Research Article

Hyperbolic Cross Truncations for Stochastic Fourier Cosine Series

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Based on our decomposition of stochastic processes and our asymptotic representations of Fourier cosine coefficients, we deduce an asymptotic formula of approximation errors of hyperbolic cross truncations for bivariate stochastic Fourier cosine series. Moreover we propose a kind of Fourier cosine expansions with polynomial factors such that the corresponding Fourier cosine coefficients decay very fast. Although our research is in the setting of stochastic processes, our results are also new for deterministic functions.

1. Introduction

For approximations of multivariate functions by algebraic/trigonometric polynomials on full grids, the approximation rate deteriorates rapidly as the dimension d increases [1, 2]; this is just so-called “dimension curse” problem. In order to solve it, hyperbolic cross approximations have received much attention in recent years [1, 3–5]. For multivariate periodic functions, Griebel and Hamaekers [4] discussed hyperbolic cross trigonometric approximation and gave the corresponding error estimate. Instead of trigonometric polynomial space on full grids, they used the hyperbolic cross space:

$$\mathbf{X}_N = \text{span} \left\{ \exp(i\mathbf{k} \cdot \mathbf{x}) : |\mathbf{k}|_{\text{mix}} = \prod_{j=1}^d \max\{|k_j|, 1\} \leq N \right\}, \quad (1)$$

as approximation space. In 2010, Shen and Wang [1] studied the hyperbolic cross Jacobi polynomial approximation for functions on the unit cube and gave various formulas on error estimates. Moreover, they also considered the hyperbolic cross Hermite/Laguerre polynomial approximation. In 2009, Boyd [6] deeply researched large-degree asymptotics for Fourier, Chebyshev, and Hermite coefficients of analytic functions. For high-dimensional polynomial interpolation

approximation, one often uses tensor product method to generalize one-dimensional interpolation polynomial approximation with Chebyshev knots. In 2000, Barthelmann et al. [3] showed that when the dimension is larger than 10, it is suggested to use sparse grids instead of full grids. Moreover, Barthelmann et al. obtained the corresponding error estimates.

Up to now, for various hyperbolic cross approximation, the asymptotic formulas of approximation errors are not available. In this paper, we will deeply study the hyperbolic cross approximation in Fourier cosine analyses and give a precise asymptotic formula of approximation error of hyperbolic cross truncation. Although our research is in the setting of stochastic processes, our results are also new for deterministic functions.

In order to obtain asymptotic formulas of approximation errors, we first decompose stochastic process ξ on $[0, 1]^2$ into a sum of three terms $\xi = P_\xi + \xi_2 + \xi_3$, where P_ξ is a stochastic polynomial determined by partial derivatives of ξ at vertexes of $[0, 1]^2$, ξ_2 is determined by partial derivative values of ξ on the boundary of $[0, 1]^2$ and is a sum of four univariate stochastic processes with simple polynomial factors, and ξ_3 is a bivariate stochastic process whose partial derivatives vanish on the boundary of $[0, 1]^2$.

Secondly, based on the above decomposition, we show that Fourier cosine coefficients of a bivariate stochastic process ξ on $[0, 1]^2$ can be approximated asymptotically by

a combination of partial derivatives of ξ at vertexes of $[0, 1]^2$ and univariate cosine coefficients on the boundary of $[0, 1]^2$.

Thirdly, for hyperbolic cross approximation, we give the following precise result: if ξ is a stochastic process on $[0, 1]^2$ with smoothness index $l \geq 2$, then its hyperbolic cross truncations $S_N^{(h)}(\xi)$ (see (16)) of stochastic Fourier cosine series of ξ satisfy the following asymptotic formula:

$$E \left[\int_{[0,1]^2} (S_N^{(h)}(\xi; t) - \xi(t))^2 dt \right] = \frac{M \log N}{N^3} (1 + o(1)) \quad (2)$$

$$(N \rightarrow \infty),$$

where

$$M = \frac{4}{3\pi^8} E \left[(\xi^{(1,1)}(0, 0))^2 + (\xi^{(1,1)}(0, 1))^2 + (\xi^{(1,1)}(1, 0))^2 + (\xi^{(1,1)}(1, 1))^2 \right], \quad (3)$$

and E is the mathematical expectation.

Finally, based on our decomposition of stochastic processes, we propose Fourier cosine expansions with polynomial factors (see (11)) whose hyperbolic cross truncations are a combination of stochastic algebraic polynomials and stochastic cosine polynomials. When the smoothness index $l \geq 3$ of the stochastic process, for partial sum approximation, hyperbolic cross approximation, and hyperbolic cross approximation with polynomial factors, the square of their approximation errors are

$$O\left(\frac{1}{N_c^{3/2}}\right), \quad O\left(\frac{\log^4 N_c}{N_c^3}\right), \quad o\left(\frac{\log^6 N_c}{N_c^5}\right), \quad (4)$$

respectively, where N_c is the number of Fourier cosine coefficients used in each approximation method. From this, we see that hyperbolic cross approximation with polynomial factors can reconstruct the stochastic process on $[0, 1]^2$ by using the least Fourier cosine coefficients.

This paper is organized as follows. In Section 2 we recall stochastic calculus and stochastic Fourier cosine series. In Section 3 we give decompositions of stochastic processes. In Section 4 we discuss univariate stochastic Fourier cosine analyses. In Section 5 we give an asymptotic formula of Fourier cosine coefficients for bivariate stochastic processes. In Section 6 we discuss partial sum approximations. In Section 7 we discuss hyperbolic cross approximations. In Section 8, we present the Fourier cosine series with polynomial factors and study its hyperbolic cross approximations.

2. Fourier Cosine Series of Stochastic Processes

We recall some concepts in calculus of stochastic processes and stochastic Fourier cosine series.

2.1. Calculus of Stochastic Processes. For a stochastic variable ξ , we denote its expectation, second-order moment, and variance by $E[\xi]$, $E[\xi^2]$, and $\text{Var}(\xi)$, respectively. If $\xi(t)$ is a

stochastic variable for each $t \in [0, 1]^d$, then we say $\xi(t)$ is a stochastic process on $[0, 1]^d$. In this paper, we always assume that a stochastic process $\xi(t)$ is real-valued and satisfies $E[\xi^2(t)] < \infty$ for each t . This ensures that its expectation, variance, and second-order moment always exist. Calculus of stochastic processes is a generalization of classical calculus. Let $\{\xi_n\}_1^\infty$ be a sequence of stochastic variables and let ξ be a stochastic variable. If $\lim_{n \rightarrow \infty} E[|\xi_n - \xi|^2] = 0$, we say that ξ is the limit of $\{\xi_n\}_1^\infty$. Starting from the concept of the limit, one defines continuity, derivatives, partial derivatives, integrals, and double integrals of stochastic processes [7]. Moreover, Newton-Leibnitz formula in calculus of stochastic processes is as follows. If ξ is a differentiable stochastic process on $[0, 1]$ and the derivative ξ' is continuous on $[0, 1]$, then

$$\int_0^1 \xi'(t) dt = \xi(1) - \xi(0). \quad (5)$$

Let $\xi(t)$ be a continuously differentiable stochastic process on $[0, 1]$ and let $f(t)$ be a continuously differentiable deterministic function on $[0, 1]$. Then [7]

$$(\xi(t)f(t))' = \xi'(t)f(t) + \xi(t)f'(t);$$

$$\int_0^1 \xi(t)f'(t) dt = \xi(1)f(1) \quad (6)$$

$$- \xi(0)f(0) - \int_0^1 \xi'(t)f(t) dt.$$

Burkardt et al. studied stochastic partial differential equations in [8]. Xiu reviewed the current state-of-the-art of numerical method for stochastic computations in [9].

2.2. Fourier Cosine Series. If $\xi(t)$ is a univariate stochastic process on $[0, 1]$ and $E[\int_0^1 \xi^2(t) dt] < \infty$, then it can be expanded into the Fourier cosine series

$$\xi(t) = \sum_{n=0}^{\infty} c_n(\xi) \cos \pi n t \quad (7)$$

in mean square sense; that is,

$$E \left[\|S_N(\xi) - \xi\|_2^2 \right] = E \left[\int_0^1 \left(\sum_{n=0}^{N-1} c_n(\xi) \cos(\pi n t) - \xi(t) \right)^2 dt \right] \rightarrow 0 \quad (8)$$

as $N \rightarrow \infty$,

where

$$c_0(\xi) = \int_0^1 \xi(t) dt, \quad (9)$$

$$c_n(\xi) = 2 \int_0^1 \xi(t) \cos(\pi n t) dt \quad (n \neq 0).$$

The corresponding Parseval identity is

$$E \left[\int_0^1 \xi^2(t) dt \right] = E \left[c_0^2(\xi) \right] + \frac{1}{2} \sum_{n=1}^{\infty} E \left[c_n^2(\xi) \right]. \quad (10)$$

If $\xi(t_1, t_2)$ is a bivariate stochastic process on $[0, 1]^2$ and $E[\int_{[0,1]^2} \xi^2(t_1, t_2) dt_1 dt_2] < \infty$, then it can be expanded into the Fourier cosine series

$$\xi(t_1, t_2) = \sum_{n_1, n_2=0}^{\infty} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) \quad (11)$$

in mean square sense, where Fourier sine coefficients

$$\begin{aligned} c_{n_1, n_2}(\xi) \\ = \lambda_{n_1, n_2} \int_{[0,1]^2} \xi(t_1, t_2) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) dt_1 dt_2 \end{aligned} \quad (12)$$

are stochastic variables and

$$\lambda_{n_1, n_2} = \begin{cases} 1, & n_1 = n_2 = 0, \\ 2, & n_1 = 0, n_2 \neq 0 \text{ or } n_2 = 0, n_1 \neq 0, \\ 4, & n_1 \neq 0, n_2 \neq 0. \end{cases} \quad (13)$$

The corresponding Parseval identity holds:

$$E \left[\int_{[0,1]^2} \xi^2(t_1, t_2) dt_1 dt_2 \right] = \sum_{n_1, n_2=0}^{\infty} \frac{E[c_{n_1, n_2}^2(\xi)]}{\lambda_{n_1, n_2}}. \quad (14)$$

2.3. Partial Sums and Hyperbolic Cross Truncations. Let ξ be a stochastic process on $[0, 1]^2$. Partial sums of its Fourier cosine series are

$$\begin{aligned} S_N^{(r)}(\xi; t_1, t_2) &= \sum_{n_1, n_2=0}^{N-1} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) \\ &\quad (N \in \mathbb{Z}_+). \end{aligned} \quad (15)$$

The number of Fourier cosine coefficients in the partial sum $S_N^{(r)}(\xi)$ is N^2 .

Hyperbolic cross truncations of its Fourier cosine series are

$$\begin{aligned} S_N^{(h)}(\xi; t_1, t_2) \\ = \sum_{n_1=0}^{N-1} c_{n_1, 0}(\xi) \cos(\pi n_1 t_1) \\ + \sum_{n_2=1}^{N-1} \sum_{n_1=0}^{[(N-1)/n_2]-1} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2), \end{aligned} \quad (16)$$

where $[\cdot]$ means the integral part. The number of Fourier cosine coefficients in the hyperbolic cross truncation $S_N^{(h)}(\xi)$ is of order $N \log N$. The hyperbolic cross approximations have been widely used in multivariate function approximation [1, 3–5].

2.4. Some Notations. For convenience, we denote vertexes of the unit square $[0, 1]^2$ by $\{0, 1\}^2$, the boundary of $[0, 1]^2$ by $\partial([0, 1]^2)$.

For a bivariate stochastic process $\xi(t_1, t_2)$, denote its mixed derivative $\partial^{l_1+l_2} \xi / \partial t_1^{l_1} \partial t_2^{l_2}$ by $\xi^{(l_1, l_2)}$. The notation $C_s([0, 1]^2)$ represents the set of continuous stochastic processes on $[0, 1]^2$. If $\xi^{(l, l)} \in C_s([0, 1]^2)$, then we say ξ has the smoothness index l on $[0, 1]^2$.

Let $\{a_n\}$ and $\{b_n\}$ be two sequences. If $K_1|a_n| \leq |b_n| \leq K_2|a_n|$ for any n , then we say $a_n \sim b_n$; if $|a_n| \leq K_3|b_n|$ for any n , then we say $a_n = O(b_n)$; here K_i are constants independent of n . If $a_n \rightarrow 0$, $b_n \rightarrow 0$, and $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$, then we say $a_n = o(b_n)$. If $a_{n_1, n_2} \rightarrow 0$ and $b_{n_1, n_2} \rightarrow 0$, and $a_{n_1, n_2}/b_{n_1, n_2} \rightarrow 0$ as $\|n\| = \sqrt{n_1^2 + n_2^2} \rightarrow \infty$, then we say $a_{n_1, n_2} = o(b_{n_1, n_2})$.

3. Decomposition of Stochastic Processes

In order to study stochastic Fourier cosine series, we give a decomposition of stochastic processes. Although this decomposition is given in the setting of stochastic processes, it is also new for deterministic functions.

Let ξ be a bivariate stochastic process on $[0, 1]^2$ and $\xi^{(l, l)} \in C_s([0, 1]^2)$ for some $l \geq 1$. First, based on a fundamental polynomial $p(t) = t^2/2 - 1/6$, we construct a stochastic polynomial P_ξ as follows:

$$\begin{aligned} P_\xi(t_1, t_2) &= \xi^{(1,1)}(0, 0) p(1-t_1) p(1-t_2) \\ &\quad - \xi^{(1,1)}(0, 1) p(1-t_1) p(t_2) \\ &\quad - \xi^{(1,1)}(1, 0) p(t_1) p(1-t_2) \\ &\quad + \xi^{(1,1)}(1, 1) p(t_1) p(t_2). \end{aligned} \quad (17)$$

This stochastic polynomial is determined by partial derivatives of ξ at vertexes $\{0, 1\}^2$.

Denote

$$\xi_1(t_1, t_2) = \xi(t_1, t_2) - P_\xi(t_1, t_2). \quad (18)$$

The following is clear.

Proposition 1. Let ξ be a bivariate stochastic process on $[0, 1]^2$ with smoothness index $l \geq 1$ and ξ_1 be stated in (18). Then

$$\xi_1^{(1,1)}(0, 0) = \xi_1^{(1,1)}(0, 1) = \xi_1^{(1,1)}(1, 0) = \xi_1^{(1,1)}(1, 1) = 0. \quad (19)$$

Now we define

$$\begin{aligned} \xi_2(t_1, t_2) &= -\xi_1^{(1,0)}(0, t_2) p(1-t_1) \\ &\quad + \xi_1^{(1,0)}(1, t_2) p(t_1) - \xi_1^{(0,1)}(t_1, 0) p(1-t_2) \\ &\quad + \xi_1^{(0,1)}(t_1, 1) p(t_2), \end{aligned} \quad (20)$$

$$\xi_3(t_1, t_2) = \xi_1(t_1, t_2) - \xi_2(t_1, t_2), \quad (21)$$

and we derive the following.

Proposition 2. Let ξ be a bivariate stochastic process on $[0, 1]^2$ with smoothness index $l \geq 1$ and ξ_3 be stated in (21). Then

$$\xi_3^{(1,1)}(t_1, t_2) = 0, \quad (t_1, t_2) \in \partial([0, 1]^2). \quad (22)$$

Proof. Consider the bottom side $0 \leq t_1 \leq 1, t_2 = 0$ of the square $[0, 1]^2$. Using Proposition 1, it follows from (20) that $\xi_2^{(1,1)}(t_1, 0) = \xi_1^{(1,1)}(t_1, 0)p'(1) = \xi_1^{(1,1)}(t_1, 0)$ ($0 \leq t_1 \leq 1$). Therefore, by $\xi_3 = \xi_1 - \xi_2$, we get $\xi_3^{(1,1)}(t_1, t_2) = 0$ ($0 \leq t_1 \leq 1, t_2 = 0$). Similarly, $\xi_3^{(1,1)}$ vanishes on other sides of the square $[0, 1]^2$. $\square$

From (18) and (21), we get a decomposition of stochastic processes on $[0, 1]^2$ as follows.

Let ξ be a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 1$. Then the decomposition

$$\xi = P_\xi + \xi_2 + \xi_3 \quad (23)$$

holds, where P_ξ , ξ_2 , and ξ_3 are stated in (17), (20), and (21), respectively.

4. Univariate Stochastic Fourier Cosine Series

Suppose that ξ is a stochastic process on $[0, 1]$ and the derivative $\xi^{(l)} \in C_s([0, 1])$ for some $l \geq 2$. Its Fourier cosine coefficients are as follows:

$$c_n(\xi) = 2 \int_0^1 \xi(t) \cos(\pi nt) dt = g_n + r_n, \quad (24)$$

where

$$\begin{aligned} g_n &= -\frac{2}{(n\pi)^2} (\xi'(0) - (-1)^n \xi'(1)), \\ r_n &= -\frac{2}{(n\pi)^2} \int_0^1 \xi''(t) \cos(\pi nt) dt. \end{aligned} \quad (25)$$

Since the expectation and the integral can be exchanged and $E[\xi''(t)] \in C([0, 1])$, using the Riemann-Lebesgue lemma [10–12], we get

$$E[r_n] = -\frac{2}{(n\pi)^2} \int_0^1 E[\xi''(t)] \cos(\pi nt) dt = o\left(\frac{1}{n^2}\right). \quad (26)$$

From (25),

$$r_n^2 = \frac{4}{(n\pi)^4} \int_{[0,1]^2} \xi''(t) \xi''(s) \cos(\pi nt) \cos(\pi ns) dt ds. \quad (27)$$

Again, by the Riemann-Lebesgue lemma, we get

$$\begin{aligned} E[r_n^2] &= \frac{4}{(n\pi)^4} \int_{[0,1]^2} E[\xi''(t) \xi''(s)] \cos(\pi nt) \cos(\pi ns) dt ds \\ &= o\left(\frac{1}{n^4}\right). \end{aligned} \quad (28)$$

Since $\text{Var}(r_n) \leq E[r_n^2]$, we have $\text{Var}(r_n) = o(1/n^4)$. By (24) and (28), we get

$$\begin{aligned} E[c_n^2(\xi)] &= E[g_n^2 + 2g_n r_n + r_n^2] \\ &= E[g_n^2] + 2E[g_n r_n] + o\left(\frac{1}{n^4}\right). \end{aligned} \quad (29)$$

By the Schwarz inequality,

$$|E[g_n r_n]| \leq (E[g_n^2])^{1/2} (E[r_n^2])^{1/2} = o\left(\frac{1}{n^4}\right). \quad (30)$$

Therefore, we have

$$\begin{aligned} E[c_{2n}^2(\xi)] + E[c_{2n+1}^2(\xi)] &= \frac{1}{2(n\pi)^4} E[(\xi'(0))^2 + (\xi'(1))^2] + o\left(\frac{1}{n^4}\right). \end{aligned} \quad (31)$$

By the Parseval identity, the partial sums of its Fourier cosine series $S_N(\xi; t) = \sum_{k=0}^{N-1} c_k(\xi) \cos(\pi kt)$ satisfy

$$\begin{aligned} E[\|S_N(\xi) - \xi\|_2^2] &= E\left[\int_0^1 (S_N(\xi; t) - \xi(t))^2 dt\right] \\ &= \frac{1}{2} \sum_{n=N}^{\infty} E[c_n^2(\xi)] \\ &= \frac{2}{3\pi^4 N^3} E[(\xi'(0))^2 + (\xi'(1))^2] \\ &\quad + o\left(\frac{1}{N^3}\right). \end{aligned} \quad (32)$$

From this, we deduce that $E[\|S_N(\xi) - \xi\|_2^2] = o(1/N^3)$ if and only if $\xi'(0) = \xi'(1) = 0$.

5. Asymptotic Representations of Bivariate Fourier Cosine Coefficients

Suppose that ξ is a bivariate stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 2$. We expand ξ into the Fourier cosine series

$$\xi(t_1, t_2) = \sum_{n_1, n_2=0}^{\infty} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) \quad (33)$$

and Fourier coefficients

$$\begin{aligned} c_{n_1, n_2}(\xi) &= \lambda_{n_1, n_2} \int_{[0,1]^2} \xi(t_1, t_2) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) dt_1 dt_2, \end{aligned} \quad (34)$$

where λ_{n_1, n_2} is stated in (13).

Based on the decomposition (23) of bivariate stochastic processes, we have

$$c_{n_1, n_2}(\xi) = c_{n_1, n_2}(P_\xi) + c_{n_1, n_2}(\xi_2) + c_{n_1, n_2}(\xi_3). \quad (35)$$

In order to obtain the asymptotic representation of Fourier cosine coefficients $c_{n_1, n_2}(\xi)$, we will precisely compute the first two terms and estimate the expectation and variance of the last term on the right-hand side of (35) as follows.

(i) For the first term $c_{n_1, n_2}(P_\xi)$, by the representation (17) of stochastic polynomial P_ξ , we get that, for $n_1 \neq 0$ and $n_2 \neq 0$,

$$c_{n_1, n_2}(P_\xi) = \frac{4}{\pi^4 n_1^2 n_2^2} \eta_{n_1, n_2}, \quad (36)$$

where

$$\begin{aligned} \eta_{n_1, n_2} = & \xi^{(1,1)}(0,0) - (-1)^{n_2} \xi^{(1,1)}(0,1) \\ & - (-1)^{n_1} \xi^{(1,1)}(1,0) + (-1)^{n_1+n_2} \xi^{(1,1)}(1,1) \end{aligned} \quad (37)$$

is an algebraic sum of values of $\xi^{(1,1)}$ at $\{0,1\}^2$ and signs for addition and subtraction are determined by oddity of n_1 and n_2 . Since $\int_0^1 p(t)dt = 0$, we get $c_{0,0}(P_\xi) = c_{n_1,0}(P_\xi) = c_{0,n_2}(P_\xi) = 0$.

(ii) For the second term $c_{n_1, n_2}(\xi_2)$, by (20), we know that ξ_2 is the sum of products of separated variables.

Denote

$$\begin{aligned} \varphi_1 &= \xi_1^{(1,0)}(0, \cdot), & \varphi_2 &= \xi_1^{(1,0)}(1, \cdot), \\ \varphi_3 &= \xi_1^{(1,0)}(\cdot, 0), & \varphi_4 &= \xi_1^{(1,0)}(\cdot, 1). \end{aligned} \quad (38)$$

The Fourier cosine coefficients of ξ_2 are equal to

$$\begin{aligned} c_{n_1, n_2}(\xi_2) = & -c_{n_2}(\varphi_1) c_{n_1}(p(1-\cdot)) + c_{n_2}(\varphi_2) c_{n_1}(p) \\ & - c_{n_1}(\varphi_3) c_{n_2}(p(1-\cdot)) + c_{n_1}(\varphi_4) c_{n_2}(p), \end{aligned} \quad (39)$$

where each $c_n(\varphi_i)$ is the Fourier cosine coefficient of the univariate stochastic process φ_i and each $c_n(p)$ and each $c_n(p(1-\cdot))$ are both Fourier cosine coefficients of univariate deterministic functions p and $p(1-\cdot)$. A direct computation shows that, for $n_1 \neq 0$ and $n_2 \neq 0$,

$$\begin{aligned} c_{n_1, n_2}(\xi_2) = & \frac{2}{\pi^2 n_1^2} (-c_{n_2}(\varphi_1) + (-1)^{n_1} c_{n_2}(\varphi_2)) \\ & + \frac{2}{\pi^2 n_2^2} (-c_{n_1}(\varphi_3) + (-1)^{n_2} c_{n_1}(\varphi_4)). \end{aligned} \quad (40)$$

(iii) For the last term $c_{n_1, n_2}(\xi_3)$, using the integration by parts, we deduce that

$$\begin{aligned} c_{n_1, n_2}(\xi_3) = & \frac{1}{\pi^2 n_1 n_2} \\ & \times \int_0^1 \left(\int_0^1 \xi_3^{(1,1)}(t_1, t_2) \sin(\pi n_1 t_1) dt_1 \right) \\ & \times \sin(\pi n_2 t_2) dt_2. \end{aligned} \quad (41)$$

By Proposition 2, $\xi_3^{(1,1)}(0, t_2) = \xi_3^{(1,1)}(1, t_2) = 0$ ($0 \leq t_2 \leq 1$), the interior integral is equal to

$$\frac{1}{\pi n_1} \int_0^1 \xi_3^{(2,1)}(t_1, t_2) \cos(\pi n_1 t_1) dt_1. \quad (42)$$

So

$$\begin{aligned} c_{n_1, n_2}(\xi_3) = & \frac{1}{\pi^3 n_1^2 n_2} \\ & \times \int_0^1 \left(\int_0^1 \xi_3^{(2,1)}(t_1, t_2) \sin(\pi n_2 t_2) dt_2 \right) \\ & \times \cos(\pi n_1 t_1) dt_1. \end{aligned} \quad (43)$$

By $\xi_3^{(1,1)}(t_1, 0) = 0$ ($0 \leq t_1 \leq 1$), we have $\xi_3^{(2,1)}(t_1, 1) = 0$ ($0 \leq t_1 \leq 1$). This implies that

$$\begin{aligned} c_{n_1, n_2}(\xi_3) = & \frac{4}{(n_1 \pi)^2 (n_2 \pi)^2} \\ & \times \int_{[0,1]^2} \xi_3^{(2,2)}(t_1, t_2) \cos(\pi n_1 t_1) \\ & \times \cos(\pi n_2 t_2) dt_1 dt_2, \end{aligned} \quad (44)$$

and so

$$\begin{aligned} c_{n_1, n_2}^2(\xi_3) = & \frac{16}{(n_1 \pi)^4 (n_2 \pi)^4} \\ & \times \int_{[0,1]^4} \xi_3^{(2,2)}(t_1, t_2) \xi_3^{(2,2)}(s_1, s_2) \\ & \times \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) \cos(\pi n_1 s_1) \\ & \times \cos(\pi n_2 s_2) dt_1 dt_2 ds_1 ds_2. \end{aligned} \quad (45)$$

Taking expectations on these two equations, we get

$$\begin{aligned} E[c_{n_1, n_2}(\xi_3)] = & \frac{4}{(n_1 \pi)^2 (n_2 \pi)^2} \\ & \times \int_{[0,1]^2} E[\xi_3^{(2,2)}(t_1, t_2)] \cos(\pi n_1 t_1) \\ & \times \cos(\pi n_2 t_2) dt_1 dt_2, \\ E[c_{n_1, n_2}^2(\xi_3)] = & \frac{16}{(n_1 \pi)^4 (n_2 \pi)^4} \\ & \times \int_{[0,1]^4} E[\xi_3^{(2,2)}(t_1, t_2) \xi_3^{(2,2)}(s_1, s_2)] \\ & \times \cos(\pi n_1 t_1) \cos(\pi n_2 t_2) \cos(\pi n_1 s_1) \\ & \times \cos(\pi n_2 s_2) dt_1 dt_2 ds_1 ds_2. \end{aligned} \quad (46)$$

It follows from $\xi_3^{(2,2)} \in C_s([0,1]^2)$ that $E[\xi_3^{(2,2)}]$ is a continuous function on $[0,1]^2$ and $E[\xi_3^{(2,2)}(t_1, t_2) \xi_3^{(2,2)}(s_1, s_2)]$ is a continuous on $[0,1]^4$. By the Riemann-Lebesgue lemma, we have

$$\begin{aligned} E[c_{n_1, n_2}(\xi_3)] = & o\left(\frac{1}{n_1^2 n_2^2}\right), & E[c_{n_1, n_2}^2(\xi_3)] = & o\left(\frac{1}{n_1^4 n_2^4}\right), \\ \text{Var}(c_{n_1, n_2}(\xi_3)) = & o\left(\frac{1}{n_1^4 n_2^4}\right) \end{aligned} \quad (47)$$

since $\text{Var}(c_{n_1, n_2}(\xi_3)) \leq E[c_{n_1, n_2}^2(\xi_3)]$.

From (i), (ii), and (iii), we get

$$\begin{aligned} c_{n_1, n_2}(\xi) &= \frac{4}{\pi^4 n_1^2 n_2^2} \eta_{n_1, n_2} \\ &+ \frac{2}{\pi^2 n_1^2} (-c_{n_2}(\varphi_1) + (-1)^{n_1} c_{n_2}(\varphi_2)) \\ &+ \frac{2}{\pi^2 n_2^2} (-c_{n_1}(\varphi_3) + (-1)^{n_2} c_{n_1}(\varphi_4)) \\ &+ c_{n_1, n_2}(\xi_3), \end{aligned} \quad (48)$$

and the error $c_{n_1, n_2}(\xi)$ satisfies (47), where η_{n_1, n_2} is stated in (37) and each φ_i is stated in (38).

Now we further estimate these four univariate Fourier cosine coefficients $c_n(\varphi_i)$.

Lemma 3. Let each φ_i be stated as in (38). Then Fourier cosine coefficients $c_n(\varphi_i)$ satisfy

$$E[c_n(\varphi_i)] = o\left(\frac{1}{n^2}\right), \quad E[c_n^2(\varphi_i)] = o\left(\frac{1}{n^4}\right), \quad (49)$$

($i = 1, 2, 3, 4$).

Proof. By similarity, we only prove the case $i = 1$, since $\varphi_1 = \xi_1^{(1,0)}(0, \cdot)$. By Proposition 1 and $\xi_1 = \xi - P_\xi$, we have $\varphi_1'(0) = \varphi_1'(1) = 0$. From this, we deduce that Fourier cosine coefficients satisfy

$$E[c_n(\varphi_1)] = o\left(\frac{1}{n^2}\right), \quad E[c_n^2(\varphi_1)] = o\left(\frac{1}{n^4}\right). \quad (50)$$

□

We will deduce these asymptotic representations of Fourier cosine coefficients.

Theorem 4. Let ξ be a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 2$, and let η_{n_1, n_2} and φ_i be stated in (37) and (38), respectively. Then

(i) for $n_1 \rightarrow \infty$, $c_{n_1, n_2}(\xi) = b_{n_1, n_2} + \tau_{n_1, n_2}$, where

$$\begin{aligned} b_{n_1, n_2} &= \frac{4}{\pi^4 n_1^2 n_2^2} \eta_{n_1, n_2} \\ &+ \frac{2}{\pi^2 n_1^2} (-c_{n_2}(\varphi_1) + (-1)^{n_1} c_{n_2}(\varphi_2)), \end{aligned} \quad (51)$$

and τ_{n_1, n_2} satisfies

$$\begin{aligned} E[\tau_{n_1, n_2}] &= o\left(\frac{1}{n_1^2}\right) \frac{1}{n_2^2}, \quad E[\tau_{n_1, n_2}^2] = o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^4}, \\ \text{Var}(\tau_{n_1, n_2}) &= o\left(\frac{1}{n_1^2}\right) \frac{1}{n_2^2}; \end{aligned} \quad (52)$$

(ii) for $n_2 \rightarrow \infty$, $c_{n_1, n_2}(\xi) = \tilde{b}_{n_1, n_2} + \tilde{\tau}_{n_1, n_2}$, where

$$\begin{aligned} \tilde{b}_{n_1, n_2} &= \frac{4}{\pi^4 n_1^2 n_2^2} \eta_{n_1, n_2} \\ &+ \frac{2}{\pi^2 n_2^2} (-c_{n_1}(\varphi_3) + (-1)^{n_2} c_{n_1}(\varphi_4)), \end{aligned} \quad (53)$$

and $\tilde{\tau}_{n_1, n_2}$ satisfies

$$\begin{aligned} E[\tilde{\tau}_{n_1, n_2}] &= o\left(\frac{1}{n_2^2}\right) \frac{1}{n_1^2}, \quad E[\tilde{\tau}_{n_1, n_2}^2] = o\left(\frac{1}{n_2^4}\right) \frac{1}{n_1^4}, \\ \text{Var}(\tilde{\tau}_{n_1, n_2}) &= o\left(\frac{1}{n_2^2}\right) \frac{1}{n_1^2}; \end{aligned} \quad (54)$$

(iii) for $n_1 \rightarrow \infty$ and $n_2 \rightarrow \infty$, $c_{n_1, n_2}(\xi) = (4/\pi^2 n_1^2 n_2^2) \eta_{n_1, n_2} + \tau_{n_1, n_2}^*$ and τ_{n_1, n_2}^* satisfies

$$\begin{aligned} E[\tau_{n_1, n_2}^*] &= o\left(\frac{1}{n_1^2 n_2^2}\right), \quad E[\tau_{n_1, n_2}^{*2}] = o\left(\frac{1}{n_1^4 n_2^4}\right), \\ \text{Var}(\tau_{n_1, n_2}^*) &= o\left(\frac{1}{n_1^4 n_2^4}\right). \end{aligned} \quad (55)$$

Proof. When $n_1 \rightarrow \infty$, denote

$$\tau_{n_1, n_2} = \frac{2}{(\pi n)^2} (-c_{n_1}(\varphi_3) + (-1)^{n_2} c_{n_1}(\varphi_4)) + c_{n_1, n_2}(\xi_3). \quad (56)$$

By (48), $c_{n_1, n_2}(\xi) = b_{n_1, n_2} + \tau_{n_1, n_2}$. Using Lemma 3 and (47), we deduce that $E[\tau_{n_1, n_2}] = o(1/n_1^2)(1/n_2^2)$ and

$$\begin{aligned} E[\tau_{n_1, n_2}^2] &\leq K \left(\frac{1}{n_2^2} E[c_{n_1}^2(\varphi_3) + c_{n_1}^2(\varphi_4)] + E[c_{n_1, n_2}^2(\xi_3)] \right) \\ &= o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^2}. \end{aligned} \quad (57)$$

So we get (i). Similarly, we can get (ii). □

From this, we can give asymptotic representations of expectation, second-order moment, and variance of Fourier cosine coefficients.

Corollary 5. Under conditions of Theorem 4, we have

(i) for $n_1 \rightarrow \infty$ and $n_2 \neq 0$,

$$\begin{aligned} E[c_{n_1, n_2}(\xi)] &= E[b_{n_1, n_2}] + o\left(\frac{1}{n_1^2}\right) \frac{1}{n_2^2}, \\ E[c_{n_1, n_2}^2(\xi)] &= E[b_{n_1, n_2}^2] + o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^4}, \end{aligned} \quad (58)$$

$$\text{Var}(c_{n_1, n_2}(\xi)) = \text{Var}(b_{n_1, n_2}) + o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^4},$$

where b_{n_1, n_2} is stated in (51) and “o” is uniform for n_2 ;

(ii) for $n_2 \rightarrow \infty$ and $n_1 \neq 0$,

$$\begin{aligned} E[c_{n_1, n_2}(\xi)] &= E[\tilde{b}_{n_1, n_2}] + o\left(\frac{1}{n_2^2}\right) \frac{1}{n_1^2}, \\ E[c_{n_1, n_2}^2(\xi)] &= E[\tilde{b}_{n_1, n_2}^2] + o\left(\frac{1}{n_2^4}\right) \frac{1}{n_1^4}, \\ \text{Var}(c_{n_1, n_2}(\xi)) &= \text{Var}(\tilde{b}_{n_1, n_2}) + o\left(\frac{1}{n_2^4}\right) \frac{1}{n_1^4}, \end{aligned} \quad (59)$$

where $\tilde{b}_{n_1, n_2}$ is stated in (53) and “o” is uniform for n_1 .

Proof. By similarity, we only prove (58).

From Theorem 4 (i), we deduce that, for $n_1 \rightarrow \infty$, $n_2 \neq 0$,

$$\begin{aligned} E[c_{n_1, n_2}^2(\xi)] &= E[b_{n_1, n_2}^2] + 2E[b_{n_1, n_2} \tau_{n_1, n_2}] + o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^2}, \\ |E[b_{n_1, n_2} \lambda_{n_1, n_2}]| &\leq \sqrt{E[b_{n_1, n_2}^2]} \cdot \sqrt{E[\tau_{n_1, n_2}^2]} = o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^2}. \end{aligned} \quad (60)$$

So we get (58). $\square$

From Corollary 5 (iii), we know that

$$E[c_{n_1, n_2}(\xi)] \sim \frac{1}{n_1^2 n_2^2}, \quad E[c_{n_1, n_2}^2(\xi)] \sim \frac{1}{n_1^4 n_2^4}. \quad (61)$$

These results cannot be improved as smoothness index l increases.

6. Approximation of Partial Sums

Let ξ be a stochastic process on $[0, 1]^2$ and $\xi^{(l, l)} \in C_s([0, 1]^2)$ for some $l \geq 2$. We expand ξ into the Fourier cosine series:

$$\xi(t_1, t_2) = \sum_{n_1, n_2=0}^{\infty} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2). \quad (62)$$

We give further the asymptotic representation of approximation error of partial sums. The partial sums $S_N^{(r)}(\xi)$ of Fourier cosine series (62) are defined as

$$S_N^{(r)}(\xi; t_1, t_2) = \sum_{n_1, n_2=0}^{N-1} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2). \quad (63)$$

Using the Parseval identity, we get

$$\begin{aligned} E[\|S_N^{(r)}(\xi) - \xi\|_2^2] &= \left(\sum_{n_1, n_2=0}^{\infty} - \sum_{n_1, n_2=0}^{N-1} \right) \frac{E[c_{n_1, n_2}^2(\xi)]}{\lambda_{n_1, n_2}} \\ &= \frac{1}{4} \left(\sum_{n_1, n_2=1}^{\infty} - \sum_{n_1, n_2=1}^{N-1} \right) E[c_{n_1, n_2}^2(\xi)] \\ &\quad + \frac{1}{2} \sum_{n_1=N}^{\infty} E[c_{n_1, 0}^2(\xi)] \\ &\quad + \frac{1}{2} \sum_{n_2=N}^{\infty} E[c_{0, n_2}^2(\xi)] \\ &=: P_N + Q_N + R_N. \end{aligned} \quad (64)$$

(i) We compute P_N . The first term P_N on the right-hand side of the formula (64) can be decomposed into three sums

$$P_N = \frac{1}{4} (P_N^1 + P_N^2 + P_N^3), \quad (65)$$

where

$$\begin{aligned} P_N^1 &= \sum_{n_2=1}^{N-1} \sum_{n_1=N}^{\infty} E[c_{n_1, n_2}^2(\xi)], \\ P_N^2 &= \sum_{n_1=1}^{N-1} \sum_{n_2=N}^{\infty} E[c_{n_1, n_2}^2(\xi)], \\ P_N^3 &= \sum_{n_1, n_2=N}^{\infty} E[c_{n_1, n_2}^2(\xi)]. \end{aligned} \quad (66)$$

First, we consider the interior sum of P_N^1 with $N = 2k$,

$$\begin{aligned} \sum_{n_1=2k}^{\infty} E[c_{n_1, n_2}^2(\xi)] &= \sum_{n_1=k}^{\infty} E[c_{2n_1, n_2}^2(\xi)] \\ &\quad + \sum_{n_1=k}^{\infty} E[c_{2n_1+1, n_2}^2(\xi)] =: X_{k, n_2} + Y_{k, n_2}. \end{aligned} \quad (67)$$

By Corollary 5 (i), we deduce that

$$\begin{aligned} X_{k, n_2} &= \sum_{n_1=k}^{\infty} \left(\frac{1}{2\pi^2 n_1^2} \right)^2 E[(b_{n_2}^{(1)})^2] + o\left(\frac{1}{k^3}\right) \frac{1}{n_2^4}, \\ Y_{k, n_2} &= \sum_{n_1=k}^{\infty} \left(\frac{1}{2\pi^2 n_1^2} \right)^2 E[(b_{n_2}^{(2)})^2] + o\left(\frac{1}{k^3}\right) \frac{1}{n_2^4}, \end{aligned} \quad (68)$$

where “ o ” is uniform for n_2 and

$$\begin{aligned} b_{n_2}^{(1)} &= \frac{2}{\pi^2 n_2^2} \left(\xi^{(1,1)}(0,0) - \xi^{(1,1)}(1,0) \right. \\ &\quad \left. - (-1)^{n_2} \left(\xi^{(1,1)}(0,1) - \xi^{(1,1)}(1,1) \right) \right) \\ &\quad - c_{n_2}(\varphi_1) + c_{n_2}(\varphi_2), \\ b_{n_2}^{(2)} &= \frac{2}{\pi^2 n_2^2} \left(\xi^{(1,1)}(0,0) + \xi^{(1,1)}(1,0) \right. \\ &\quad \left. - (-1)^{n_2} \left(\xi^{(1,1)}(0,1) + \xi^{(1,1)}(1,1) \right) \right) \\ &\quad - c_{n_2}(\varphi_1) - c_{n_2}(\varphi_2). \end{aligned} \quad (69)$$

Again, by

$$\sum_{n_1=k}^{\infty} \frac{1}{(2\pi^2 n_1^2)^2} = \frac{1}{12\pi^4 k^3} + O\left(\frac{1}{k^4}\right) \quad (70)$$

and (67), we obtain that

$$\begin{aligned} \sum_{n_1=2k}^{\infty} E[c_{n_1, n_2}^2(\xi)] &= \frac{1}{12\pi^4 k^3} E\left[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2\right] \\ &\quad + o\left(\frac{1}{k^3}\right) \frac{1}{n_2^4}, \end{aligned} \quad (71)$$

where “ o ” is uniform for n_2 . By the convergence of the series $\sum_{n=1}^{\infty} (1/n^4)$ and (65), we have

$$P_{2k}^1 = \frac{1}{12\pi^4 k^3} \left(\sum_{n_2=1}^{2k-1} E\left[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2\right] \right) + o\left(\frac{1}{k^3}\right). \quad (72)$$

By (69), we deduce that there exists a constant $K_1 > 0$ such that

$$\begin{aligned} &E\left[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2\right] \\ &\leq K_1 E\left[\frac{1}{n_2^4} \sum_{\lambda \in \{0,1\}^2} \left(\xi^{(1,1)}(\lambda)\right)^2 + c_{n_2}^2(\varphi_1) + c_{n_2}^2(\varphi_2)\right]. \end{aligned} \quad (73)$$

Again, by Lemma 3, we have

$$E\left[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2\right] = O\left(\frac{1}{n_2^4}\right), \quad (74)$$

and so the series $\sum_{n_2=1}^{\infty} E[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2]$ converges, and denote its sum by A . So

$$\sum_{n_2=1}^{2k-1} E\left[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2\right] = A + o(1) \quad (k \rightarrow \infty). \quad (75)$$

By (72), we get $P_{2k}^1 = (A/12\pi^4 k^3) + o(1/k^3)$. Notice that

$$P_{2k+1}^1 = P_{2k}^1 + \sum_{n_1=2k+1}^{\infty} E[c_{n_1, 2k}^2(\xi)] - \sum_{n_2=1}^{2k-1} E[c_{2k, n_2}^2(\xi)] \quad (76)$$

and $E[c_{n_1, n_2}^2(\xi)] = O(1/n_1^4 n_2^4)$ (see (61)). We have $P_{2k+1}^1 = P_{2k}^1 + o(1/k^3)$. This implies that

$$P_N^1 = \frac{2A}{3\pi^4 N^3} + o\left(\frac{1}{N^3}\right), \quad (77)$$

$$\text{where } A = \sum_{n_2=1}^{\infty} E\left[(b_{n_2}^{(1)})^2 + (b_{n_2}^{(2)})^2\right].$$

Here $b_{n_2}^{(1)}$ and $b_{n_2}^{(2)}$ are stated in (69).

Similarly,

$$P_N^2 = \frac{2B}{3\pi^4 N^3} + o\left(\frac{1}{N^3}\right), \quad (78)$$

$$\text{where } B = \sum_{n_1=1}^{\infty} E\left[(\tilde{b}_{n_1}^{(1)})^2 + (\tilde{b}_{n_1}^{(2)})^2\right].$$

Here “ o ” is uniform for n_1 and

$$\begin{aligned} \tilde{b}_{n_1}^{(1)} &= \frac{2}{\pi^2 n_1^2} \left(\xi^{(1,1)}(0,0) - \xi^{(1,1)}(0,1) \right. \\ &\quad \left. - (-1)^{n_1} \left(\xi^{(1,1)}(1,0) - \xi^{(1,1)}(1,1) \right) \right) \\ &\quad - c_{n_1}(\varphi_3) + c_{n_1}(\varphi_4), \\ \tilde{b}_{n_1}^{(2)} &= \frac{2}{\pi^2 n_1^2} \left(\xi^{(1,1)}(0,0) + \xi^{(1,1)}(0,1) \right. \\ &\quad \left. - (-1)^{n_1} \left(\xi^{(1,1)}(1,0) + \xi^{(1,1)}(1,1) \right) \right) \\ &\quad - c_{n_1}(\varphi_3) - c_{n_1}(\varphi_4). \end{aligned} \quad (79)$$

From $E[c_{n_1, n_2}^2(\xi)] = O(1/n_1^4 n_2^4)$, we have $P_N^3 = o(1/N^3)$. Finally, by (65), we get

$$P_N = \frac{A+B}{6\pi^4 N^3} + o\left(\frac{1}{N^3}\right), \quad (80)$$

where A, B are stated in (77) and (78).

(ii) We compute Q_N and R_N in (64). By the decomposition formula (23),

$$c_{n_1, 0}(\xi) = c_{n_1, 0}(P_{\xi}) + c_{n_1, 0}(\xi_2) + c_{n_1, 0}(\xi_3). \quad (81)$$

By (17) and $\int_0^1 p(t)dt = 0$, we have $c_{n_1, 0}(P_{\xi}) = 0$. By (39), we have

$$c_{n_1, 0}(\xi_2) = \frac{2}{(\pi n_1)^2} (-c_0(\varphi_1) + (-1)^{n_1} c_0(\varphi_2)). \quad (82)$$

From this, we get

$$\begin{aligned} &E[c_{2n_1, 0}^2(\xi_2)] + E[c_{2n_1+1, 0}^2(\xi_2)] \\ &= \frac{1}{2(\pi n_1)^4} (E[c_0^2(\varphi_1) + c_0^2(\varphi_2)]) + o\left(\frac{1}{n_1^4}\right). \end{aligned} \quad (83)$$

Similar to (47), we have

$$\begin{aligned} E \left[c_{n_1,0}^2(\xi_3) \right] &= \frac{4}{(\pi n_1)^4} \\ &\times \int_0^1 E \left[\xi_3(t_1, t_2) \xi_3(s_1, t_2) \right] \\ &\times \cos(\pi n_1 t_1) \cos(\pi n_1 s_1) dt_1 ds_1 \\ &= o\left(\frac{1}{n_1^4}\right). \end{aligned} \quad (84)$$

Again, by (81), we have

$$\begin{aligned} \sum_{n_1=2k}^{\infty} E \left[c_{n_1,0}^2(\xi) \right] &= \sum_{n_1=k}^{\infty} \left(E \left[c_{2n_1,0}^2(\xi) \right] + E \left[c_{2n_1+1,0}^2(\xi) \right] \right) \\ &= \frac{1}{6\pi^4 k^3} \left(E \left[c_0^2(\varphi_1) + c_0^2(\varphi_2) \right] + o(1) \right). \end{aligned} \quad (85)$$

This implies by (64) that

$$Q_N = \frac{2C}{3\pi^4 N^3} + o\left(\frac{1}{N^3}\right), \quad (86)$$

$$\text{where } C = E \left[c_0^2(\varphi_1) + c_0^2(\varphi_2) \right].$$

Similarly, we get

$$R_N = \frac{2D}{3\pi^4 N^3} + o\left(\frac{1}{N^3}\right), \quad (87)$$

$$\text{where } D = E \left[c_0^2(\varphi_3) + c_0^2(\varphi_4) \right].$$

From this and (80), we get by (64) the following.

Theorem 6. Let ξ be a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 2$. Then the partial sums $S_N(\xi)$ of its Fourier cosine series satisfy

$$E \left[\|S_N^{(r)}(\xi) - \xi\|_2^2 \right] = \frac{M}{N^3} + o\left(\frac{1}{N^3}\right), \quad (88)$$

where $M = (1/6\pi^4)(A+B+4C+4D)$ and A, B, C, D are stated in (77), (78), (86), and (87).

7. Approximation of Hyperbolic Cross Truncations

Suppose that ξ is a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 2$. We consider hyperbolic cross truncations of its Fourier cosine series:

$$\begin{aligned} S_N^{(h)}(\xi; t_1, t_2) &= \sum_{n_1=0}^{N-1} c_{n_1,0}(\xi) \cos(\pi n_1 t_2) \\ &+ \sum_{n_2=1}^{N-1} \sum_{n_1=0}^{[(N-1)/n_2]-1} c_{n_1, n_2}(\xi) \cos(\pi n_1 t_1) \\ &\times \cos(\pi n_2 t_2). \end{aligned} \quad (89)$$

By the Parseval identity and (61), (86), and (87), we have

$$\begin{aligned} E \left[\|S_N^{(h)}(\xi) - \xi\|_2^2 \right] &= \frac{1}{4} J_N + \frac{1}{2} \sum_{n_1=N}^{\infty} E \left[c_{n_1,0}^2(\xi) \right] \\ &+ \frac{1}{4} \sum_{n_2=N}^{\infty} \sum_{n_1=0}^{\infty} E \left[c_{n_1, n_2}^2(\xi) \right] \\ &= \frac{1}{4} J_N + O\left(\frac{1}{N^3}\right), \end{aligned} \quad (90)$$

where $J_N = \sum_{n_2=1}^{N-1} \sum_{n_1=[(N-1)/n_2]}^{\infty} E \left[c_{n_1, n_2}^2(\xi) \right]$. We rewrite J_N in the form

$$\begin{aligned} J_N &= \sum_{n_2=1}^{[(N-1)/2]} \sum_{n_1=[(N-1)/4n_2]}^{\infty} \left(E \left[c_{2n_1, 2n_2}^2(\xi) + c_{2n_1-1, 2n_2}^2(\xi) \right] \right) \\ &+ \sum_{n_2=1}^{[(N-1)/2]} \sum_{n_1=[(N-1)/(4n_2-2)]}^{\infty} \\ &\times \left(E \left[c_{2n_1, 2n_2-1}^2(\xi) + c_{2n_1-1, 2n_2-1}^2(\xi) \right] \right) \\ &+ O\left(\frac{1}{N^3}\right) = J_N^1 + J_N^2 + O\left(\frac{1}{N^3}\right). \end{aligned} \quad (91)$$

We first consider J_N^1 . By Corollary 5 (i), we have

$$\begin{aligned} E \left[c_{2n_1, 2n_2}^2(\xi) \right] &= \frac{1}{4\pi^4 n_1^4} E \left[\left(\frac{1}{2\pi^2 n_2^2} \omega_1 - c_{2n_2}(\varphi_1) + c_{2n_2}(\varphi_2) \right)^2 \right] \\ &+ o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^4}, \\ E \left[c_{2n_1-1, 2n_2}^2(\xi) \right] &= \frac{1}{4\pi^4 n_1^4} E \left[\left(\frac{1}{2\pi^2 n_2^2} \omega_2 - c_{2n_2}(\varphi_1) - c_{2n_2}(\varphi_2) \right)^2 \right] \\ &+ o\left(\frac{1}{n_1^4}\right) \frac{1}{n_2^4}, \end{aligned} \quad (92)$$

where “ o ” is uniform for n_2 and

$$\begin{aligned} \omega_1 &= \xi^{(1,1)}(0, 0) - \xi^{(1,1)}(0, 1) \\ &\quad - \xi^{(1,1)}(1, 0) + \xi^{(1,1)}(1, 1), \\ \omega_2 &= \xi^{(1,1)}(0, 0) - \xi^{(1,1)}(0, 1) \\ &\quad + \xi^{(1,1)}(1, 0) - \xi^{(1,1)}(1, 1). \end{aligned} \quad (93)$$

Therefore,

$$\begin{aligned} & \sum_{n_1=[(N-1)/4n_2]}^{\infty} E[c_{2n_1, 2n_2}^2(\xi)] \\ &= \left(\sum_{n_1=[(N-1)/4n_2]}^{\infty} \frac{1}{4\pi^4 n_1^4} \right) \\ & \times E \left[\left(\frac{1}{2\pi^2 n_2^2} \omega_1 - c_{2n_2}(\varphi_1) + c_{2n_2}(\varphi_2) \right)^2 \right] \\ &+ o\left(\frac{1}{N^3}\right) \frac{1}{n_2}. \end{aligned} \quad (94)$$

Here

$$\begin{aligned} & \sum_{n_1=[(N-1)/4n_2]}^{\infty} \frac{1}{4\pi^4 n_1^4} = \frac{16n_2^3}{3\pi^4 N^3} + O\left(\frac{n_2^4}{N^4}\right), \\ & E \left[\left(\frac{1}{2\pi^2 n_2^2} \omega_1 - c_{n_2}(\varphi_1) + c_{n_2}(\varphi_2) \right)^2 \right] \\ &= \frac{1}{4\pi^4 n_2^4} E[\omega_1^2] + \mu_{n_2}, \end{aligned} \quad (95)$$

where $\mu_{n_2} = -(1/\pi^2 n_2^2) E[\omega_1(c_{n_2}(\varphi_1) - c_{n_2}(\varphi_2))] + E[(c_{n_2}(\varphi_1) - c_{n_2}(\varphi_2))^2]$.

By using the Schwarz inequality in Probability theory,

$$\begin{aligned} & |E[\omega_1(c_{n_2}(\varphi_1) - c_{n_2}(\varphi_2))]| \\ & \leq (E[\omega_1^2])^{1/2} (E[(c_{n_2}(\varphi_1) - c_{n_2}(\varphi_2))^2])^{1/2}. \end{aligned} \quad (96)$$

By Lemma 3, $E[(c_{n_2}(\varphi_1) - c_{n_2}(\varphi_2))^2] \leq 2E[c_{n_2}^2(\varphi_1) + c_{n_2}^2(\varphi_2)] = o(1/n_2^4)$. Therefore, $\mu_{n_2} = o(1/n_2^4)$. From this, we have

$$\begin{aligned} & E \left[\left(\frac{1}{2\pi^2 n_2^2} \omega_1 - c_{n_2}(\varphi_1) + c_{n_2}(\varphi_2) \right)^2 \right] \\ &= \frac{1}{4\pi^4 n_2^4} E[\omega_1^2] + o\left(\frac{1}{n_2^4}\right). \end{aligned} \quad (97)$$

Substituting (95) and (97) into (94), we get that, for $n_2 \leq N$,

$$\sum_{n_1=[(N-1)/4n_2]}^{\infty} E[c_{2n_1, 2n_2}^2(\xi)] = \frac{4E[\omega_1^2]}{3\pi^8 N^3 n_2} + o\left(\frac{1}{N^3 n_2}\right). \quad (98)$$

Similarly, we have

$$\sum_{n_1=[(N-1)/4n_2]}^{\infty} E[c_{2n_1-1, 2n_2}^2(\xi)] = \frac{4E[\omega_2^2]}{3\pi^8 N^3 n_2} + o\left(\frac{1}{N^3 n_2}\right). \quad (99)$$

Denote $\omega = (1/2)(\omega_1^2 + \omega_2^2)$. We get

$$\omega = (\xi^{(1,1)}(0,0) - \xi^{(1,1)}(0,1))^2 + (\xi^{(1,1)}(1,0) - \xi^{(1,1)}(1,1))^2. \quad (100)$$

This implies that

$$\begin{aligned} & \sum_{n_1=[(N-1)/4n_2]}^{\infty} E[c_{2n_1, 2n_2}^2(\xi) + c_{2n_1-1, 2n_2}^2(\xi)] \\ &= \frac{8E[\omega]}{3\pi^8 N^3 n_2} + o\left(\frac{1}{N^3 n_2}\right). \end{aligned} \quad (101)$$

So we have

$$\begin{aligned} J_N^1 &= \frac{8E[\omega]}{3\pi^8 N^3} \sum_{n_2=1}^{[(N-1)/2]} \frac{1}{n_2} + o\left(\frac{1}{N^3}\right) \sum_{n_2=1}^{[(N-1)/2]} \frac{1}{n_2} \\ &= \frac{8 \log N}{3\pi^8 N^3} (E[\omega] + o(1)). \end{aligned} \quad (102)$$

Similarly, we have

$$J_N^2 = \frac{8 \log N}{3\pi^8 N^3} (E[\tilde{\omega}] + o(1)), \quad (103)$$

where $\tilde{\omega} = (\xi^{(1,1)}(0,0) + \xi^{(1,1)}(0,1))^2 + (\xi^{(1,1)}(1,0) + \xi^{(1,1)}(1,1))^2$. From this and (91), we deduce by (90) that

$$\begin{aligned} J_N &= \sum_{n_2=1}^{[(N-1)/2]} \left(\frac{8E[\omega + \tilde{\omega}]}{3\pi^8 N^3 n_2} + o\left(\frac{1}{N^3 n_2}\right) \right) \\ &= \frac{16 \log N}{3\pi^8 N^3} (E[\omega^*] + o(1)), \end{aligned} \quad (104)$$

where $\omega^* = (\xi^{(1,1)}(0,0))^2 + (\xi^{(1,1)}(0,1))^2 + (\xi^{(1,1)}(1,0))^2 + (\xi^{(1,1)}(1,1))^2$. Finally, we get by (90) that

$$E[\|S_N^{(h)}(\xi) - \xi\|_2^2] = \frac{4 \log N}{3\pi^8 N^3} (E[\omega^*] + o(1)). \quad (105)$$

So we get the following.

Theorem 7. Suppose that ξ is a stochastic process on $[0, 1]^2$ and $\xi^{(l,1)} \in C_s([0, 1]^2)$ for some $l \geq 2$. Let $S_N^{(h)}(\xi)$ be the hyperbolic cross truncations of its Fourier cosine series, which is stated in (89). Then the following asymptotic formula holds:

$$E[\|S_N^{(h)}(\xi) - \xi\|_2^2] = \widetilde{M} \frac{\log N}{N^3} (1 + o(1)), \quad (106)$$

where $\widetilde{M} = (4/3\pi^8)E[(\xi^{(1,1)}(0,0))^2 + (\xi^{(1,1)}(0,1))^2 + (\xi^{(1,1)}(1,0))^2 + (\xi^{(1,1)}(1,1))^2]$.

From Theorem 7, we know that $E[\|S_N^{(h)}(\xi) - \xi\|_2^2] \sim (\log N/N^3)$. This result cannot be improved as the smoothness index l increases.

By (89), the number of Fourier cosine coefficients in hyperbolic cross truncation $S_N^{(h)}(\xi)$ is

$$N_c = N + \sum_{n_1=1}^{N-1} \left[\frac{N-1}{n_1} \right] \sim N \log N \quad (107)$$

and $\log N_c \sim \log N$. From this, we get

$$E \left[\left\| S_N^{(h)}(\xi) - \xi \right\|_2^2 \right] \sim \frac{\log^4 N_c}{N_c^3}. \quad (108)$$

Since the number of Fourier cosine coefficients in the partial sum $S_N^{(r)}(\xi)$ is $N_c = N^2$, by Theorem 6, we have

$$E \left[\left\| S_N^{(r)}(\xi) - \xi \right\|_2^2 \right] \sim \frac{1}{N_c^{3/2}}. \quad (109)$$

8. Approximation of Hyperbolic Cross Truncations with Polynomial Factors

In order to use the least Fourier cosine coefficients to reconstruct stochastic processes, we introduce the Fourier cosine expansion with polynomial factors. Suppose that ξ is a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 3$. By using notation (38), the decomposition (23) can be rewritten in the form

$$\begin{aligned} \xi(t_1, t_2) &= P_\xi(t_1, t_2) + \xi_2(t_1, t_2) + \xi_3(t_1, t_2) \\ &= P_\xi(t_1, t_2) - \varphi_1(t_2)p(1-t_1) + \varphi_2(t_2)p(t_1) \\ &\quad - \varphi_3(t_1)p(1-t_2) + \varphi_4(t_1)p(t_2) + \xi_3(t_1, t_2). \end{aligned} \quad (110)$$

In this decomposition, we expand each φ_i into univariate Fourier cosine series and expand $\xi_3(t_1, t_2)$ into bivariate Fourier cosine series. Finally, we obtain the Fourier cosine expansion of ξ with polynomial factors:

$$\begin{aligned} \xi(t_1, t_2) &= P_\xi(t_1, t_2) \\ &\quad + \left(\sum_{n=0}^{\infty} c_n(\varphi_1) \cos(\pi n t_2) \right) \\ &\quad \times (-p(1-t_1)) + \left(\sum_{n=0}^{\infty} c_n(\varphi_2) \cos(\pi n t_2) \right) p(t_1) \\ &\quad + \left(\sum_{n=0}^{\infty} c_n(\varphi_3) \cos(\pi n t_1) \right) \\ &\quad \times (-p(1-t_2)) + \left(\sum_{n=0}^{\infty} c_n(\varphi_4) \cos(\pi n t_1) \right) p(t_2) \\ &\quad + \sum_{n_1, n_2=0}^{\infty} c_{n_1, n_2}(\xi_3) \cos(\pi n_1 t_1) \cos(\pi n_2 t_2), \end{aligned} \quad (111)$$

where $p(t) = (t^2/2) - (1/6)$. Notice that $\varphi_1(t_2) = \xi_1^{(1,0)}(0, t_2)$ and $\xi_1^{(3,3)} \in C_s([0, 1]^2)$, where $\xi_1 = \xi - P_\xi$. Using Proposition 1, we have $\varphi_1''' \in C_s([0, 1])$ and $\varphi_1'(0) = \varphi_1'(1) = 0$, and so

$$c_n(\varphi_1) = \frac{2}{(n\pi)^3} \int_0^1 \varphi_1'''(t) \sin(\pi n t) dt. \quad (112)$$

Therefore, we have $E[c_n(\varphi_1)] = o(1/n^3)$ and

$$\begin{aligned} E[c_n^2(\varphi_1)] &= \frac{4}{(n\pi)^6} \\ &\quad \times \int_{[0,1]^2} E[\varphi_1'''(t) \varphi_1'''(s)] \sin(\pi n t) \\ &\quad \times \sin(\pi n s) dt ds. \end{aligned} \quad (113)$$

Similarly, for $c_n(\varphi_2)$, $c_n(\varphi_3)$, and $c_n(\varphi_4)$, we get

$$\begin{aligned} E[c_n(\varphi_i)] &= o\left(\frac{1}{n^3}\right), \quad E[c_n^2(\varphi_i)] = o\left(\frac{1}{n^6}\right) \\ (i &= 1, 2, 3, 4). \end{aligned} \quad (114)$$

From this, we see that these univariate Fourier coefficients decay fast.

Now we show that bivariate Fourier cosine coefficients $c_{n_1, n_2}(h_3)$ decay fast. By $\xi^{(l,l)} \in C_s([0, 1]^3)$ ($l \geq 3$) and Proposition 2, using the integration by parts, we get

$$\begin{aligned} E[c_{n_1, n_2}(\xi_3)] &= \frac{4}{(n_1\pi)^3(n_2\pi)^3} \\ &\quad \times \int_{[0,1]^2} E[\xi_3^{(3,3)}(t_1, t_2)] \sin(\pi n_1 t_1) \\ &\quad \times \sin(\pi n_2 t_2) dt_1 dt_2, \\ E[c_{n_1, n_2}^2(\xi_3)] &= \frac{16}{(n_1\pi)^6(n_2\pi)^6} \\ &\quad \times \int_{[0,1]^4} E[\xi_3^{(3,3)}(t_1, t_2) \xi_3^{(3,3)}(s_1, s_2)] \\ &\quad \times \sin(\pi n_1 t_1) \sin(\pi n_2 t_2) \\ &\quad \times \sin(\pi n_1 s_1) \\ &\quad \times \sin(\pi n_2 s_2) dt_1 dt_2 ds_1 ds_2. \end{aligned} \quad (115)$$

So we get

$$\begin{aligned} E[c_{n_1, n_2}(\xi_3)] &= o\left(\frac{1}{n_1^3 n_2^3}\right), \\ E[c_{n_1, n_2}^2(\xi_3)] &= o\left(\frac{1}{n_1^6 n_2^6}\right). \end{aligned} \quad (116)$$

Take the hyperbolic cross truncations of the expansion (111),

$$\begin{aligned} T_N^{(h)}(\xi; t_1, t_2) &= P_\xi(t_1, t_2) + S_N(\varphi_1; t_1) p(1-t_1) \\ &\quad + S_N(\varphi_4, t_1) p(t_1) \\ &\quad + S_N(\xi_1^{(0,1)}(t_1, 0)) p(1-t_2) \\ &\quad + S_N(\xi_1^{(0,1)}(t_1, 1)) p(t_2) + S_N^{(h)}(\xi_3; t_1, t_2), \end{aligned} \quad (117)$$

where each $S_N(\varphi_i)$ is the first N terms partial sums of Fourier cosine series of φ_i and $S_N^{(h)}(\xi_3)$ is the hyperbolic cross truncation of Fourier cosine series of bivariate stochastic process of ξ_3 which is stated in (21). The hyperbolic cross truncation $T_N^{(h)}(\xi)$ is a combination of stochastic polynomials and cosine polynomials. By (111) and (117),

$$\begin{aligned} T_N^{(h)}(\xi; t_1, t_2) - \xi(t_1, t_2) &= (S_N(\varphi_1; t_2) - \varphi_1(t_2))p(t_1) \\ &\quad + (S_N(\varphi_2; t_2) - \varphi_2(t_2))p(t_1) \\ &\quad + (S_N(\varphi_3; t_1) - \varphi_3(t_1))p(t_2) \\ &\quad + (S_N(\varphi_4; t_1) - \varphi_4(t_1))p(t_2) \\ &\quad + (S_N^{(h)}(\xi_3; t_1, t_2) - \xi_3(t_1, t_2)). \end{aligned} \quad (118)$$

Again, by the formula $(\sum_{n=1}^k a_n)^2 \leq k^2 \sum_{n=1}^k a_n^2$, we get

$$\begin{aligned} \frac{1}{25} E[\|T_N^{(h)}(\xi) - \xi\|_2^2] &\leq \left(\sum_{i=1}^4 E[\|S_N(\varphi_i) - \varphi_i\|_2^2] \right) \|p\|_2^2 \\ &\quad + E[\|S_N^{(h)}(\xi_3) - \xi_3\|_2^2]. \end{aligned} \quad (119)$$

By using Parseval identity and (114), we get

$$\begin{aligned} E[\|S_N(\varphi_i) - \varphi_i\|_2^2] &= \frac{1}{2} \sum_{n=N}^{\infty} E[c_n^2(\varphi_i)] = o\left(\frac{1}{N^5}\right) \\ (i &= 1, 2, 3, 4). \end{aligned} \quad (120)$$

By Proposition 2, we have $\xi_3^{(1,1)}(0, 0) = \xi_3^{(1,1)}(0, 1) = \xi_3^{(1,1)}(1, 0) = \xi_3^{(1,1)}(1, 1) = 0$. Based on (114) and (116), the argument similar to Theorem 7 shows that

$$E[\|S_N^{(h)}(\xi_3) - \xi_3\|_2^2] = o\left(\frac{\log N}{N^5}\right). \quad (121)$$

From this and (119), we deduce that $E[\|T_N^{(h)}(\xi) - \xi\|_2^2] = o((\log N)/N^5)$. Noticing that the number of Fourier cosine coefficients in $T_N(t_1, t_2) : N_c \sim N \log N$, we get the following.

Theorem 8. Suppose that ξ is a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 3$. Then

$$E[\|T_N^{(h)}(\xi) - \xi\|_2^2] = o\left(\frac{\log^6 N_c}{N_c^5}\right), \quad (122)$$

where N_c is the number of Fourier cosine coefficients in $T_N^{(h)}(\xi)$.

Comparing Theorem 8 with (108) and (109), we obtain the following.

Theorem 9. Let ξ be a stochastic process on $[0, 1]^2$ and $\xi^{(l,l)} \in C_s([0, 1]^2)$ for some $l \geq 3$. Then

$$\begin{aligned} E[\|S_N^{(r)}(\xi) - \xi\|_2^2] &\sim \frac{1}{N_c^{3/2}}, \\ E[\|S_N^{(h)}(\xi) - \xi\|_2^2] &\sim \frac{\log^4 N_c}{N_c^3}, \\ E[\|T_N^{(h)}(\xi) - \xi\|_2^2] &= o\left(\frac{\log^6 N_c}{N_c^5}\right), \end{aligned} \quad (123)$$

where $S_N^{(r)}(\xi)$ and $S_N^{(h)}(\xi)$ are partial sums and hyperbolic cross truncations of Fourier cosine series of ξ , respectively, $T_N^{(h)}(\xi)$ is the hyperbolic cross truncations of Fourier expansion of ξ with polynomial factors, and N_c is the number of Fourier cosine coefficients in each sum.

From this, we see that if we reconstruct ξ by $T_N^{(h)}(\xi)$, we need the least Fourier cosine coefficients.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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