Research Article

Numerical Analysis of an $H^1$-Galerkin Mixed Finite Element Method for Time Fractional Telegraph Equation

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We discuss and analyze an $H^1$-Galerkin mixed finite element (H1-GMFE) method to look for the numerical solution of time fractional telegraph equation. We introduce an auxiliary variable to reduce the original equation into lower-order coupled equations and then formulate an $H^1$-GMFE scheme with two important variables. We discretize the Caputo time fractional derivatives using the finite difference methods and approximate the spatial direction by applying the $H^1$-GMFE method. Based on the discussion on the theoretical error analysis in $L^2$-norm for the scalar unknown and its gradient in one dimensional case, we obtain the optimal order of convergence in space-time direction. Further, we also derive the optimal error results for the scalar unknown in $H^1$-norm.

Moreover, we derive and analyze the stability of $H^1$-GMFE scheme and give the results of a priori error estimates in two- or three-dimensional cases. In order to verify our theoretical analysis, we give some results of numerical calculation by using the Matlab procedure.

1. Introduction

In this paper, our purpose is to present and discuss a mixed finite element method for the time fractional telegraph equation

$$
\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) + 2\kappa \frac{\partial}{\partial t} u(x,t) - \Delta u(x,t) + \beta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times J,
$$

with boundary condition

$$
u(x,t) = 0, \quad (x,t) \in \partial \Omega \times J,
$$

and initial conditions

$$
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,
$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ is a bounded domain with boundary $\partial \Omega$ and $J = (0,T]$ is the time interval with $0 < T < \infty$. The coefficients $\kappa > 0$ and $\beta \geq 0$ are two constants, $f(x,t)$ is a given source function, $u_0(x)$ and $u_1(x)$ are two given initial functions, and the time Caputo fractional-order derivatives $\frac{\partial^\alpha}{\partial t^\alpha} u(x,t)$ and $\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} u(x,t)$ are defined, respectively, by

$$
\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} (t-\tau)^{\alpha-1} d\tau,
$$

$$
\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} u(x,t) = \frac{1}{\Gamma(2-2\alpha)} \int_0^t \frac{\partial^2 u(x,\tau)}{\partial \tau^2} (t-\tau)^{2\alpha-1} d\tau,
$$

(4)

where $1/2 < \alpha < 1$.

In the current literatures, we can see that some numerical methods for solving fractional partial differential equations (PDEs), which include finite element methods [1–8], mixed finite element methods [9], finite difference methods [10–24], finite volume methods [25, 26], spectral methods [27], and discontinuous Galerkin methods [28–31], have been considered and analyzed. In 2014, Liu et al. [9] gave some theoretical error analysis for a class of fractional PDE based on a nonstandard mixed method in spatial direction and a finite difference scheme in time direction. In [32], Zhao and Li discussed finite element method for the fractional telegraph equation. In 2014, Wei et al. [31] studied the numerical
solution for time fractional telegraph equation based on the
LDG method. But, we have not seen any related studies
on mixed finite element methods for solving the fractional
telegraph equation.

Recently, some people have made use of the method to
obtain the numerical solution for some partial differential
equations since Pani (in 1998) [33] proposed an $H^1$-GMFE
method. This method includes some advantages, such as
avoiding the LBB consistency condition, allowing different
polynomial degrees of the finite element spaces, and obtaining
the optimal a priori estimates in both $H^1$ and $L^2$-norms.

In [34], Panì and Fairweather discussed some detailed a
priori error results of two numerical schemes based on the
$H^1$-GMFE method for linear parabolic integrodifferential
equations. In [35], Panì et al. gave some error analysis based
on $H^1$-GMFE scheme for the partial differential equation of
hyperbolic type. At the same time, a modified $H^1$-GMFE
method for RLW equation and Sobolev equation. In 2013,
Li et al. [38] introduced another auxiliary variable, which is
different from the one in [36], and then proposed and studied
an explicit multistep $H^1$-GMFE scheme for a RLW equation.

Liu and Li [39] and Zhou [40] gave some different discussions
on $H^1$-GMFE method for pseudohyperbolic equations (heat
transport equation), respectively. In [41], Che et al. studied
the $H^1$-GMFE method for a nonlinear integrodifferential
equation. Recently, Shi et al. [42, 43] proposed some noncon-
forming mixed scheme based on the $H^1$-GMFE method.

Based on the above review on $H^1$-GMFE method, we eas-
ily see that the method was studied based on the integer-order
partial differential equations. However, the theoretical results
of $H^1$-GMFE method for fractional telegraph equation have
not been presented and analyzed.

In this paper, our aim is to give some detailed a
priori error analysis and numerical results on the $H^1$-GMFE
method for time fractional telegraph equation. We apply
the difference schemes to approximate the time fractional
derivatives and use the $H^1$-GMFE method to discretize
spatial direction. We obtain some optimal a priori error
results of one dimension for the scalar unknown in $L^2$ and
$H^1$-norms. Moreover, we also get an a priori error result of
the optimal $L^2$-norm for the auxiliary variable. At the same
time, we use the $H^1$-GMFE method to deal with the cases in
several dimensions and analyze the stable results for the $H^1$-
GMFE scheme. We calculate some numerical results to verify
the theoretical analysis of $H^1$-GMFE method for fractional
telegraph equation.

The layout of the paper is as follows. In Section 2, we
formulate an $H^1$-GMFE scheme for time fractional telegraph
equation (5). In Section 3, we introduce some lemmas of two
important projections and two difference approximations for
time fractional derivatives and then analyze some a priori
error results. In Section 4, some theoretical results are given
in the cases of two and three dimensions. In Section 5, we
choose a numerical example to verify the theoretical analysis
of our method. In Section 6, we give some remarks and
extensions about the $H^1$-GMFE method for fractional PDEs.

For the need of study, we denote the natural inner product
as $(\cdot, \cdot)$ in $(L^2(\Omega))^d$, $d = 1, 2, 3$. Further, we write
the classical Sobolev spaces $W^{m,s}_0(\Omega)$ as $H^m_0$
with norm $\|z\|_{s,m} = \sum_{\alpha \leq \beta \leq m} \|\partial_{\alpha} \partial_{\beta} z \|^2_{L^2(\Omega)}^{1/2}$. When $m = 0$, we simply write the
norm $\|z\|_0$ as $\|z\|$

2. An $H^1$-GMFE Scheme in
One Space Dimension

In this section, we first consider the $H^1$-GMFE method for
the following time fractional telegraph equation in 1D case:

$$
\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} u(x,t) + 2k\partial_x^\alpha u(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + \beta u(x,t) = f(x,t), \ (x,t) \in \Omega \times J,
$$

with boundary condition

$$
u(x_L,t) = u(x_R,t) = 0, \ t \in J,
$$

and initial conditions

$$
u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) , \ x \in \Omega,
$$

where $\Omega = [x_L, x_R] \subset R$.

In order to get the $H^1$-GMFE formulation, we first
introduce an auxiliary variable $\sigma = \partial u(x,t)/\partial x$ and split (5)
into the following first-order system by

$$
\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \sigma(x,t) + 2k\partial_x^\alpha \sigma(x,t) - \frac{\partial \sigma(x,t)}{\partial x} + \beta \sigma(x,t) = f(x,t), \ 
$$

$$
\sigma - \frac{\partial u(x,t)}{\partial x} = 0.
$$

Multiplying (8) by $-\partial \sigma/\partial x$, $\sigma \in H^1$, integrating with respect to space from $x_L$ to $x_R$, and using an integration by
parts with $\partial u(x_L,t)/\partial t = \partial u(x_R,t)/\partial t = 0$ and $\partial^2 u(x,t)/\partial t^2 = \partial^2 u(x_R,t)/\partial t^2 = 0$, we easily get

$$
\left(\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \sigma, \omega\right) + 2k\left(\partial_x^\alpha \sigma, \omega\right) + \left(\frac{\partial \sigma}{\partial x}, \frac{\partial w}{\partial x}\right)
$$

$$
= \beta \left(\frac{\partial w}{\partial x}, \sigma\right) - \left(f, \frac{\partial w}{\partial x}\right).
$$

Multiply (9) by $\partial v/\partial x$, $v \in H^1_0$, and integrate with respect to
space from $x_L$ to $x_R$ to obtain

$$
\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) = \left(\sigma, \frac{\partial v}{\partial x}\right), \ \forall v \in H^1_0.
$$

For formulating finite element scheme, we now choose the
finite element spaces $V_h \subset H^1_0$ and $W_h \subset H^1$, which satisfy
the following approximation properties: for $1 \leq p \leq \infty$ and $k, r$ positive integers [33],

$$
\inf_{v_h \in V_h} \left\{ \| v - v_h \|_{L^p} + h^r \| v - v_h \|_{W^{k+1,p}} \right\} \\
\leq C h^{k+1} \| v \|_{W^{k+1,p}}, \quad v \in H^1 \cap W^{k+1,p},
$$

(12)

$$
\inf_{w_h \in W_h} \left\{ \| w - w_h \|_{L^p} + h^r \| w - w_h \|_{W^{k+1,p}} \right\} \\
\leq C h^{r+1} \| w \|_{W^{r+1,p}}, \quad w \in W^{r+1,p}.
$$

Based on the chosen finite element spaces, the semidiscrete $H^1$-GMFE scheme is described by

$$
\frac{\partial u_h}{\partial t} - \left( \sigma_h, \frac{\partial w_h}{\partial x} \right) = \beta \left( u_h, \frac{\partial w_h}{\partial x} \right) - \left( f, \frac{\partial w_h}{\partial x} \right) + \left( E_n, w \right), \quad \forall w \in H^1,
$$

(24)

where $E_n = E_n^\alpha + E_n'$. In the following analysis, for deriving the convenience of theoretical process, we now denote

$$
B_{n-k}^{\alpha} = (n-k+1)^{2-\alpha} - (n-k)^{2-\alpha}, \quad j = 1, 2
$$

and then holds

$$
|E_n^\alpha| \leq C_0 \Delta t^{2-\alpha}.
$$

(21)

3. Full Discrete Scheme and A Priori Error Estimates

3.1. Two Projection Lemmas. For a priori error estimates for fully discrete scheme, we introduce two projection operators [33, 44] in Lemmas 1 and 2.

**Lemma 1.** One defines a Ritz projection $P_h u \in V_h$ for the variable $u$ by

$$
(u - P_h u, v_h) = 0, \quad v_h \in V_h.
$$

(14)

Then the following estimates hold, for $j = 0, 1$:

$$
\| u - P_h u \|_j \leq C_h^{1-j} \| u \|_{k+1}.
$$

(15)

**Lemma 2.** Further, one also defines an elliptic projection $R_n \sigma \in W_h$ of $\sigma$ as the solution of

$$
\mathfrak{B} (\sigma - R_n \sigma, w_h) = 0, \quad w_h \in W_h,
$$

(16)

where $\mathfrak{B}(\sigma, w) = (\sigma_x, w_x) + \lambda(\sigma, w)$. Here $\lambda > 0$ is chosen to satisfy

$$
\mathfrak{B}(w, w) \geq \mu_0 \| w \|_{1}^2, \quad w \in H^1, \quad \mu_0 > 0.
$$

(17)

Then the following estimates are found: for $j = 0, 1$,

$$
\| \sigma - R_n \sigma \|_j \leq C_h^{1-j} \| \sigma \|_{r+1}.
$$

(18)

3.2. Approximation of Time-Fractional Derivative. For formulating fully discrete scheme, let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a given partition of the time interval $[0, T]$ with step length $\Delta t = T/M$ and nodes $t_n = n\Delta t$, for some positive integer $M$. For a smooth function $\phi$ on $[0, T]$, define $\phi^n = \phi(t_n)$. In the following analysis, for deriving the time fractional order derivative

$$
\partial_{\alpha}^a \sigma (x, t_n) = \frac{\Delta t^{2-\alpha}}{\Gamma (2-\alpha)} \sum_{k=1}^{n} B_{n-k}^\alpha D_t \sigma^k + E_n^\alpha,
$$

(20)

and then holds

$$
|E_n^\alpha| \leq C_0 \Delta t^{2-\alpha}.
$$

(23)

In the next analysis, we will derive and prove some a priori error results for $u$ and $\sigma$.

3.3. Error Estimates for Fully Discrete Scheme. Based on the approximation formulas (20) and (22) of time-fractional derivatives, we obtain the time semidiscrete scheme of (10) and (11):

$$
\left( \frac{\partial u^n}{\partial x}, \frac{\partial v}{\partial x} \right) = \left( \sigma^n, \frac{\partial v}{\partial x} \right), \quad \forall v \in H^1,
$$

$$
\frac{\Delta t^{2-\alpha}}{\Gamma (3-2\alpha)} \sum_{k=1}^{n} B_{n-k}^\alpha (D_t \sigma^k, w) + \frac{2\kappa \Delta t^{1-\alpha}}{\Gamma (2-\alpha)} \sum_{k=1}^{n} B_{n-k}^\alpha \left( \frac{\sigma^n}{\partial x}, \frac{\partial w}{\partial x} \right)
$$

$$
= \beta \left( u^n, \frac{\partial u^n}{\partial x} \right) - \left( f^n, \frac{\partial w}{\partial x} \right) + \left( E_0^n, w \right), \quad \forall w \in H^1,
$$

(24)

where $E_0^n = E_0^\alpha + E_2^\alpha$. 


Now, we formulate a fully discrete procedure: find 
\((u^n_h, \sigma^n_h) \in V_h \times W_h (n = 1, \ldots, M - 1)\) such that
\[
\frac{\Delta t^{2-2\alpha}}{\Gamma (3-2\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t \delta_k, w_h) 
+ \frac{2k \Delta t^{1-\alpha}}{\Gamma (2-\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t \sigma_k, w_h) + \left( \frac{\partial \sigma^n_h}{\partial x} - \frac{\partial v^n_h}{\partial x} \right) 
= \beta \left( u^n_h - \frac{\partial w_h}{\partial x} \right) - \left( f^n, \frac{\partial w_h}{\partial x} \right), \quad \forall w_h \in W_h.
\]
(25)

Making a combination of (24)-(25) with two projections (14) and (16), we get the following error equations:
\[
\frac{\Delta t^{2-2\alpha}}{\Gamma (3-2\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t \delta_k^{k-1}, w_h) 
+ \frac{2k \Delta t^{1-\alpha}}{\Gamma (2-\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t \delta_k^{k}, w_h) 
= \beta \left( u^n_h - \frac{\partial w_h}{\partial x} \right) - \left( f^n + \delta^n, \frac{\partial w_h}{\partial x} \right), \quad \forall w_h \in W_h.
\]
(26)

where
\[
\delta^n = (u^n - (u^n_h - \frac{\partial w_h}{\partial x})), \quad \sigma^n = (\sigma^n - \frac{\partial w_h}{\partial x}),
\]
(28)

In the following discussion, we will derive the proof for
the fully discrete a priori error estimates.

**Theorem 5.** With \(\sigma_0 = u_0\) and \(\sigma_1 = u_1\), suppose that \(u^n \in H_0^1 \cap H^{k+1}\), \(\sigma^n \in H^{k+1}\), \(\sigma^n = R_h \sigma(0)\), and \(\sigma_2(0) = L_h \sigma(0)\),
where \(L_h\) is the \(L^2\) projection defined by \((w - L_h w, w_h) = 0, w_h \in W_h\). Then there exists a positive constant \(C(u, \sigma, \alpha, \beta)\)
free of space-time discrete parameters \(h\) and \(\Delta t\) such that, for \(1/2 < \alpha < 1\),
\[
\|
\begin{align*}
\sigma^n - \sigma^n_h & \leq C (u, \sigma, \alpha, \beta) T^{2-1} \min \left[ \Delta t^{1-(k+1)}, \Delta t^{3-2\alpha} \right] \\
& \leq C (u, \sigma, \alpha, \beta) T^{2-1} (\Delta t^{1-(k+1)}, \Delta t^{3-2\alpha}) \\
& \quad + C (\sigma) h^{r+1}, \quad j = 0, 1,
\end{align*}
\]
(30)

and for \(\alpha \to 1\)
\[
\|
\begin{align*}
\sigma^n - \sigma^n_h & \leq C (u, \sigma, \beta) \left( \Delta t^{1-(k+1)} + \Delta t \right) \\
& \quad + C (\sigma) h^{r+1}, \quad j = 0, 1,
\end{align*}
\]
(31)

Proof. For the need of error analysis, we first consider the \(L^2\)-norm \(||\delta^n||\) and the \(H^1\)-norm \(||\sigma^n||_1\). Taking \(v_h = \sigma^n\) in (26) and using Poincaré inequality based on the \(H_0^1\)-space and Cauchy-Schwarz inequality, we easily get
\[
||\delta^n|| \leq ||\sigma^n||_1 \leq C \left( ||\frac{\partial \delta^n}{\partial x}|| \right) \leq C \left( ||\sigma^n|| + ||\delta^n|| \right).
\]
(33)

In the next analysis, we will give the estimates of \(||\delta^n||\) and \(||\sigma^n||\) in \(L^2\)-norm. Noting that
\[
\sum_{k=1}^{n-1} B_{n-k}^{\alpha} D_t \delta_k = \sum_{k=0}^{n-1} B_{n-k}^{\alpha} D_t \delta^{n-k},
\]
(34)
then (27) may be rewritten as
\[
\frac{\Delta t^{2-2\alpha}}{\Gamma (3-2\alpha)} \sum_{k=0}^{n-1} B_k^{2\alpha} (D_{t\alpha}^b \delta^{n-k-1}, w_h)
\]
\[
+ \frac{2\kappa \Delta t^{1-\alpha}}{\Gamma (2 - \alpha)} n \sum_{k=0}^{n-1} B_k^{2\alpha} (D_{t\alpha}^b \delta^{n-k}, w_h) + \mathfrak{B} (\delta^n, w_h)
\]
\[
= - \frac{\Delta t^{2-2\alpha}}{\Gamma (3-2\alpha)} \sum_{k=1}^{n} B_k^{2\alpha} (D_{t\alpha}^b \delta^{k-1}, w_h)
\]
\[
- \frac{2\kappa \Delta t^{1-\alpha}}{\Gamma (2 - \alpha)} n \sum_{k=1}^{n} B_k^{2\alpha} (D_{t\alpha}^b \delta^{k}, w_h)
\]
\[
+ \lambda (\partial_x \delta^n, w_h) + (E_\alpha, w_h) + \beta (\psi^n + \delta^n, \frac{\partial w_h}{\partial x}).
\]
We take \( w_h = \delta^n \) in (35) and multiply by
\[
\Gamma_\alpha \doteq \Gamma (3 - 2\alpha) \Gamma (2 - \alpha) \Delta t^{3-2\alpha},
\]

to arrive at
\[
\Gamma (2 - \alpha) \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - 2\delta^{n-k-1} + \delta^{n-k-2}, \delta^n)
\]
\[
+ 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - \delta^{n-k-1}, \delta^n)
\]
\[
+ \Gamma_\alpha \mathfrak{B} (\delta^n, \delta^n)
\]
\[
= - \Gamma (2 - \alpha) \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - 2\delta^{n-k-1} + \delta^{n-k-2}, \delta^n)
\]
\[
- 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - \delta^{n-k-1}, \delta^n)
\]
\[
+ \lambda \Gamma_\alpha (\psi^n, \delta^n) + \Gamma_\alpha (E_\alpha, \delta^n) + \beta \Gamma_\alpha (\psi^n + \delta^n, \frac{\partial \phi^n}{\partial x}).
\]
(35)

By the simple calculation, we get the following equality:
\[
\Gamma (2 - \alpha) \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - 2\delta^{n-k-1} + \delta^{n-k-2}, \delta^n)
\]
\[
+ 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - \delta^{n-k-1}, \delta^n)
\]
\[
= (\Gamma (2 - \alpha) + 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha}) \| \delta^n \|^2 + \Gamma (2 - \alpha)
\]
\[
\times \left[ (B_0^{2\alpha} - 2B_1^{2\alpha}) (\delta^{n-1} - \delta^n)
\right.
\]
\[
- B_n^{2\alpha} (\delta^n, \delta^n) + B_{n-1}^{2\alpha} (\delta^{-1}, \delta^n)
\]
\[
+ \left( \sum_{k=1}^{n-1} (B_{k+1}^{2\alpha} - 2B_k^{2\alpha} + B_{k-1}^{2\alpha}) \delta^{n-k-1}, \delta^n \right)
\]
\[
+ 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} \left[ \left( \sum_{k=1}^{n-1} (B_k^{2\alpha} - B_{k-1}^{2\alpha}) \delta^{n-k}, \delta^n \right) - B_{n-1}^{2\alpha} (\delta^n, \delta^n) \right].
\]
(37)

By applying the similar process of calculation to (38), we use Cauchy-Schwarz inequality to get
\[
- \Gamma (2 - \alpha) \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - 2\delta^{n-k-1} + \delta^{n-k-2}, \delta^n)
\]
\[
- 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} \sum_{k=0}^{n-1} B_k^{2\alpha} (\delta^{n-k} - \delta^{n-k-1}, \delta^n)
\]
\[
\leq \left( \Gamma (2 - \alpha) + 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} \right) \| \delta^n \| + \Gamma (2 - \alpha)
\]
\[
\times \left[ \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) \left( \| \delta^{n-1} \| + B_0^{2\alpha} \| \delta^n \| + B_{n-1}^{2\alpha} \| \delta^{n-1} \| \right) \| \delta^n \|
\right.
\]
\[
+ \sum_{k=1}^{n-1} |B_k^{2\alpha} - B_{k-1}^{2\alpha}| \left( \| \delta^{n-k} \| + B_{n-1}^{2\alpha} \| \delta^{n-k} \| \right) \| \delta^n \|.
\]
(39)

Noting (33), we use Cauchy-Schwarz inequality and Young inequality to have
\[
\beta \Gamma_\alpha \left( \psi^n + \delta^n, \frac{\partial \phi^n}{\partial x} \right)
\]
\[
\leq \beta \Gamma_\alpha \| \psi^n \| \| \frac{\partial \phi^n}{\partial x} \| + \beta \Gamma_\alpha \| \delta^n \| \| \frac{\partial \phi^n}{\partial x} \|
\]
\[
\leq C (\beta \Gamma_\alpha) \| \psi^n \| \| \frac{\partial \phi^n}{\partial x} \| + \| \frac{\partial \phi^n}{\partial x} \| \| \delta^n \|
\]
\[
+ C (\beta \Gamma_\alpha) \| \psi^n \|^2 + \| \frac{\partial \phi^n}{\partial x} \|^2 + C (\beta \Gamma_\alpha) \| \delta^n \|^2.
\]
(40)

Substitute (38), (39), and (40) into (37) and use Cauchy-Schwarz inequality to arrive at
\[
(\Gamma (2 - \alpha) + 2\kappa \Gamma (3 - 2\alpha) \Delta t^{\alpha} - C (\beta \Gamma_\alpha) \| \delta^n \|^2
\]
\[
+ \Gamma_\alpha \mathfrak{B} (\delta^n, \delta^n)\]
\[
\begin{align*}
&\leq \left( (\Gamma (2 - \alpha) + 2\kappa\Gamma (3 - 2\alpha) \Delta t^\alpha) \| \varepsilon^n \| + \Gamma (2 - \alpha) \right) \\
&\times \left[ \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) \left( \| \delta^{n-1} \| + \| \varepsilon^{n-1} \| \right) \\
&+ B_n^{2\alpha} \left( \| \delta^0 \| + \| \varepsilon^0 \| \right) + B_{n-1}^{2\alpha} \left( \| \delta^{-1} \| + \| \varepsilon^{-1} \| \right) \\
&+ \sum_{k=1}^{n-1} \left( B_{k+1}^{2\alpha} - 2B_k^{2\alpha} + B_k^{2\alpha} \right) \\
&\times \left( \| \delta^{n-k-1} \| + \| \varepsilon^{n-k-1} \| \right) \\
&+ \lambda \Gamma_n \| \varepsilon^n \| + \Gamma_n \| E_0^n \| \\
&+ 2\kappa (3 - 2\alpha) \Delta t^\alpha \\
&\times \left[ \left( \Gamma (2 - \alpha) + 2\kappa\Gamma (3 - 2\alpha) \Delta t^\alpha \right) \| \varepsilon^n \| + \Gamma (2 - \alpha) \right) \\
&\times \left[ \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) \left( \| \delta^{n-1} \| + \| \varepsilon^{n-1} \| \right) \\
&+ B_n^{2\alpha} \left( \| \delta^0 \| + \| \varepsilon^0 \| \right) + B_{n-1}^{2\alpha} \left( \| \delta^{-1} \| + \| \varepsilon^{-1} \| \right) \\
&+ \sum_{k=1}^{n-1} \left( B_{k+1}^{2\alpha} - 2B_k^{2\alpha} + B_k^{2\alpha} \right) \\
&\times \left( \| \delta^{n-k-1} \| + \| \varepsilon^{n-k-1} \| \right) \\
&+ \lambda \Gamma_n \| \varepsilon^n \| + \Gamma_n \| E_0^n \| \\
&+ 2\kappa (3 - 2\alpha) \Delta t^\alpha
\end{align*}
\]

By an application of Young inequality, we get

\[
\sqrt{\Gamma (2 - \alpha)} \| \delta^n \| + \sqrt{\Gamma_n} \| \varepsilon^n \| 
\leq \frac{1}{\sqrt{\Gamma (2 - \alpha)}} \left( (\Gamma (2 - \alpha) + 2\kappa\Gamma (3 - 2\alpha) \Delta t^\alpha) \| \varepsilon^n \| + \Gamma (2 - \alpha) \right) \\
\times \left[ \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) \left( \| \delta^{n-1} \| + \| \varepsilon^{n-1} \| \right) \\
+ B_n^{2\alpha} \left( \| \delta^0 \| + \| \varepsilon^0 \| \right) + B_{n-1}^{2\alpha} \left( \| \delta^{-1} \| + \| \varepsilon^{-1} \| \right) \\
+ \sum_{k=1}^{n-1} \left( B_{k+1}^{2\alpha} - 2B_k^{2\alpha} + B_k^{2\alpha} \right) \\
\times \left( \| \delta^{n-k-1} \| + \| \varepsilon^{n-k-1} \| \right) \\
+ \lambda \Gamma_n \| \varepsilon^n \| + \Gamma_n \| E_0^n \| \\
+ 2\kappa (3 - 2\alpha) \Delta t^\alpha
\right)
\]

Noting that \( B_k^{2\alpha} - B_{k+1}^{2\alpha} > 0 \), we arrive at

\[
\sum_{k=1}^{n-1} \left| B_k^{2\alpha} - B_{k+1}^{2\alpha} \right| \leq \sum_{k=1}^{n-1} \left( B_k^{2\alpha} - B_{k+1}^{2\alpha} + B_{k-1}^{2\alpha} - B_k^{2\alpha} \right)
\]

\[
= B_1^{2\alpha} - B_n^{2\alpha} + B_0^{2\alpha} - B_{n-1}^{2\alpha}.
\]

At the same time, noting that \( B_k^{2\alpha} - B_{k+1}^{2\alpha} > 0 \), we use the similar method to have

\[
\sum_{k=1}^{n-1} \left| B_k^{2\alpha} - B_{k+1}^{2\alpha} \right| = B_0^{2\alpha} - B_{n-1}^{2\alpha}.
\]

By a combination of (43) and (44) with (18) and noting that \( B_0^{2\alpha} = 1 \), we arrive at

\[
\frac{1}{\sqrt{\Gamma (2 - \alpha)}} \left( (\Gamma (2 - \alpha) + 2\kappa\Gamma (3 - 2\alpha) \Delta t^\alpha) \| \varepsilon^n \| + \Gamma (2 - \alpha) \right) \\
\times \left[ \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) \left( \| \delta^{n-1} \| + \| \varepsilon^{n-1} \| \right) \\
+ B_n^{2\alpha} \left( \| \delta^0 \| + \| \varepsilon^0 \| \right) + B_{n-1}^{2\alpha} \left( \| \delta^{-1} \| + \| \varepsilon^{-1} \| \right) \\
+ \sum_{k=1}^{n-1} \left( B_{k+1}^{2\alpha} - 2B_k^{2\alpha} + B_k^{2\alpha} \right) \\
\times \left( \| \delta^{n-k-1} \| + \| \varepsilon^{n-k-1} \| \right) \\
+ \lambda \Gamma_n \| \varepsilon^n \| + \Gamma_n \| E_0^n \| \\
+ 2\kappa (3 - 2\alpha) \Delta t^\alpha
\right)
\]

\[
\leq \frac{C}{\sqrt{\Gamma (2 - \alpha)}} \left( (\Gamma (2 - \alpha) + 2\kappa\Gamma (3 - 2\alpha) \Delta t^\alpha + \Gamma (2 - \alpha) \right) \\
\times \left[ \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) + B_n^{2\alpha} + B_{n-1}^{2\alpha} + B_0^{2\alpha} - B_{n-1}^{2\alpha} \right] \\
+ \Gamma (3 - 2\alpha) \sqrt{\Gamma (2 - \alpha) \Delta t^{1+2\alpha}}
\]

\[
= \frac{C}{\sqrt{\Gamma (2 - \alpha)}} \left( (\Gamma (2 - \alpha) + 2\kappa\Gamma (3 - 2\alpha) \Delta t^\alpha \right) h^{r+1}
\]

\[
+ \sqrt{C (u, \sigma, \beta)} \frac{\Gamma_n}{\Gamma (2 - \alpha)} \left( h^{r+1} + h^{k+1} \right)
\]

\[
+ \Gamma (3 - 2\alpha) \sqrt{\Gamma (2 - \alpha) \Delta t^{1+2\alpha}}.
\]
Substitute (45) into (42) and note that $\mu_0 \|\delta^n\|_1 \leq \mathcal{B}(\delta^n, \delta^n)/\|\delta^n\|_1 \leq \mathcal{B}(\delta^n, \delta^n)/\|\delta^n\|$ to get

$$
\sqrt{\Gamma(2-\alpha)} \|\delta^n\|_1 + \mu_0 \sqrt{\Gamma(2-\alpha)} \|\delta^n\|_1 \\
\leq \sqrt{\Gamma(2-\alpha)} \left( (2B_0^{2\alpha} - B_1^{2\alpha}) \|\delta^{n-1}\|_1 + B_1^{2\alpha} \|\delta^0\| + B_n^{2\alpha} \|\delta^{n-1}\|_1 \\
+ \sum_{k=1}^{n-1} B_{k+1}^{2\alpha} - 2B_k^{2\alpha} + B_k^{2\alpha} \|\delta^{n-k-1}\|_1 \right) \\
+ \frac{2\kappa \Gamma(3-2\alpha) \Delta t^\alpha}{\sqrt{\Gamma(2-\alpha)}} \\
\times \left[ \sum_{k=1}^{n-1} |B_k^{\alpha} - B_{k-1}^{\alpha}| \|\delta^{n-k}\| + B_{n-1}^{\alpha} \|\delta^0\| \right] \\
+ \frac{C}{\sqrt{\Gamma(2-\alpha)}} (\Gamma(2-\alpha) + 2\kappa \Gamma(3-2\alpha) \Delta t^\alpha) h^{r+1} \\
+ \sqrt{\Gamma(2-\alpha)} \Gamma(\alpha) (h^{r+1} + h^{k+1}) \\
+ C \Gamma(3-2\alpha) \sqrt{\Gamma(2-\alpha)} \Delta t^\alpha \leq \frac{C(u, \sigma, \alpha, \beta)}{B_n^{2\alpha}} \left( h^{r+1} + h^{k+1} + \Delta t^2 + \Delta t^{1+2\alpha} \right). \tag{46}
$$

In the following discussion, we apply mathematical induction to obtain the error result

$$
\|\delta^n\| \leq \frac{C(u, \sigma, \alpha, \beta)}{B_n^{2\alpha}} \left( h^{r+1} + h^{k+1} + \Delta t^2 + \Delta t^{1+2\alpha} \right). \tag{47}
$$

Take $n = 1$ in (46) and note that $B_0^{2\alpha} = 1$; it is easy to find that the following inequality holds:

$$
\|\delta^1\| + \Delta t^\alpha \|\delta^1\|_1 \\
\leq C(u, \sigma, \alpha, \beta) \left( h^{k+1} + h^{r+1} + \Delta t^2 + \Delta t^{1+2\alpha} \right) \\
\leq \frac{C(u, \sigma, \alpha, \beta)}{B_0^{2\alpha}} \left( h^{k+1} + h^{r+1} + \Delta t^2 + \Delta t^{1+2\alpha} \right). \tag{48}
$$

Based on the process of mathematical induction, we claim that (47) holds.

For the need of the next proof, we have to estimate the term $n^{1-2\alpha}/B_n^{2\alpha}$. We now use Taylor formula to arrive at

$$
n^{1-2\alpha}/B_n^{2\alpha} = \frac{n^{1-2\alpha}}{n^{2-2\alpha} - (n-1)^{2-2\alpha}} \\
= \frac{1}{n - (n-1)(1 + 1/(n-1))^{2\alpha-1}} \\
= \left( \frac{n}{n - (n-1)} \right) \left( 1 + \frac{(2\alpha - 1)}{n-1} \right) \\
\times \left( \frac{1 + \theta \frac{1}{n-1}}{n-1} \right)^{2\alpha-3} \left( \frac{1}{n-1} \right)^{\alpha-1}.
$$

Assuming that the inequalities $\|\delta^n\| \leq (C(u, \sigma, \alpha, \beta)/B_{j-1}^{2\alpha})(h^{k+1} + h^{r+1} + \Delta t^2 + \Delta t^{1+2\alpha})$ hold for $j = 1, 2, \ldots, n-1$, we now prove that the inequality (47) holds. Noting that $1/B_{k+1}^{2\alpha} < 1/B_k^{2\alpha}$, $j = 1, 2$, and combining (43) and (44) with (46), we have
where 0 < θ < 1 and 1/2 < α < 1. Noting that (nΔt)2α−1 ≤ T2α−1 and (50), we have
\[
\left\| \delta^n \right\| \leq \frac{C (u, \sigma, \alpha, \beta)}{B_{2n}^{2\alpha}} \left( h^{r+1} + h^{k+1} + \Delta t^2 + \Delta t^{1+2\alpha} \right)
\]
(50)
 substituting (51) into (33) and using (18) to get
\[
\left\| \delta^n \right\| \leq C \left( \left\| e^n \right\| + \left\| \delta^n \right\| \right)
\]
(51)
 similarly, the estimate \( \left\| u^n - u^n_0 \right\| \) in the \( L^2 \)-norm also are obtained by a combination of (52) and (15) with triangle inequality.

Now we mainly analyze the case \( \alpha \to 1 \). We can find that the error inequality (29) in this case has no meaning since the coefficient \( C(u, \sigma, \alpha, \beta)/(2 - 2\alpha) \to \infty \) as \( \alpha \to 1 \). So, we have to look for another error estimate's process. Noting the fact that \( n\Delta t \leq T \) (n = 1, 2, ..., M), we can obtain the following error inequality:
\[
\left\| \delta^n \right\| \leq \frac{C (u, \sigma, \alpha, \beta)}{B_{2n}^{2\alpha}} n\Delta t \left( h^{r+1} + h^{k+1} + \Delta t^2 + \Delta t^{3-2\alpha} \right)
\]
(53)

Now we can use induction to prove the inequality (53). The detailed proof is similar to the above process of analysis, so we do not give the detailed proof again. Making a combination of (53) and (18) with triangle inequality, we can get the estimate (31). A similar discussion for (32) can also be made.

4. An \( H^1 \)-GMFE Scheme for Several Spaces Variables

In this section, we consider (1) with two and three spaces variables.

Let \( L^2(\Omega) = (L^2(\Omega))^d \) (d = 2 or 3) with inner product and norm
\[
(\mathbf{q}, \mathbf{w}) = \sum_{i=1}^{d} (q_i, w_i), \quad \left\| \mathbf{w} \right\| = \left( \sum_{i=1}^{d} |w_i|^2 \right)^{1/2}
\]
(54)

Further, let
\[
\mathbf{W} = H(\text{div}; \Omega) = \left\{ \mathbf{w} \in L^2(\Omega) \mid \nabla \cdot \mathbf{w} \in L^2(\Omega) \right\}
\]
(55)
with norm
\[
\left\| \mathbf{w} \right\|_{H(\text{div};\Omega)} = \left( \left\| \mathbf{w} \right\|^2 + \left\| \nabla \cdot \mathbf{w} \right\|^2 \right)^{1/2}
\]
(56)

Taking \( \mathbf{q} = \nabla u \), we use a similar process to the system (10) and (11) to get the \( H^1 \)-Galerkin mixed weak formulation \( [u, \mathbf{q}] : [0, T] \to H^1_0 \times \mathbf{W} \) by
\[
(\nabla u, \nabla v) = (\mathbf{q}, \nabla v), \quad \forall v \in H^1_0,
\]
\[
(\partial_t^{2\alpha} \mathbf{q}, \mathbf{w}) + 2\alpha (\partial_t^{2\alpha} \mathbf{q}, \mathbf{w}) + (\nabla \cdot \mathbf{q}, \nabla \cdot \mathbf{w}) = \beta (u, \nabla \cdot \mathbf{w}) - (f, \nabla \cdot \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{W}
\]
(57)

The corresponding time semidiscrete system is defined by
\[
(\nabla u^n, \nabla v) = (\mathbf{q}^n, \nabla v), \quad \forall v \in H^1_0
\]
\[
\frac{\Delta t^{2-2\alpha}}{\Gamma(3-2\alpha)} \sum_{k=1}^{n} B_{2n-k}^{2\alpha} (D_k u, \mathbf{q}^{k-1}, \mathbf{w}) + \frac{2\alpha \Delta t^{2-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} B_{2n-k}^{2\alpha} (D_k \mathbf{q}^{k-1}, \mathbf{w}) + (\nabla \cdot \mathbf{q}^n, \nabla \cdot \mathbf{w}) = \beta (u^n, \nabla \cdot \mathbf{w}) - (f^n, \nabla \cdot \mathbf{w}) + (\mathbf{R}_n^0, \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{W}
\]
(58)

where \( \mathbf{R}_n^0 = O(\Delta t) \), which can be estimated by a similar process to \( E_n^0 \).
In order to get fully discrete mixed finite element scheme, we now choose the finite element spaces \( V_h \subset H^1 \) and \( W_h \subset W \), which satisfy the following approximation properties: for \( 1 \leq p \leq \infty \) and \( k, r \) positive integers [33],

\[
\inf_{v_h \in V_h} \| v - v_h \| + h \| v - v_h \|_1 \leq C h^{k+1} \| v \|_{k+1}, \quad v \in H^{k+1} \cap H^1 \cap H^1(\Omega),
\]

\[
\inf_{w_h \in W_h} \| w - w_h \| + h \| w - w_h \|_{H^1(\text{div}; \Omega)} \leq C h^{r+1} \| w \|_{r+1}, \quad w \in H^r+1.
\]

(59)

The fully discrete \( H^1 \)-GMFE scheme is to find \( \{u^n_h, q^n_h\} : [0, T] \mapsto V_h \times W_h \) such that

\[
(\nabla u^n_h, \nabla v_h) = (q^n_h, \nabla v_h), \quad \forall v_h \in V_h,
\]

(60)

\[
\frac{\Delta t^{2-2\alpha}}{\Gamma(3-2\alpha)} \sum_{k=1}^{n} B_{n-k}^{2\alpha} (D_t q^{k-1}_h, w_h) + \frac{2k \Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t q^{k}_h, w_h) + (\nabla \cdot q^n_h, \nabla \cdot w_h) = \beta (u^n_0, \nabla \cdot w_h) - (f^n, \nabla \cdot w_h), \quad \forall w \in W_h.
\]

(61)

4.1. The Analysis of Stability. In the following discussion, we will give the stability for the system (60) and (61). First, we need to obtain an important lemma on two initial value conditions.

**Lemma 6.** With \( q_0 = \nabla u_0 \) and \( q(0) = \nabla u_1 \), the following inequality holds:

\[
\| q_0^{-1} \| \leq C (\| u_0 \| + \Delta t \| u_1 \|)
\]

(62)

**Proof.** By a similar discussion as in [32], we have

\[
\| q_0^{-1} \| \leq C (\| q_0 \| + \Delta t \| q_0(0) \|)
\]

(63)

Based on the given initial value conditions and (63), we arrive at

\[
\| q_0^{-1} \| \leq C (\| \nabla u_0 \| + \Delta t \| \nabla u_1 \|)
\]

(64)

\[
\leq C (\| u_0 \|_1 + \Delta t \| u_1 \|_1).
\]

\(\square\)

**Theorem 7.** The following stable inequality for the system (60) and (61) holds:

\[
\| u^n_0 \| + \| q^n_0 \| + \sqrt{\kappa \Gamma_a} \| \nabla \cdot q^n_0 \|
\]

\[
\leq C (\alpha) \left( \sqrt{\gamma_0 \max_{0 \leq \alpha \leq n} \| f^n \| + \| u_0 \|_1 + \Delta t \| u_1 \|_1} \right).
\]

(65)

**Proof.** In (60), we take \( v_h = \nabla u^n_0 \) and use Cauchy-Schwarz inequality and Poincaré inequality to arrive at

\[
\| u^n_0 \| \leq \| q^n_0 \| \leq C \| \nabla u^n_0 \| \leq C \| q^n_0 \|.
\]

(66)

In (61), we choose \( w_h = q^n_0 \) and use Cauchy-Schwarz inequality, Young inequality, and (66) to get

\[
\frac{\Delta t^{2-2\alpha}}{\Gamma(3-2\alpha)} \sum_{k=1}^{n} B_{n-k}^{2\alpha} (D_t q^{k-1}_h, q^n_0) + \frac{2k \Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t q^{k}_h, q^n_0) + \| \nabla \cdot q^n_0 \|^2
\]

\[
\leq \beta \| u^n_0 \| \| \nabla \cdot q^n_0 \| + \| f^n \| \| \nabla \cdot q^n_0 \|
\]

\[
\leq C \| q^n_0 \| \| \nabla \cdot q^n_0 \| + \| f^n \|^2
\]

(67)

By the simple simplification for (67), we easily get

\[
\frac{\Delta t^{2-2\alpha}}{\Gamma(3-2\alpha)} \sum_{k=1}^{n} B_{n-k}^{2\alpha} (D_t q^{k-1}_h, q^n_0)
\]

\[
+ \frac{2k \Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} B_{n-k}^{\alpha} (D_t q^{k}_h, q^n_0) + \frac{1}{2} \| \nabla \cdot q^n_0 \|^2
\]

\[
\leq C (\| q^n_0 \|^2 + \| f^n \|^2).
\]

(68)

For the case \( n = 1 \) in (68), we multiply (36) to get easily

\[
\Gamma(2-\alpha) B_0^{2\alpha} (q_0^1 - 2q_0^0 + q_0^{-1}, q_0^1)
\]

\[
+ 2k \Delta t^{1-\alpha} \Gamma(3-2\alpha) B_0^{\alpha} (q_0^1 - 2q_0^0 + q_0^{-1}) + \kappa \Gamma_a \| \nabla \cdot q_0^0 \|^2
\]

\[
\leq C \Gamma_a \| f_0^1 \|^2 + C (\| q_0^{-1} \|^2 + \| q_0^0 \|^2)
\]

(69)

Noting that \( B_0^0 = 1 = B_0^{2\alpha} \) and Lemma 6, we use Cauchy-Schwarz inequality to get

\[
\left( \frac{1}{2} \Gamma(2-\alpha) + \kappa \Gamma_a \Gamma(3-2\alpha) - C \Gamma_a \right) \| q_0^0 \|^2
\]

\[
+ \kappa \Gamma_a \| \nabla \cdot q_0^1 \|^2
\]

\[
\leq C \Gamma_a \| f_0^1 \|^2 + C (\| q_0^{-1} \|^2 + \| q_0^0 \|^2)
\]

\[
\leq C \Gamma_a \| f_0^1 \|^2 + C (\| q_0^{-1} \|^2 + \| q_0^0 \|^2 + \| u_0 \|^2_1 + \Delta t \| u_1 \|^2_1)
\]

(70)

By the simple calculation for (70), we arrive at

\[
\| q_0^0 \| + \sqrt{\kappa \Gamma_a} \| \nabla \cdot q_0^1 \|
\]

\[
\leq C (\alpha) \left( \sqrt{\gamma_0 \max_{0 \leq \alpha \leq n} \| f_0^1 \| + \| u_0 \|_1 + \Delta t \| u_1 \|_1} \right).
\]

(71)
For the case \( n \geq 1 \), using a similar process to the proof of Theorem 5 based on the mathematical induction, we can get
\[
\|q^n_0\|_n + \sqrt{k} \|q^n_0\|_n \leq C(\alpha) \left( \max_{0 \leq i \leq n} f_i + \|u_0\| + \Delta t \|u_1\| \right).
\] (72)

By a combination of (72) and (66), we get the conclusion of Theorem 7.

4.2. A Priori Error Results. For deriving the a priori error analysis, we define the Ritz projection \( \tilde{u}_h \in V_h \) by
\[
(\nabla (u - \tilde{u}_h), \nabla v_h) = 0, \quad v_h \in V_h.
\] (73)

Further, let \( q_h \in W_h \) be the standard finite element interpolant of \( q \).
Let \( \rho = q - q_h, \eta = u - u_h \); see \[33, 44\]; we obtain
\[
\|\eta\|_j \leq CH^{k+1-j}\|u\|_{k+1}, \quad j = 0, 1,
\]
\[
\|\rho\| + h\|\rho\|_{H(div, \Omega)} \leq CH^{k+1}\|q\|_{l+1}.
\] (74)

Now, we can get the following theorem of a priori error estimates based on the above contents.

**Theorem 8.** With \( q_h = \nabla u_0, q_0 = \nabla u_j, u^n \in H^1 \cap H^{k+1} \), and \( \tilde{q}_h \in H^{l+1} \), there exists a positive constant \( C(u, q, \alpha, \beta) \) free of space-time meshes \( h \) and \( \Delta t \) such that, for \( 1/2 < \alpha < 1 \),
\[
\|q^n - q^n_h\| \leq C(u, q, \alpha, \beta) \frac{T^{2\alpha - 1}}{2 - 2\alpha} \left( \Delta t^{1 - 2\alpha} h^{\min[1 + k, l]} + \Delta t^{3 - 2\alpha} \right) + C(q) H^{l+1},
\]
\[
\|u^n - u^n_h\|_j \leq C(u, q, \alpha, \beta) \frac{T^{2\alpha - 1}}{2 - 2\alpha} \left( \Delta t^{1 - 2\alpha} h^{\min[1 + k, l]} + \Delta t^{3 - 2\alpha} \right) + C(q) H^{l+1} + C(u) H^{k+1-j}, \quad j = 0, 1,
\] (75)
and for \( \alpha \to 1 \)
\[
\|q^n - q^n_h\| \leq C(u, q, \beta) \left( \Delta t^{-1} h^{\min[1 + k, l]} + \Delta t \right) + C(q) H^{l+1},
\]
\[
\|u^n - u^n_h\|_j \leq C(u, q, \beta) \left( \Delta t^{-1} h^{\min[1 + k, l]} + \Delta t \right) + C(q) H^{l+1} + C(u) H^{k+1-j}, \quad j = 0, 1.
\] (76)

**Proof.** We can use a similar proof as in Theorem 5 to get the conclusion of Theorem 8, so we do not discuss that again.

\[\square\]

5. Some Numerical Results

Here, in order to show the numerical performances on the rate of convergence and a priori error estimates, we now choose \( \kappa = 1/2 \) and \( \beta = 0 \), the exact solution \( u(x,t) = t^{2\alpha} \sin(2\pi x) \), for all \( (x,t) \in [0,1] \times [0,1] \), and the determined source term \( f(x,t) \) by the exact solution in (5) and then calculate some numerical results by using Matlab procedure. In the numerical calculation, the time direction is approximated by finite difference schemes and the spatial direction is discretized by the \( H^1 \)-GMFE method. Now, we divide interval \([0,1]\) into equal-length intervals \( e_i = [x_i, x_{i+1}] \), \( 0 \leq i \leq N - 1 \). Now we choose the piecewise linear spaces \( V_h = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{N-1}\} \) and \( W_h = \text{span}\{\psi_0, \psi_1, \ldots, \psi_N\} \) with index \( k = r = 1 \).

Let
\[
u^n_h = \sum_{k=1}^{N-1} \nu_k \varphi_k, \quad \sigma^n_h = \sum_{k=0}^{N} \sigma_k \psi_k.
\] (77)

Based on the above expressions (77), we can get the following equivalent algebraic equation of (25):
\[
[(a + b) E + F] \ddot{\omega}^n = a \sum_{k=1}^{n-1} \left( -B_{n-k+1}^{2\alpha} + 2B_{n-k}^{2\alpha} - B_{n-k-1}^{2\alpha} \right) E \ddot{\omega}^k - b \sum_{k=1}^{n} (-B_{n-k+1}^{\alpha} + B_{n-k}^{\alpha}) E \ddot{\omega}^k
\]
\[+ b \left( 2B_0^{2\alpha} - B_1^{2\alpha} \right) E \ddot{\omega}^{n-1} + b \sum_{k=1}^{n} \left( -B_{n-k+1}^{\alpha} + B_{n-k}^{\alpha} \right) E \ddot{\omega}^{k-1} - ab_1^{2\alpha} E \ddot{\omega} - f^n,
\]
\[G \ddot{\omega}^n = H \ddot{\omega}^n,
\] (78)
where \( a = \Delta t^{-2\alpha}/\Gamma(3 - 2\alpha), b = \Delta t^{-2\alpha}/\Gamma(2 - \alpha) \), \( \ddot{\omega}^n = (\ddot{u}_1, \ddot{u}_2, \ldots, \ddot{u}_{N-1})^T, \ddot{\omega}^{n-1} = (\sigma_0, \sigma_1, \ldots, \sigma_{N-1})^T, f^n = (f^n, \psi_0, \ldots, (f^n, \psi_N))^T \),
\[
E = \begin{pmatrix}
h & h & \frac{h}{6} & \ldots & \frac{h}{6} \\
h & 2h & h & \ldots & \frac{h}{6} \\
\frac{h}{6} & \frac{h}{3} & \frac{h}{3} & \ldots & \frac{h}{6} \\
\frac{h}{6} & \frac{h}{3} & \frac{h}{3} & \ldots & \frac{h}{6} \\
\frac{h}{6} & \frac{h}{3} & \frac{h}{3} & \ldots & \frac{h}{6}
\end{pmatrix}_{(n+1) \times (n+1)},
\]
\[
F = \begin{pmatrix}
\frac{1}{h} & -\frac{1}{h} & \ldots & \ldots & \ldots \\
\frac{1}{h} & -\frac{1}{h} & \ldots & \ldots & \ldots \\
\frac{1}{h} & -\frac{1}{h} & \ldots & \ldots & \ldots \\
\frac{1}{h} & -\frac{1}{h} & \ldots & \ldots & \ldots \\
\frac{1}{h} & -\frac{1}{h} & \ldots & \ldots & \ldots
\end{pmatrix}_{(n+1) \times (n+1)}.
\]
Table 1: Spatial convergence results in $L^2$-norm with fixed $\Delta t = 1/4000$ and changed $\alpha$.

<table>
<thead>
<tr>
<th>$L^2$-norm</th>
<th>$\alpha$</th>
<th>$h_1 = 1/20$</th>
<th>$h_2 = 1/40$</th>
<th>$h_3 = 1/80$</th>
<th>Order $\frac{h_1}{h_2}$</th>
<th>Order $\frac{h_2}{h_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u - u_h|$</td>
<td>0.6</td>
<td>5.163644E - 03</td>
<td>1.28056E - 03</td>
<td>3.129308E - 04</td>
<td>2.013469</td>
<td>2.039022</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>5.051593E - 03</td>
<td>1.251162E - 03</td>
<td>3.044436E - 04</td>
<td>2.013469</td>
<td>2.039022</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>4.916797E - 03</td>
<td>1.215362E - 03</td>
<td>2.936933E - 04</td>
<td>2.016333</td>
<td>2.049004</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>4.758728E - 03</td>
<td>1.17217E - 03</td>
<td>2.795694E - 04</td>
<td>2.021459</td>
<td>2.067839</td>
</tr>
<tr>
<td>$|\sigma - \sigma_h|$</td>
<td>0.6</td>
<td>4.396500E - 03</td>
<td>1.122373E - 03</td>
<td>3.219747E - 04</td>
<td>1.969804</td>
<td>1.801533</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>5.14035E - 03</td>
<td>1.31204E - 03</td>
<td>3.759918E - 04</td>
<td>1.970247</td>
<td>1.803038</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>6.036700E - 03</td>
<td>1.543010E - 03</td>
<td>4.441222E - 04</td>
<td>1.968013</td>
<td>1.795777</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>7.087004E - 03</td>
<td>1.82201E - 03</td>
<td>5.343041E - 04</td>
<td>1.959645</td>
<td>1.769798</td>
</tr>
</tbody>
</table>

Table 2: Spatial convergence results in $H^1$-norm with fixed $\Delta t = 1/4000$ and changed $\alpha$.

<table>
<thead>
<tr>
<th>$H^1$-norm</th>
<th>$\alpha$</th>
<th>$h_1 = 1/20$</th>
<th>$h_2 = 1/40$</th>
<th>$h_3 = 1/80$</th>
<th>Order $\frac{h_1}{h_2}$</th>
<th>Order $\frac{h_2}{h_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u - u_h|$</td>
<td>0.6</td>
<td>6.93346E - 01</td>
<td>3.484812E - 01</td>
<td>1.744206E - 01</td>
<td>0.992495</td>
<td>0.99810</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>6.933489E - 01</td>
<td>3.484816E - 01</td>
<td>1.744207E - 01</td>
<td>0.992499</td>
<td>0.99811</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>6.933533E - 01</td>
<td>3.484823E - 01</td>
<td>1.744208E - 01</td>
<td>0.992505</td>
<td>0.99813</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>6.933591E - 01</td>
<td>3.484833E - 01</td>
<td>1.744210E - 01</td>
<td>0.992513</td>
<td>0.99816</td>
</tr>
<tr>
<td>$|\sigma - \sigma_h|$</td>
<td>0.6</td>
<td>4.374796E + 00</td>
<td>2.191164E + 00</td>
<td>1.096078E + 00</td>
<td>0.997386</td>
<td>0.999347</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>4.374658E + 00</td>
<td>2.191198E + 00</td>
<td>1.096083E + 00</td>
<td>0.997450</td>
<td>0.999363</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>4.374979E + 00</td>
<td>2.191241E + 00</td>
<td>1.096089E + 00</td>
<td>0.997528</td>
<td>0.999383</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>4.375365E + 00</td>
<td>2.191294E + 00</td>
<td>1.096098E + 00</td>
<td>0.997620</td>
<td>0.999406</td>
</tr>
</tbody>
</table>

Based on the algebraic equation (78), we calculate and get the detailed numerical results listed in Tables 1–4.

In Table 1, we calculate the numerical results of a priori error results and orders of convergence in $L^2$-norm for both $u$ and $\sigma$ with different space-time step length $h = 5\Delta t = 1/M (M = 20, 40, 80)$. From the calculated data, we can find that the rates of convergence, which are higher than the results $O(\Delta t^{3\alpha - 2\alpha})$ of theory, gradually decrease with the increased $\alpha$ (which is taken from 0.6 to 0.9 with interval 0.1). In Table 4, some first-order convergence results for both $u$ and $\sigma$ in $H^1$-norm, which are unchanged with the changed $\alpha = 0.6, 0.7, 0.8, 0.9$, are given.

In view of the above analysis on the numerical results, we now announce that the time fractional telegraph equation can be well solved by the $H^1$-GMFE method.

6. Some Concluding Remarks and Extensions

As far as we know, more and more people have proposed and analyzed a lot of numerical methods for fractional partial differential equations. However, the discussions on mixed finite element methods for solving fractional partial differential equations are fairly limited. The a priori error analysis of $H^1$-GMFE method for time fractional telegraph equation, especially, has not been made and discussed. In this paper, we give the detailed proof’s process of the error analysis on $H^1$-GMFE method for time fractional telegraph equation. Further, we provide a numerical procedure to verify the theoretical results of the studied method.
Table 3: Space-time convergence results in $L^2$-norm with $h = 5\Delta t = 1/M$ and changed $\alpha$.

<table>
<thead>
<tr>
<th>$L^2$-norm</th>
<th>$\alpha$</th>
<th>$M_1 = 20$</th>
<th>$M_2 = 40$</th>
<th>$M_3 = 80$</th>
<th>Order ($M_1/M_2$)</th>
<th>Order ($M_2/M_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u - u_h|$</td>
<td>0.6</td>
<td>4.781196E - 03</td>
<td>1.095478E - 03</td>
<td>2.262375E - 04</td>
<td>2.125881</td>
<td>2.275651</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>4.587916E - 03</td>
<td>1.028885E - 03</td>
<td>2.011905E - 04</td>
<td>2.156757</td>
<td>2.354448</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>4.395990E - 03</td>
<td>9.315674E - 04</td>
<td>1.630376E - 04</td>
<td>2.213162</td>
<td>2.514455</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>3.941100E - 03</td>
<td>7.841911E - 04</td>
<td>2.154685E - 04</td>
<td>2.329321</td>
<td>1.863729</td>
</tr>
<tr>
<td>$|\sigma - \sigma_h|$</td>
<td>0.6</td>
<td>6.937717E - 03</td>
<td>2.316455E - 03</td>
<td>8.737355E - 04</td>
<td>1.582542</td>
<td>1.406650</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>8.229880E - 03</td>
<td>2.746089E - 03</td>
<td>1.031348E - 03</td>
<td>1.582108</td>
<td>1.410332</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>1.000490E - 02</td>
<td>3.373945E - 03</td>
<td>1.275972E - 03</td>
<td>1.568199</td>
<td>1.402840</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>1.251982E - 02</td>
<td>4.324761E - 03</td>
<td>1.669151E - 03</td>
<td>1.533521</td>
<td>1.373505</td>
</tr>
</tbody>
</table>

Table 4: Space-time convergence results in $H^1$-norm with $h = 5\Delta t = 1/M$ and changed $\alpha$.

<table>
<thead>
<tr>
<th>$H^1$-norm</th>
<th>$\alpha$</th>
<th>$M_1 = 20$</th>
<th>$M_2 = 40$</th>
<th>$M_3 = 80$</th>
<th>Order ($M_1/M_2$)</th>
<th>Order ($M_2/M_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u - u_h|_1$</td>
<td>0.6</td>
<td>6.933582E - 01</td>
<td>3.484856E - 01</td>
<td>1.744222E - 01</td>
<td>0.992502</td>
<td>0.998515</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>6.933669E - 01</td>
<td>3.484880E - 01</td>
<td>1.744230E - 01</td>
<td>0.992510</td>
<td>0.998519</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>6.933819E - 01</td>
<td>3.484924E - 01</td>
<td>1.744244E - 01</td>
<td>0.992523</td>
<td>0.998525</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>6.943089E - 01</td>
<td>3.485010E - 01</td>
<td>1.744275E - 01</td>
<td>0.992543</td>
<td>0.998536</td>
</tr>
<tr>
<td>$|\sigma - \sigma_h|_1$</td>
<td>0.6</td>
<td>4.375309E + 00</td>
<td>2.191391E + 00</td>
<td>1.096134E + 00</td>
<td>0.997538</td>
<td>0.999422</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>4.375792E + 00</td>
<td>2.191479E + 00</td>
<td>1.096153E + 00</td>
<td>0.997639</td>
<td>0.999456</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>4.376484E + 00</td>
<td>2.191614E + 00</td>
<td>1.096183E + 00</td>
<td>0.997779</td>
<td>0.999505</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>4.377505E + 00</td>
<td>2.191830E + 00</td>
<td>1.096235E + 00</td>
<td>0.997973</td>
<td>0.999579</td>
</tr>
</tbody>
</table>

In the near future, our aim is to study an $H^1$-Galerkin moving mixed finite element method, which is based on a combination of $H^1$-GMFE methods and moving finite element methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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