theorem 4. *Let $s_0 \in \mathbb{N}$ with $s_0 > (3/2) + 1$. Let $u_0^I \in C^\infty(\Omega)$ satisfy $\operatorname{div} u_0^I = 0$ and let $(u_0, E_0, B_0) \in C^\infty(\Omega)$ satisfy (8). Assume that*

$$\|(u_0 - u_0^I, B_0)\|_{H^{s_0}(\Omega)} \leq C\gamma \quad (9)$$

for some positive constant C independent of γ . Let T_ be the maximal existence time of the smooth solution $(u^I, p^I) \in C^\infty(\Omega \times [0, T_*))$ of the incompressible Euler equations (2). Then, for any $T_0 < T_*$, there exist constants $\gamma_0(T_0) > 0$ and $C(T_0) > 0$, depending only on T_0 and the initial data, such that the e-MHD system (1) has a classical smooth solution (u, E, B) , defined on $[0, T_0]$, satisfying*

$$\|(u - u^I, B - \gamma b^I)\|_{H^{s_0}(\Omega)} \leq C\gamma \quad (10)$$

for all $0 < \gamma \leq \gamma_0$ and $0 < t \leq T_0$, where the function b^I satisfies $-\nabla \times b^I = u^I$ and $\operatorname{div} b^I = 0$.

3. Proof of Theorem 4

Now we begin to justify the convergence of e-MHD equations to incompressible Euler equations when $\gamma \rightarrow 0$. To this end, by the local existence theory and extension method, it suffices to obtain the uniform estimates of the smooth solutions to (1) with respect to the parameter γ so as to guarantee $T^\gamma > T_*$ for any given $T_0 < T_*$. Denote by $T = \min\{T_*, T^\gamma\}$ and by $C > 0$ a constant which depends on T_0 .

3.1. Derivation of Error Equations and L^2 Estimate. Let (u, E, B) be the unknown solution to the problem (1) and let (u^I, p^I) be the solution to the incompressible Euler equations (2) defined on $[0, T_*)$ given by Proposition 3. Denote this by

$$(U, F, G) = (u - u^I, E - \nabla p^I, B - \gamma b^I), \quad (11)$$

which satisfies the following problem:

$$\begin{aligned} \partial_t U + (U + u^I) \cdot \nabla U &= -F - U \times (G + \gamma b^I) \\ &\quad - u^I \times (G + \gamma b^I) - U \cdot \nabla u^I, \\ \partial_t G - \frac{1}{\gamma} \nabla \times F &= -\gamma \partial_t b^I, \end{aligned} \quad (12)$$

$$U = \frac{1}{\gamma} \nabla \times G, \quad \operatorname{div} G = 0$$

with the initial data

$$(U, G)|_{t=0} = (u_0 - u_0^I, B_0 - \gamma b^I(t=0)). \quad (13)$$

Based on L^2 -conservation of solutions to the e-MHD system, we obtain L^2 estimates of the error function (U, F, G) . Our basic idea is to cancel the oscillations of the electric field F and the magnetic field G by using the special structures of the e-MHD system.

Lemma 5. *For all $0 < t < T$ and sufficiently small γ , it holds that*

$$\begin{aligned} \int_\Omega (|U|^2 + |G|^2)(t) dx &\leq \int_\Omega (|U|^2 + |G|^2)(t=0) dx \\ &\quad + C \int_0^t (\|U\|^2 + \|G\|^2)(s) ds \\ &\quad + C\gamma^2. \end{aligned} \quad (14)$$

Proof. Taking the L^2 inner product of the first equation in the error system (12) for U , by integration by parts, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (U, U) &= -(F, U) - u^I \times (G + \gamma b^I, U) - (U \cdot \nabla u^I, U) \\ &= A_1 + A_2 + A_3, \end{aligned} \quad (15)$$

where $(\cdot, \cdot)$ stands for the L^2 inner product of two scalar or vector functions in Ω . Now we estimate each term on the right-hand side of (15).

For A_1 , by using the equation $U = (1/\gamma)\nabla \times G$ in (12) and vector analysis formula

$$\operatorname{div}(f \times g) = \nabla \times f \cdot g - \nabla \times g \cdot f, \quad (16)$$

we have

$$A_1 = -\frac{1}{\gamma} (F, \nabla \times G) = -\frac{1}{\gamma} (\nabla \times F, G). \quad (17)$$

For A_2 and A_3 , using the property of the approximate solution u^I , Cauchy-Schwarz's inequality, and Minkowski's inequality, we have

$$\begin{aligned} A_2 &\leq C(\gamma + \|G\|) \|U\| \leq C\gamma^2 + C(\|G\|^2 + \|U\|^2), \\ A_3 &\leq C\|U\|^2. \end{aligned} \quad (18)$$

Combining (17) with (18), we have

$$\frac{1}{2} \frac{d}{dt} (U, U) \leq -\frac{1}{\gamma} (\nabla \times F, G) + C(\|U\|^2 + \|G\|^2) + C\gamma^2. \quad (19)$$

Multiplying the second equation in the error system (12) by G , Cauchy-Schwarz's inequality, we get

$$\frac{1}{2} \frac{d}{dt} (G, G) \leq \frac{1}{\gamma} (\nabla \times F, G) + C\|G\|^2 + C\gamma^2. \quad (20)$$

It follows from (19) and (20) that

$$\frac{d}{dt} \int_{\Omega} (|U|^2 + |G|^2) (t) dx \leq C(\|U\|^2 + \|G\|^2) + C\gamma^2. \quad (21)$$

This completes the proof of Lemma 5. $\square$

3.2. High Order Energy Estimates. We differentiate (12) with ∂_x^α for a multi-index $\alpha \in \mathbb{N}^3$ satisfying $|\alpha| \leq s_0$ with $s_0 > (3/2) + 1$ to get that

$$(U_\alpha, F_\alpha, G_\alpha) = (\partial_x^\alpha U, \partial_x^\alpha F, \partial_x^\alpha G) \quad (22)$$

satisfies the following problem:

$$\begin{aligned} \partial_t U_\alpha + (U + u^I) \cdot \nabla U_\alpha &= -F_\alpha + \sum_{i=1}^4 \mathcal{H}_{1i}, \\ \partial_t G_\alpha - \frac{1}{\gamma} \nabla \times F_\alpha &= -\gamma \partial_t b_\alpha^I, \\ U_\alpha &= \frac{1}{\gamma} \nabla \times G_\alpha, \quad \operatorname{div} G_\alpha = 0, \end{aligned} \quad (23)$$

where

$$\begin{aligned} \mathcal{H}_{11} &= (U + u^I) \cdot \nabla U_\alpha - [(U + u^I) \cdot \nabla U]_\alpha, \\ \mathcal{H}_{12} &= -[U \times (G + \gamma b^I)]_\alpha, \\ \mathcal{H}_{13} &= -[u^I \times (G + \gamma b^I)]_\alpha, \\ \mathcal{H}_{14} &= -[U \cdot \nabla u^I]_\alpha. \end{aligned} \quad (24)$$

Before performing the energy estimate, we set

$$\mathcal{E}_{s_0}(t) = \|(U, G)\|_{H^{s_0}(\Omega)}. \quad (25)$$

Lemma 6. For all $0 < t < T$ and sufficiently small γ , it holds that

$$\begin{aligned} \int_{\Omega} (|U_\alpha|^2 + |G_\alpha|^2) (t) dx &\leq \int_{\Omega} (|U_\alpha|^2 + |G_\alpha|^2) (t=0) dx \\ &\quad + C \int_0^t (1 + \mathcal{E}_{s_0}(s)) \mathcal{E}_{s_0}^2(s) ds \\ &\quad + C\gamma^2. \end{aligned} \quad (26)$$

Proof. Taking the L^2 inner product of the first equation in (23) with U_α , one gets, by using the third equation in (23) and integration by parts, that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (U_\alpha, U_\alpha) &= -\frac{1}{\gamma} (F_\alpha, \nabla \times G) + \sum_{i=1}^4 (\mathcal{H}_{1i}, U_\alpha) \\ &= -\frac{1}{\gamma} (\nabla \times F_\alpha, G_\alpha) + \sum_{i=1}^4 (\mathcal{H}_{1i}, U_\alpha). \end{aligned} \quad (27)$$

By using Cauchy-Schwarz's inequality, the Moser-type calculus inequalities in Lemma 1, and Sobolev's lemma, we have that

$$\begin{aligned} (\mathcal{H}_{11}, U_\alpha) &\leq C(\|\nabla(U + u^I)\|_{L^\infty} \|D^{|\alpha|-1} \nabla U\| \\ &\quad + \|D^{|\alpha|}(U + u^I)\| \|\nabla U\|_{H^{s_0-1}}) \|U_\alpha\| \\ &\leq C(\|\nabla(U + u^I)\|_{H^{s_0-1}} \|D^{|\alpha|-1} \nabla U\| \\ &\quad + \|D^{|\alpha|}(U + u^I)\| \|\nabla U\|_{L^\infty}) \|U_\alpha\| \\ &\leq C(1 + \|U\|_{H^{s_0}}) \|U\|_{H^{s_0}}^2 \\ &\leq C(1 + \mathcal{E}_{s_0}(t)) \mathcal{E}_{s_0}^2(t), \\ (\mathcal{H}_{12}, U_\alpha) &\leq C(1 + \mathcal{E}_{s_0}(t)) \mathcal{E}_{s_0}^2(t), \\ (\mathcal{H}_{13}, U_\alpha) &\leq C\mathcal{E}_{s_0}^2(t) + C\gamma^2, \\ (\mathcal{H}_{14}, U_\alpha) &\leq C\mathcal{E}_{s_0}^2(t). \end{aligned} \quad (28)$$

Combining (27) with (28) together, one gets

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (U_\alpha, U_\alpha) &\leq -\frac{1}{\gamma} (F_\alpha, \nabla \times G_\alpha) + C (1 + \mathcal{E}_{s_0}(t)) \mathcal{E}_{s_0}^2(t) \\ &\quad + C\gamma^2. \end{aligned} \quad (29)$$

Taking the L^2 inner product of the second equation in (23) with G_α , one gets, by using Cauchy-Schwarz's inequality,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (G_\alpha, G_\alpha) &= \frac{1}{\gamma} (\nabla \times F_\alpha, G_\alpha) - (\gamma \partial_t b_\alpha^I, G_\alpha) \\ &\leq \frac{1}{\gamma} (\nabla \times F_\alpha, G_\alpha) + C \|G_\alpha\|^2 + C\gamma^2 \\ &\leq \frac{1}{\gamma} (\nabla \times F_\alpha, G_\alpha) + C \mathcal{E}_{s_0}^2(t) + C\gamma^2. \end{aligned} \quad (30)$$

Combining (29) and (30), one gets

$$\begin{aligned} \frac{d}{dt} \int_\Omega (|U_\alpha|^2 + |G_\alpha|^2)(t) dx &\leq C (1 + \mathcal{E}_{s_0}(t)) \mathcal{E}_{s_0}^2(t) \\ &\quad + C\gamma^2, \end{aligned} \quad (31)$$

which yields (26). $\square$

3.3. *The End of Proof of Theorem 4.* Now we let

$$z(t) = \mathcal{E}_{s_0}^2(t). \quad (32)$$

Then it follows from Lemmas 5 and 6 that there exists $\gamma_0 > 0$, depending only on T_0 , such that, for any $0 < \gamma \leq \gamma_0$ and any $0 < t < T$,

$$z(t) \leq Cz(t=0) + C \int_0^t ((1 + \sqrt{z})z)(s) ds + C\gamma^2. \quad (33)$$

Since $z(t=0) \leq C\gamma^2$ for some positive constant C , now applying Gronwall's inequality to (33), one can conclude that there exists an $\gamma_0 > 0$ sufficiently small such that, for any $0 < \gamma \leq \gamma_0$ and any $0 < t < T$,

$$z(t) \leq C\gamma^2. \quad (34)$$

Thus, using the a priori estimate (10), by the extension argument, we can conclude that $T^\gamma > T_0$ for any $T_0 < T_*$.

The proof of Theorem 4 is complete.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] A. Kingsep, K. Chukbar, and V. Yankov, *Electron Magnetohydrodynamics*, vol. 16, Review of Plasma Physics, New York, NY, USA, 1990, edited by B.B. Kadomtsev.
- [2] Y. Brenier, N. J. Mauser, and M. Puel, "Incompressible Euler and e-MHD as scaling limits of the Vlasov-Maxwell system," *Communications in Mathematical Sciences*, vol. 1, no. 3, pp. 437–447, 2003.
- [3] C. Rohde and W.-A. Yong, "The nonrelativistic limit in radiation hydrodynamics: I. Weak entropy solutions for a model problem," *Journal of Differential Equations*, vol. 234, no. 1, pp. 91–109, 2007.
- [4] L. P. Horwitz and F. C. Rotbart, "Nonrelativistic limit of relativistic quantum mechanics," *Physical Review D*, vol. 24, no. 8, pp. 2127–2131, 1981.
- [5] N. Masmoudi and K. Nakanishi, "Nonrelativistic limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger," *International Mathematics Research Notices*, no. 13, pp. 697–734, 2003.
- [6] Y.-J. Peng and W. Shu, "Rigorous derivation of incompressible e-mhd equations from compressible euler-maxwell equations," *SIAM Journal on Mathematical Analysis*, vol. 40, no. 2, pp. 540–565, 2008.
- [7] S. Wang, "Asymptotic limits of compressible Euler-Maxwell system in plasma physics," *Proceedings in Applied Mathematics and Mechanics*, vol. 7, no. 1, pp. 1041005–1041006, 2008.
- [8] S. Calogero and H. Lee, "The non-relativistic limit of the Nordström-Vlasov system," *Communications in Mathematical Sciences*, vol. 2, no. 1, pp. 19–34, 2004.
- [9] Y. Li and Y. Geng, "Non-relativistic global limits of entropy solutions to the isentropic relativistic Euler equations," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 57, no. 6, pp. 960–983, 2006.
- [10] C. Rohde and W.-A. Yong, "The nonrelativistic limit in radiation hydrodynamics: I. Weak entropy solutions for a model problem," *Journal of Differential Equations*, vol. 234, no. 1, pp. 91–109, 2007.
- [11] Y. Peng and S. Wang, "Convergence of compressible Euler-Maxwell equations to compressible Euler-Poisson equations," *Chinese Annals of Mathematics B*, vol. 28, no. 5, pp. 583–602, 2007.
- [12] J. Yang and S. Wang, "The non-relativistic limit of Euler-Maxwell equations for two-fluid plasma," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 3-4, pp. 1829–1840, 2010.
- [13] S. Klainerman and A. Majda, "Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids," *Communications on Pure and Applied Mathematics*, vol. 34, no. 4, pp. 481–524, 1981.
- [14] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer, New York, NY, USA, 1984.
- [15] F. J. McGrath, "Nonstationary plane flow of viscous and ideal fluids," *Archive for Rational Mechanics and Analysis*, vol. 27, pp. 229–348, 1968.
- [16] T. Kato, "Nonstationary flow of viscous and ideal fluids in R^3 ," *Journal of Functional Analysis*, vol. 9, pp. 296–305, 1972.
- [17] Y.-J. Peng and W. Shu, "Rigorous derivation of incompressible e-mhd equations from compressible euler-maxwell equations," *SIAM Journal on Mathematical Analysis*, vol. 40, no. 2, pp. 540–565, 2008.