Research Article

Strongly Semicontinuous Domains and Semi-FS Domains

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We are mainly concerned with some special kinds of semicontinuous domains and relationships between them. New concepts of strongly semicontinuous domains, meet semicontinuous domains and semi-FS domains are introduced. It is shown that a dcpo L is strongly semicontinuous if and only if L is semicontinuous and meet semicontinuous. It is proved that semi-FS domains are strongly semicontinuous. Some interpolation properties of semiway-below relations in (strongly) semicontinuous bc-domains are given. In terms of these properties, it is proved that strongly semicontinuous bc-domains, in particular strongly semicontinuous lattices, are all semi-FS domains.

1. Introduction

The theory of continuous lattices [1] and the more general theory of domains [2] initiated by Scott provide a mathematical foundation for the denotational semantics of programming languages and are closely linked to theoretical computer science, general topology and logics [3–5]. The purpose of the theory of domains is to give models for spaces to define computable functions. The idea is that the semantics of a programming language should be formally specified in terms of a small number of basic mathematical constructions on partial orders of information. Intuitively, we say that a state x approximates a state y if any computation of y yields the information of x at some finite stage. A logic-oriented approach to domain theory to formalize the properties of computation is provided in [2, 6].

So far, continuous lattices were generalized to other types of order structures, such as quasicontinuous posets [7, 8], Z -continuous posets [9], and Z -precontinuous posets [10]. Motivated by the concept of semiprime ideals studied by Rav in [11], Zhao in [12] first introduced the concept of semicontinuous lattices and showed that semicontinuous lattices have many properties similar to that of continuous lattices. Li and Wu [13] studied properties of semicontinuous lattices. Bi and Xu [14] introduced the semi-Scott topology and the semi-Lawson topology on semicontinuous lattices. Jiang and Shi [15] discussed characterizations of pseudoprimes and studied

strong retracts of (stable) semicontinuous lattices. In [16], Li introduced semiprime sets and generalized semicontinuous lattices to semicontinuous domains.

Note that, in the definition of a semicontinuous lattice L , the condition used is that, for every element $x \in L$, $\bigvee \downarrow x \geq x$. This condition is too weak to guarantee that every element can be approximated by elements which are below and semiway-below it in a semicontinuous lattice. To guarantee the mentioned approximation property in a suitable class of semicontinuous lattices, or even of semicontinuous domains, we in this paper introduce the concept of strongly semicontinuous domains. It turns out that strongly semicontinuous domains have domain-like features that every element can be approximated by elements below and semiway-below it. The concept of meet semicontinuous domains is also introduced. It is obtained that a dcpo is strongly semicontinuous if and only if it is semicontinuous and meet semicontinuous. Moreover, inspired by the definition of FS-domains posed by Jung [17] in the realm of domains, we introduce the concept of semi-FS domains which are defined by the existence of some approximating identities consisting of finitely separated functions in the realm of dcpos. It is proved that every semi-FS domain is a strongly semicontinuous domain and that every strongly semicontinuous bc-domain, in particular every strongly semicontinuous lattice, is a semi-FS domain. A counterexample is constructed to show that a strongly semicontinuous domain need not be a semi-FS domain.

We organize the paper as follows: Section 2 gives preliminaries; Section 3 investigates strongly semicontinuous domains; Section 4 introduces semi-FS domains and discusses their properties.

2. Preliminaries

We give some basic concepts and results which will be used in the sequel. Most of them come from [4, 12]. For other unstated concepts please refer to [16].

A subset D of a poset L is called *directed* (resp., *filtered*) if it is nonempty and every finite subset of D has an upper (resp., a lower) bound in D . A poset in which every directed set has a supremum is called a *dcpo*. For a subset A of a poset L , let $\downarrow A = \{y \in L : \exists x \in A, y \leq x\}$ and $\uparrow A = \{y \in L : \exists x \in A, y \geq x\}$. We say that A is a *lower set* (resp., an *upper set*) if $\downarrow A = A$ (resp., $\uparrow A = A$). A subset of L is an *ideal* (resp., a *filter*) if and only if it is a directed lower set (resp., filtered upper set). A *principal ideal* (resp., *principal filter*) is a set of the form $\downarrow x = \{y \in L : y \leq x\}$ (resp., $\uparrow x = \{y \in L : x \leq y\}$). The set of all ideals (resp. filters) in L is denoted by $Id(L)$ (resp., $Filt(L)$). For an ideal $P \in Id(L)$, P is said to be *prime* if $L \setminus P = \emptyset$ or $L \setminus P \in Filt(L)$. We denote the family of all prime ideals of L by $PI(L)$. A poset is called *bounded complete*, if every subset that is bounded above has a supremum. A bounded complete dcpo is called a *bc-dcpo*.

Let $S \subseteq L$. We denote $S \downarrow = \{y \in L : \forall x \in S, y \leq x\}$ and $S \uparrow = \{y \in L : \forall x \in S, y \geq x\}$ the set of upper bounds and lower bounds of S , respectively.

Lemma 1 (see [16, Lemma 3.2.8]). *Let L be a poset, $A, B \subseteq L$. If $A \subseteq B$, then we have*

$$A \uparrow \downarrow \subseteq B \uparrow \downarrow, \quad A \downarrow \uparrow \subseteq B \downarrow \uparrow. \quad (1)$$

Definition 2 (see [11]). Let L be a lattice. An ideal I of L is said to be *semiprime* if for any $x, y, z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$.

Lemma 3 (see [12, Lemma 1.2]). *An ideal P of a lattice L is semiprime if and only if there exist prime ideals P_i ($i \in I$) of L such that $P = \bigcap_{i \in I} P_i$.*

Thus every prime ideal is semiprime, and $PI(L) \subseteq Rd(L)$. And by Lemma 3, we can generalize the concept of semiprime ideals to the setting of posets.

Definition 4. Let L be a poset and $S \subseteq L$ an ideal of L . If there is a family of prime ideals $\{P_i\}_{i \in I}$ such that $S = \bigcap_{i \in I} P_i$, then S is called a *semiprime ideal*. The family of all semiprime ideals of L is denoted by $Rd(L)$.

Li in [16] generalized semiprime ideals to semiprime sets.

Definition 5 (see [16]). Let L be a poset. A set $S \subseteq L$ is said to be semiprime if there exists a family of prime ideals P_i ($i \in I$) of L such that $S = \bigcap_{i \in I} P_i$.

We denote the family of all semiprime sets of L with $SP(L)$. Clearly, a semiprime set need not to be directed. If a

semiprime set S is directed, then S is a semiprime ideal. For a dcpo, we have the following relation:

$$PI(L) \subseteq Rd(L) = SP(L) \cap Idl(L) \subseteq SP(L). \quad (2)$$

Proposition 6. *Let L be a bc-dcpo. Then $SP(L) = Rd(L)$.*

Proof. It suffices to show that $SP(L) \subseteq Rd(L)$. Let $A \in SP(L)$. Then by Definition 5 there exists a family of prime ideals P_i ($i \in I$) of L such that $A = \bigcap_{i \in I} P_i$. For any $y_1, y_2 \in A$, we have $y_1, y_2 \in P_i$ for all $i \in I$. Since P_i is directed, there exists $y \in P_i$ such that $y_1, y_2 \leq y$. As L is a bc-dcpo, we see that $y_1 \vee y_2$ exists and $y_1 \vee y_2 \in P_i$ for all $i \in I$. Therefore, $y_1 \vee y_2 \in A$. So, A is directed and $A \in Rd(L)$, as desired. $\square$

Lemma 7. *For a bc-dcpo L , let $M \subseteq L$ and $L^\top = L \cup \{\top\}$ be the complete lattice obtained from L by adjoining a top element $\top$. Then we have*

- (i) $M \in PI(L)$ if and only if $M \in PI(L^\top)$;
- (ii) $M \in Rd(L)$ if and only if $M \in Rd(L^\top)$.

Proof. (i) Suppose that $M \in PI(L^\top)$. Since $M \subseteq L$, we have $L \setminus M = L^\top \setminus M \in Filt(L)$ and $M \in PI(L)$.

Conversely, let $M \in PI(L)$. Then $M \in Idl(L)$ and $L \setminus M \in Filt(L)$. So, $L^\top \setminus M = (L \setminus M) \cup \{\top\} \in Filt(L^\top)$ and $M \in PI(L^\top)$.

(ii) Suppose that $M \in Rd(L)$. Then by Definition 4, $M = \bigcap_{i \in I} P_i$, where $P_i \in PI(L)$ ($i \in I$). It follows from (i) above that $P_i \in PI(L^\top)$ for all $i \in I$; thus, $M \in Rd(L^\top)$.

Conversely, let $M \in Rd(L^\top)$. By Lemma 3, there exist prime ideals P_i ($i \in I$) of $L^\top$ such that $M = \bigcap_{i \in I} P_i$. Since $\top \notin M$, $\top \notin P_{i_0}$ for some $i_0 \in I$. Let $J = \{j \in I : \top \notin P_j\}$. Then for any $i \in I \setminus J$, we see that $\top \in P_i = L^\top$. By (i) above, we have $P_j \in PI(L)$ for each $j \in J$. Then $M = \bigcap_{j \in J} P_j$, and $M \in SP(L)$. By Proposition 6, $M \in Rd(L)$. $\square$

In a poset L , we say that x is *way-below* y , or x *approximates* y , written $x \ll y$, and if D is directed with existing $\sup D$ and $\sup D \geq y$, then $x \leq d$ for some $d \in D$. Equivalently, $x \ll y$ iff $x \in I$ for every ideal I of L such that $y \leq \sup I$ whenever $\sup I$ exists. We use $\downarrow x$ to denote the set $\{a \in L : a \ll x\}$. If L is a dcpo and, for every element $x \in L$, the set $\downarrow x$ is directed and $\sup \downarrow x = x$, then L is called a *domain*. A domain L is called an *L-domain* if for each $x \in L$, the principal ideal $\downarrow x$ is a complete lattice. A complete lattice which is a domain is called a *continuous lattice*.

For complete lattices, replacing ideals with semiprime ideals, Zhao in [12] defined a weak form of the way-below relation.

Definition 8 (see [12]). Let L be a complete lattice. Define the semiway-below relation $\Leftarrow$ on L as follows: for $x, y \in L$, $x \Leftarrow y$ if for any semiprime ideal I of L , $y \leq \vee I$ implies $x \in I$. For each $x \in L$, we write $\Downarrow x = \{y \in L : y \Leftarrow x\}$, $\Uparrow x = \{y \in L : x \Leftarrow y\}$.

Definition 9 (see [12]). A complete lattice L is said to be *semicontinuous*, if for any $x \in L$, $x \leq \vee(\Downarrow x)$.

Zhao [12] showed that the interpolation property holds in semicontinuous lattices.

Theorem 10 (see [12, Theorem 1.8]). *If L is a semicontinuous lattice, then $x \Leftarrow y$ implies the existence of a $z \in L$ such that $x \Leftarrow z \Leftarrow y$.*

3. Strongly Semicontinuous Domains

In terms of semiprime sets, semicontinuous lattices can be generalized to semicontinuous domains. And then strongly semicontinuous domains will be defined.

Definition 11. Let L be a poset. Define the relation $\Leftarrow$ on L as follows: for any $x, y \in L$, $y \Leftarrow x$ if for any semiprime set S of L , $x \in S \uparrow \downarrow$ implies $y \in S$. An element k of L is said to be *semicompact* if $k \Leftarrow k$. For each $x \in L$, we write $\Downarrow x = \{y \in L : y \Leftarrow x\}$ and $\Uparrow x = \{y \in L : x \Leftarrow y\}$.

Proposition 12. *Let L be a dcpo; then,*

$$\begin{aligned} \Downarrow x &= \cap \{S \in SP(L) : x \in S \uparrow \downarrow\} = \cap \{S \in Rd(L) : x \leq \vee S\} \\ &= \cap \{P \in PI(L) : x \leq \vee P\}. \end{aligned} \quad (3)$$

So, for each $x \in L$, $\Downarrow x \in SP(L)$. If L is a bc-dcpo, then $\Downarrow x \in Rd(L)$.

Proof. Note that $J \uparrow \downarrow = \downarrow (\vee J)$ for each $J \in Id(L)$. So, by Lemma 1, we have that

$$\begin{aligned} \Downarrow x &= \cap \{S \in SP(L) : x \in S \uparrow \downarrow\} \subseteq \cap \{S \in Rd(L) : x \in \downarrow (\vee S)\} \\ &\subseteq \cap \{P \in PI(L) : x \in \downarrow (\vee P)\}. \end{aligned} \quad (4)$$

Let $A = \cap \{P \in PI(L) : x \in \downarrow (\vee P)\}$. Then $A \in SP(L)$.

Next we show that $A \subseteq \Downarrow x$. Let $y \in A$. For any $M \in SP(L)$ with $x \in M \uparrow \downarrow$, by Definition 5 there exists a family of prime ideals $\{P_i\}_{i \in I}$ such that $M = \bigcap_{i \in I} P_i$. It follows from Lemma 1 that $x \in M \uparrow \downarrow \subseteq \bigcap_{i \in I} P_i \uparrow \downarrow = \downarrow (\vee P_i)$ for each $i \in I$. Since $y \in A = \cap \{P \in PI(L) : x \in \downarrow (\vee P)\}$, we have that $y \in P_i$ for each $i \in I$ and $y \in \bigcap_{i \in I} P_i = M$. By Definition 11, $y \Leftarrow x$. So, $A \subseteq \Downarrow x$, as desired. $\square$

In [16], the semiway-below relation $\Leftarrow_s$ on a poset was defined [16, Definition 3.2.2] in a different way. For a dcpo L and $x \in L$, it is established in [16, Proposition 3.2.5] that $\Downarrow_s x = \cap \{P \in PI(L) : x \leq \vee P\}$. So, by Proposition 12 above, we see that $\Downarrow x = \Downarrow_s x$. This means that for a dcpo L and $x, y \in L$, $y \Leftarrow_s x \Leftrightarrow y \Leftarrow x$. Thus, in the setting of dcpos, Definition 11 is equivalent to Definition 3.2.2 in [16].

Note that for a poset L and $x, y \in L$, $y \Leftarrow_s x$ need not imply $y \Leftarrow x$.

Remark 13. Let L be a dcpo and $a, b, c, d \in L$. Then it is easy to check that

- (1) $a \Leftarrow b$ does not imply $a \leq b$, the typical modular lattice M_5 is a counterexample;
- (2) if $a \leq b \Leftarrow c \leq d$, then $a \Leftarrow d$. If $a \Leftarrow b \Leftarrow c$, then $a \Leftarrow c$;

(3) if $a \vee b$ exists and $a, b \Leftarrow c$, then $a \vee b \Leftarrow c$;

(4) for any $a, b \in L$, $a \Leftarrow b$ implies $a \Leftarrow b$.

Definition 14 (see [16]). A dcpo L is called a *semicontinuous dcpo* if for all $x \in L$, $x \in (\Downarrow x) \uparrow \downarrow$. A semicontinuous dcpo will also be called a *semicontinuous domain*. A bc-dcpo which is semicontinuous will be called a *semicontinuous bc-domain*.

By Proposition 12, one can immediately have the following.

Proposition 15. *Let L be a dcpo. Then L is semicontinuous iff for any $x \in L$, $\Downarrow x$ is the smallest semiprime set S such that $x \in S \uparrow \downarrow$.*

Proposition 16. *Let L be a dcpo. If, for any $x \in L$, there exists a subset $A \subseteq \Downarrow x$ and $x \in A \uparrow \downarrow$, then L is semicontinuous.*

Proof. It follows from Lemma 1 and Definition 14. $\square$

Strengthening the condition in Definition 14, we give the following.

Definition 17. A dcpo L is said to be *strongly semicontinuous* if for each $x \in L$,

$$x \in (\Downarrow x \cap \downarrow x) \uparrow \downarrow. \quad (5)$$

A dcpo (bc-dcpo) which is strongly semicontinuous will be called a *strongly semicontinuous domain* (*strongly semicontinuous bc-domain*). A complete lattice which is strongly semicontinuous will be called a *strongly semicontinuous lattice*.

Clearly, every strongly semicontinuous domain is semicontinuous, a semicontinuous domain L satisfying the condition $x \in (\Downarrow x \cap \downarrow x) \uparrow \downarrow$ ($\forall x \in L$) is strongly semicontinuous. It is easy to see that, for a dcpo L without proper prime ideals, every pair of elements in L has the semiway-below relation $\Leftarrow$ and L is strongly semicontinuous. However, a semicontinuous domain need not be strongly semicontinuous. The following counterexample first appeared in [18].

Example 18 (see [18]). Let $L = \{\perp, a, b, x, \top\} \cup \{x_n : n = 0, 1, \dots\}$. The partial order on L is defined by $\perp \leq a, b \leq x_0 \leq \dots \leq x_n \leq \dots \leq \top$, $\perp \leq x \leq \top$. It is clear that the prime ideals of L are $L \setminus \uparrow x$ and L . We observe that, for any $t \in L$, $\Downarrow t = L \setminus \uparrow x$ and $t \leq \vee \Downarrow t$. Thus, L is a semicontinuous domain. However, note that, in this example $x \notin (\Downarrow x \cap \downarrow x) \uparrow \downarrow$, which yields that L is not strongly semicontinuous.

Proposition 19. *Every domain is a strongly semicontinuous domain.*

Proof. It follows from Remark 13(4) that $\forall x \in L$, $\downarrow x \subseteq \Downarrow x \cap \downarrow x$. By Lemma 1, $x \in \downarrow x = \downarrow (\vee \downarrow x) = (\downarrow x) \uparrow \downarrow \subseteq (\Downarrow x \cap \downarrow x) \uparrow \downarrow$. Thus, L is a strongly semicontinuous. $\square$

Wu and Li in [19] introduced the following concept of meet semicontinuous lattices.

Definition 20 (see [19]). A complete lattice L is said to be *meet semicontinuous* if for any $x \in L$ and $I \in Rd(L)$, $x \wedge (\vee I) = \vee(x \wedge I)$.

It is known that a semicontinuous lattice need not be a meet semicontinuous lattice. However, it is proved in [18] that strongly semicontinuous lattices are all meet semicontinuous. Generalize meet semicontinuous lattices, we have the following

Definition 21. A dcpo L is said to be *meet semicontinuous* if for any $x \in L$ and $S \in SP(L)$, $(\downarrow x \cap S) \uparrow \downarrow = \downarrow x \cap S \uparrow \downarrow$.

It is easy to check that, for complete lattices, the meet semicontinuity in Definitions 20 and 21 are equivalent.

Proposition 22. Every strongly semicontinuous domain is meet semicontinuous.

Proof. Suppose that L is a strongly semicontinuous domain. For any $x \in L$ and $S \in SP(L)$, by Lemma 1, it is easy to see that $\downarrow x \cap S \uparrow \downarrow \supseteq (\downarrow x \cap S) \uparrow \downarrow$. So, it suffices to show that $\downarrow x \cap S \uparrow \downarrow \subseteq (\downarrow x \cap S) \uparrow \downarrow$. To this end, let $t \in \downarrow x \cap S \uparrow \downarrow$. Then $t \in S \uparrow \downarrow$ and $t \in (\downarrow t \cap \downarrow x) \uparrow \downarrow$ since L is strongly semicontinuous. For any $r \in \downarrow t \cap \downarrow x$, we have that $r \in S \uparrow \downarrow$ and $r \leq t \leq x$. Thus, by Definition 11, $r \in S$ and $r \in \downarrow x \cap S$. By the arbitrariness of $r \in \downarrow t \cap \downarrow x$, we see that $\downarrow t \cap \downarrow x \subseteq \downarrow x \cap S$. Therefore, $t \in (\downarrow t \cap \downarrow x) \uparrow \downarrow \subseteq (\downarrow x \cap S) \uparrow \downarrow$. So, $\downarrow x \cap S \uparrow \downarrow \subseteq (\downarrow x \cap S) \uparrow \downarrow$, as desired. $\square$

Remark 23. Note that, for the complete lattice in Example 18 and the prime ideal $L \setminus \uparrow x$, we have $\downarrow x \cap (L \setminus \uparrow x) \uparrow \downarrow \neq (\downarrow x \cap (L \setminus \uparrow x)) \uparrow \downarrow$, revealing that L is not meet semicontinuous.

Theorem 24. A dcpo L is strongly semicontinuous iff L is semicontinuous and meet semicontinuous.

Proof. $\Rightarrow$: By Proposition 22.

$\Leftarrow$: For each $x \in L$, by the semicontinuity of L we have that $x \in (\downarrow x) \uparrow \downarrow$. It follows from meet semicontinuity of L that $x \in \downarrow x \cap (\downarrow x) \uparrow \downarrow = (\downarrow x \cap \downarrow x) \uparrow \downarrow$. So, by Definition 17, L is strongly semicontinuous. $\square$

Proposition 25. For a (strongly) semicontinuous bc-domain L , let $L^\top = L \cup \{\top\}$ be the complete lattice obtained from L by adjoining a top element $\top$. Then $L^\top$ is a (strongly) semicontinuous lattice.

Proof. Firstly, we show that $\downarrow_{L^\top} x \supseteq \downarrow_L x$ for each $x \in L$. Let $y \in \downarrow_L x$. Then for any $P \in Rd(L^\top)$ with $\vee_{L^\top} P \geq x$, if $P = L^\top$, then $y \in P$. If $P \neq L^\top$, then $\top \notin P$ and $P \subseteq L$. By Lemma 7(ii), $P \in Rd(L)$. Thus, $\vee_L P = \vee_{L^\top} P \geq x$ and $y \in P$. So, $y \in \downarrow_{L^\top} x$ and $\downarrow_L x \subseteq \downarrow_{L^\top} x$ for each $x \in L$.

If L itself is a (strongly) semicontinuous lattice, then $\top$ is isolated in $L^\top$ and $L^\top$ is trivially a (strongly) semicontinuous lattice.

For a bc-dcpo L without the biggest element, we see that $\vee_{L^\top} L = \top$. So, if L is semicontinuous, then $\vee_{L^\top} \downarrow_{L^\top} x \geq \vee_{L^\top} \downarrow_L x \geq x$ for each $x \in L$. It follows from

$$\downarrow_{L^\top} \top \supseteq \cup_{x \in L} \downarrow_{L^\top} x \supseteq \cup_{x \in L} \downarrow_L x \quad (6)$$

that $\vee_{L^\top} (\downarrow_{L^\top} \top \cap \downarrow \top) = \vee_{L^\top} \downarrow_{L^\top} \top \geq \vee_{L^\top} \cup_{x \in L} \downarrow_L x \geq \vee_{L^\top} L = \top$. If L is strongly semicontinuous, then for all $x \in L$, $x \geq \vee_{L^\top} (\downarrow_{L^\top} x \cap \downarrow x) \geq \vee_{L^\top} (\downarrow_L x \cap \downarrow x) \geq x$ and $\vee_{L^\top} (\downarrow_{L^\top} x \cap \downarrow x) = x$.

To sum up, if L is a semicontinuous bc-domain, then $\forall y \in L^\top$, $y \leq \vee_{L^\top} \downarrow_{L^\top} y$ and $L^\top$ is a semicontinuous lattice; if L is a strongly semicontinuous bc-domain, then $\forall y \in L^\top$, $\vee_{L^\top} (\downarrow_{L^\top} y \cap \downarrow y) = y$, and $L^\top$ is a strongly semicontinuous lattice. $\square$

Corollary 26. If L is semicontinuous bc-domain, then $\forall x \in L$, $\downarrow_{L^\top} x = \downarrow_L x$.

Proof. By the proof of Proposition 25, we have that $\downarrow_{L^\top} x \supseteq \downarrow_L x$ and $L^\top$ is a semicontinuous lattice. Then it follows from Propositions 12 and 15 and Lemma 7(ii) that $\downarrow_{L^\top} x \subseteq \downarrow_L x$. So, $\downarrow_{L^\top} x = \downarrow_L x$. $\square$

Whether the interpolation property holds or not in semicontinuous domains is still unknown. So, by Theorem 10 and Corollary 26, we immediately have the following.

Corollary 27. If L is a (strongly) semicontinuous bc-domain, then $x \Leftarrow y$ implies the existence of a $z \in L$ such that $x \Leftarrow z \Leftarrow y$.

Next we will show that (strongly) semicontinuous bc-domains also exhibit some strong types of interpolation properties.

Proposition 28. In a (strongly) semicontinuous bc-domain L ,

(i) for all $x, y \in L$ with $x \leq y$, one has

$$(SI^{\leq}) \quad x \Leftarrow y \text{ implies } (\exists z \in L) \quad (x \Leftarrow z \Leftarrow y, x \leq z); \quad (7)$$

(ii) for all $x, y \in L$ with $x < y$, one has

$$(SI^{<}) \quad x \Leftarrow y \text{ implies } (\exists z \in L) \quad (x \Leftarrow z \Leftarrow y, x < z) \quad (8)$$

Proof. (i) For any $x, y \in L$, $x \leq y$, and $x \Leftarrow y$, it follows from Corollary 27 that there exists a $z_0 \in L$ such that $x \Leftarrow z_0 \Leftarrow y$. Noticing that $\downarrow y \in Rd(L)$, by Proposition 6, there exists a $w \in Rd(L)$ such that $x, z_0 \leq w$. Since L is a bc-dcpo, $x \vee z_0$ exists and $x \Leftarrow z_0 \leq x \vee z_0 \Leftarrow y$. Thus, $(SI^{\leq})$ holds.

(ii) Let $x, y \in L$ with $x < y$, and $x \Leftarrow y$. By Corollary 27 there exists a $z_0 \in L$ such that $x \Leftarrow z_0 \Leftarrow y$. Since $\downarrow y \in Rd(L)$ and $y \leq \vee \downarrow y$, there exists $z_1 \in \downarrow y$ such that $z_1 \not\Leftarrow x$. Noticing that $z_0, z_1 \in \downarrow y \in Rd(L)$, we see that $z_0 \vee z_1$ exists. Set $z = z_0 \vee z_1$; then, $x < z$. Therefore, $(SI^{<})$ holds. $\square$

4. Semi-FS Domains

In this section, we introduce semi-FS domains which are counterparts of FS-domains posed by Jung [17] in the setting of strongly semicontinuous domains. It is proved that every strongly semicontinuous bc-domain, in particular every strongly semicontinuous lattice, is a semi-FS domain.

Definition 29 (see [16]). Let L, M be dcpos. A function $f : L \rightarrow M$ is said to *preserve suprema of prime ideals* if it is order-preserving and for any $P \in PI(L)$, $f(\bigvee P) = \bigvee f(P)$.

Let L, M be dcpos. We use $[L, M]$ to denote all order-preserving functions from L to M , use $[L \hookrightarrow M]$ to denote all the functions preserving suprema of prime ideals from L to M and use $[L \rightarrow M]$ to denote all the Scott-continuous functions from L to M . All of them are under the pointwise order. It is easy to see that $[L \rightarrow M] \subseteq [L \hookrightarrow M] \subseteq [L, M]$.

Proposition 30. Let L, M be dcpos. Then $[L \hookrightarrow M]$ is a dcpo.

Proof. Let $\mathcal{G}$ be a directed subset of $[L \hookrightarrow M]$ and $f(x) = \bigvee_{g \in \mathcal{G}} g(x)$ for all $x \in L$. Then it is easy to see that f is order-preserving. For any prime ideal $P \in PI(L)$, we have

$$\begin{aligned} \bigvee f(P) &= \bigvee_{x \in P} f(x) = \bigvee_{x \in P} \bigvee_{g \in \mathcal{G}} g(x) = \bigvee_{g \in \mathcal{G}} \bigvee g(P) \\ &= \bigvee_{g \in \mathcal{G}} g(\bigvee P) = f(\bigvee P). \end{aligned} \quad (9)$$

Thus, $f \in [L \hookrightarrow M]$, showing that $[L \hookrightarrow M]$ is a dcpo. $\square$

Definition 31. Let L be a dcpo. If $\mathcal{D} \subseteq [L, L]$ is directed and $\sup_{\delta \in \mathcal{D}} \delta = Id_L$, then we say that $\mathcal{D}$ is an *approximate identity* for L .

Proposition 32. Let L be a dcpo. If L has an approximate identity $\mathcal{D} \subseteq [L \hookrightarrow L]$ such that $\delta(x) \leftarrow x$ for all $\delta \in \mathcal{D}$ and for all $x \in L$, then L is a strongly semicontinuous domain.

Proof. For any $x \in L$, let $A = \{\delta(x) : \delta \in \mathcal{D}\}$. Then $A \subseteq \downarrow x \cap \downarrow x$. Note that $\mathcal{D}$ is directed and $x \in \downarrow (\bigvee A) = A \uparrow \downarrow$. By the Proposition 16 and Definition 17, it follows that L is a strongly semicontinuous domain. $\square$

Definition 33 (see [17]). Let L be a dcpo. A function $\delta : L \rightarrow L$ on L is finitely separating if there is a finite set F_δ such that, for each $x \in L$, there exists $y \in F_\delta$ such that $\delta(x) \leq y \leq x$.

A dcpo L is called a *semi-FS domain* if there is an approximate identity $\mathcal{D} \subseteq [L \hookrightarrow L]$ consisting of finitely separating functions.

For an FS-domain L , there is an approximate identity $\mathcal{D} \subseteq [L \rightarrow L] \subseteq [L \hookrightarrow L]$ for L consisting of finitely separating functions. So, an FS-domain is a semi-FS domain.

Proposition 34. Let L be a dcpo. If $\delta \in [L \hookrightarrow L]$ is finitely separating, then, for all $x \in L$, $\delta(x) \leftarrow x$. Thus a semi-FS domain is a strongly semicontinuous domain.

Proof. Suppose that $x \in L$ and $P \in PI(L)$ with $x \leq \bigvee P$. Since δ is finitely separating, there exists a finite set F_δ such that for

FIGURE 1: L is strongly semicontinuous but not a semi-FS domain.

each $d \in P$ there exists $y_d \in F_\delta$ with $\delta(d) \leq y_d \leq d$. Let $F'_\delta = \{y_d \in F_\delta : d \in P\}$, a nonempty finite subset of F_δ . Then for each $y \in F'_\delta$, we can get $d_y \in P$ such that $\delta(d_y) \leq y \leq d_y$. As P is a prime ideal, there exists $d_0 \in P$ such that $y \leq d_y \leq d_0$ for all $y \in F'_\delta$. Hence for all $d \in P$, $\delta(d) \leq y_d \leq d_0$ and $\delta(x) \leq \delta(\bigvee P) = \bigvee_{d \in P} \delta(d) \leq d_0$. Therefore, $\delta(x) \in P$. It follows from Proposition 12 that $\delta(x) \leftarrow x$.

By Proposition 32 we see that L is a strongly semicontinuous domain. $\square$

The next example gives a strongly semicontinuous domain which is not a semi-FS domain, showing that the reverse of Proposition 34 is not true.

Example 35. Let $L = \{0, b_1, b_2\} \cup \{a_j : j = 1, 2, \dots\}$ be the domain showing in Figure 1, where $0 < b_i < a_j$ for $i = 1, 2$ and $j = 1, 2, \dots$. It is clear that L is an L -domain but not compact in the Lawson topology. By [20, Corollary 2.2], we immediately see that L is not an FS-domain. Note that, in L , every directed set is finite. So, an order-preserving function from L to L preserves directed sups in L . And $[L \rightarrow L] \subseteq [L \hookrightarrow L] \subseteq [L, L] = [L \rightarrow L]$. So, $[L \rightarrow L] = [L \hookrightarrow L]$. Since L is not an FS-domain, L is not a semi-FS domain either.

Proposition 36. Every strongly semicontinuous bc-domain is a semi-FS domain.

Proof. Let L be a strongly semicontinuous bc-domain. For each $x \in L$, $S \in \mathcal{P}_{\text{fin}}(L)$, define $\delta_S : L \rightarrow L$ by $\delta_S(x) = \bigvee \{y \in S \cap \downarrow x : y \leftarrow x\}$. If $\{y \in S \cap \downarrow x : y \leftarrow x\} = \emptyset$, then $\delta_S(x) = \perp$, the least element of L . So, $\delta_S(x)$ is well-defined. It is easy to see that δ_S is order-preserving with $\delta_S(x) \leq x$ for all $x \in L$. Next we show that δ_S preserves suprema of prime ideals. For each $P \in PI(L)$, it suffices to show that $\delta_S(\bigvee P) = \delta_S(k) \leq \bigvee_{p \in P} \delta_S(p)$, where $k = \bigvee P$. If $\{y \in S \cap \downarrow k : y \leftarrow k\} = \emptyset$, then, by the definition of δ_S , we see that $\delta_S(k) = \perp \leq \bigvee_{p \in P} \delta_S(p)$. Let $\{y_1, \dots, y_l\} = \{y \in S \cap \downarrow k : y \leftarrow k\} \neq \emptyset$, then $m = \bigvee_{i=1}^l y_i$ exists in L . By the definition of δ_S , $\delta_S(k) = m \leq k$. Since $y_i \leftarrow k$ for $i = 1, 2, \dots, l$, we have that $m \leftarrow k$. By $(SI^\leq)$ in Proposition 28, there is a $m^* \geq m$ such that $m \leftarrow m^* \leftarrow k$. It follows from $m^* \leftarrow k = \bigvee P$ that $m, m^* \in P$. Noticing that $m \leq m^*$ and $y_i \leq m \leftarrow m^* \in P$, we have $\{y_1, \dots, y_l\} \subseteq \{y \in S \cap \downarrow m^* : y \leftarrow m^*\}$ which yields that $\bigvee_{p \in P} \delta_S(p) \geq \delta_S(m^*) \geq \delta_S(k)$. So, δ_S preserves suprema of prime ideals, and $\delta \in [L \hookrightarrow L]$.

Suppose that $S, T \in \mathcal{P}_{\text{fin}}(L)$ with $S \subseteq T$. It is easy to see that $\delta_S \leq \delta_T$ and $\mathcal{D} = \{\delta_S\}_{S \in \mathcal{P}_{\text{fin}}(L)}$ is directed. Let $F_{\delta_S} = \{\delta_S(x) : x \in L\}$. Then F_{δ_S} is finite by the finiteness of S . So, δ_S is a finitely separating function. Since L is strongly semicontinuous, for each $x \in L$ and $\delta_S \in \mathcal{D}$, $\delta_S(x) = \vee\{y \in S \downarrow x : y \Leftarrow x\} \leq x$ and

$$\begin{aligned} \vee_{\delta_S \in \mathcal{D}} \delta_S(x) &= \vee_{\delta_S \in \mathcal{D}} (\vee\{y \in S \downarrow x : y \Leftarrow x\}) \\ &\geq \vee_{z \Leftarrow x} (\vee\{y \in \{z\} \cap \downarrow x : y \Leftarrow x\}) \quad (10) \\ &= \vee(\downarrow x \cap \downarrow x) = x. \end{aligned}$$

Therefore, $\mathcal{D} = \{\delta_S\}_{S \in \mathcal{P}_{\text{fin}}(L)} \subseteq [L \hookrightarrow L]$ is an approximate identity for L consisting of finitely separating functions, and L is a semi-FS domain. $\square$

Corollary 37. *Every strongly semicontinuous lattice is a semi-FS domain.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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