Research Article

Strong Convergence Algorithm for Split Equilibrium Problems and Hierarchical Fixed Point Problems

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The purpose of this paper is to investigate the problem of finding the approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed point problem in a real Hilbert space. We establish the strong convergence of the proposed method under some mild conditions. Several special cases are also discussed. Our main result extends and improves some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed convex subset of H . We introduce the following definitions which are useful in the following analysis.

Definition 1. The mapping $T: C \rightarrow H$ is said to be

(a) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C; \quad (1)$$

(b) strongly monotone, if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C; \quad (2)$$

(c) α -inverse strongly monotone, if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C; \quad (3)$$

(d) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C; \quad (4)$$

(e) k -Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k \|x - y\|, \quad \forall x, y \in C; \quad (5)$$

(f) contraction on C , if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k \|x - y\|, \quad \forall x, y \in C. \quad (6)$$

It is easy to observe that every α -inverse strongly monotone T is monotone and Lipschitz continuous. It is well known that every nonexpansive operator $T: H \rightarrow H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\begin{aligned} & \langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \\ & \leq \frac{1}{2} \|(T(x) - x) - (T(y) - y)\|^2 \end{aligned} \quad (7)$$

and therefore, we get, for all $(x, y) \in H \times \text{Fix}(T)$,

$$\langle x - T(x), y - T(x) \rangle \leq \frac{1}{2} \|T(x) - x\|^2. \quad (8)$$

See, for example, [1, Theorem 1], and [2, Theorem 3].

The fixed point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \quad (9)$$

We denote by $F(T)$ the set of solutions of (9). It is well known that $F(T)$ is closed and convex and $P_F(T)$ is well defined (see [3]).

The equilibrium problem denoted by EP is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \quad (10)$$

The solution set of (10) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to finding a solution of (10); see [4–7]. In 1997, Combettes and Hirstoaga [8] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is non-empty. In 2007, Plubtieng and Punpaeng [6] introduced an iterative method for finding the common element of the set $F(T) \cap EP(F)$.

Recently, Censor et al. [9] introduced a new variational inequality problem which we call the split variational inequality problem (SVIP). Let H_1 and H_2 be two real Hilbert spaces. Given operators $f: H_1 \rightarrow H_1$ and $g: H_2 \rightarrow H_2$, a bounded linear operator $A: H_1 \rightarrow H_2$, and nonempty, closed, and convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$, the SVIP is formulated as follows: find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C \quad (11)$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Q. \quad (12)$$

In [10], Moudafi introduced an iterative method which can be regarded as an extension of the method given by Censor et al. [9] for the following split monotone variational inclusions:

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*) \quad (13)$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \quad (14)$$

where $B_i: H_i \rightarrow 2^{H_i}$ is a set-valued mapping for $i = 1, 2$. Later Byrne et al. [11] generalized and extended the work of Censor et al. [9] and Moudafi [10].

Very recently, Kazmi and Rizvi [12] studied the following pair of equilibrium problems called split equilibrium problem: let $F_1: C \times C \rightarrow R$ and $F_2: Q \times Q \rightarrow R$ be nonlinear bifunctions and let $A: H_1 \rightarrow H_2$ be a bounded linear operator; then, the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \quad (15)$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \quad \forall y \in Q. \quad (16)$$

The solution set of SEP (15)-(16) is denoted by $\Lambda = \{p \in EP(F_1): Ap \in EP(F_2)\}$.

Let $S: C \rightarrow H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \quad (17)$$

It is known that the hierarchical fixed point problem (17) links with some monotone variational inequalities and convex programming problems; see [13, 14]. Various methods [15–20] have been proposed to solve the hierarchical fixed point problem. In 2010, Yao et al. [14] introduced the following strong convergence iterative algorithm to solve the problem (17):

$$y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \quad (18)$$

$$x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n], \quad \forall n \geq 0,$$

where $f: C \rightarrow H$ is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on parameters, Yao et al. proved that the sequence $\{x_n\}$ generated by (18) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f)z, y - z \rangle \geq 0, \quad \forall y \in F(T). \quad (19)$$

In 2011, Ceng et al. [21] investigated the following iterative method:

$$x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0, \quad (20)$$

where U is a Lipschitzian mapping and F is a Lipschitzian and strongly monotone mapping. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (20) converges strongly to the unique solution of the variational inequality

$$\langle \rho U(z) - \mu F(z), x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(T). \quad (21)$$

In the present paper, inspired by the above cited works and by the recent works going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (15)-(16) and (17) in real Hilbert space. Strong convergence of the iterative algorithm is obtained in the framework of Hilbert space. We would like to mention that our proposed method is quite general and flexible and includes many known results for solving split equilibrium problems and hierarchical fixed point problems; see, for example, [13, 14, 17–19, 21–23] and relevant references cited therein.

2. Preliminaries

In this section, we recall some basic definitions and properties, which will be frequently used in our later analysis. Some useful results proved already in the literature are also summarized. The first lemma provides some basic properties of projection onto C .

Lemma 2. Let P_C denote the projection of H onto C . Then, one has the following inequalities:

$$\langle z - P_C[z], P_C[z] - v \rangle \geq 0, \quad \forall z \in H, v \in C;$$

$$\langle u - v, P_C[u] - P_C[v] \rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H;$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H;$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H, u \in C. \quad (22)$$

Assumption 3 (see [24]). Let $F: C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying the following assumptions:

- (i) $F(x, x) = 0$, for all $x \in C$;
- (ii) F is monotone; that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$;
- (iii) for each $x, y, z \in C$, $\lim_{t \rightarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (iv) for each $x \in C$, $y \rightarrow F(x, y)$ is convex and lower semi-continuous;
- (v) for fixed $r > 0$ and $z \in C$, there exists a bounded subset K of H_1 and $x \in C \cap K$ such that

$$F(y, x) + \frac{1}{r} \langle y - x, x - z \rangle \geq 0, \quad \forall y \in C \setminus K. \quad (23)$$

Lemma 4 (see [8]). Assume that $F_1: C \times C \rightarrow \mathbb{R}$ satisfies Assumption 3. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{F_1}: H_1 \rightarrow C$ as follows:

$$T_r^{F_1}(x) = \left\{ z \in C: F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \right. \\ \left. \forall y \in C \right\}. \quad (24)$$

Then the following hold:

- (i) $T_r^{F_1}$ is nonempty and single-valued;
- (ii) $T_r^{F_1}$ is firmly nonexpansive; that is,

$$\|T_r^{F_1}(x) - T_r^{F_1}(y)\|^2 \leq \langle T_r^{F_1}(x) - T_r^{F_1}(y), x - y \rangle, \quad (25) \\ \forall x, y \in H_1;$$

- (iii) $F(T_r^{F_1}) = EP(F_1)$;
- (iv) $EP(F_1)$ is closed and convex.

Assume that $F_2: Q \times Q \rightarrow \mathbb{R}$ satisfies Assumption 3. For $s > 0$ and for all $u \in H_2$, define a mapping $T_s^{F_2}: H_2 \rightarrow Q$ as follows:

$$T_s^{F_2}(u) = \left\{ v \in Q: F_2(v, w) + \frac{1}{s} \langle w - v, v - u \rangle \right. \\ \left. \geq 0, \forall w \in Q \right\}. \quad (26)$$

Then $T_s^{F_2}$ satisfies conditions (i)–(iv) of Lemma 4. Consider $F(T_s^{F_2}) = EP(F_2, Q)$, where $EP(F_2, Q)$ is the solution set of the following equilibrium problem:

$$\text{find } y^* \in Q \text{ such that } F_2(y^*, y) \geq 0, \quad \forall y \in Q. \quad (27)$$

Lemma 5 (see [25]). Assume that $F_1: C \times C \rightarrow \mathbb{R}$ satisfies Assumption 3, and let $T_r^{F_1}$ be defined as in Lemma 4. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then

$$\|T_{r_2}^{F_1}(y) - T_{r_1}^{F_1}(x)\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \|T_{r_2}^{F_1}(y) - y\|. \quad (28)$$

Lemma 6 (see [26]). Let C be a nonempty closed convex subset of a real Hilbert space H . If $T: C \rightarrow C$ is a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, then the mapping $I - T$ is demiclosed at 0; that is, if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.

Lemma 7 (see [21]). Let $U: C \rightarrow H$ be τ -Lipschitzian mapping and let $F: C \rightarrow H$ be a k -Lipschitzian and η -strongly monotone mapping; then for $0 \leq \rho\tau < \mu\eta$, $\mu F - \rho U$ is $\mu\eta - \rho\tau$ -strongly monotone; that is,

$$\langle (\mu F - \rho U)x - (\mu F - \rho U)y, x - y \rangle \geq (\mu\eta - \rho\tau) \|x - y\|^2, \\ \forall x, y \in C. \quad (29)$$

Lemma 8 (see [27]). Suppose that $\lambda \in (0, 1)$ and $\mu > 0$. Let $F: C \rightarrow H$ be an k -Lipschitzian and η -strongly monotone operator. In association with nonexpansive mapping $T: C \rightarrow C$, define the mapping $T^\lambda: C \rightarrow H$ by

$$T^\lambda x = Tx - \lambda \mu FT(x), \quad \forall x \in C. \quad (30)$$

Then T^λ is a contraction provided that $\mu < (2\eta/k^2)$; that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\nu) \|x - y\|, \quad \forall x, y \in C, \quad (31)$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$.

Lemma 9 (see [28]). Assume that $\{a_n\}$ is a sequence of non-negative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad (32)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and δ_n is a sequence such that

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 10 (see [29]). Let C be a closed convex subset of H . Let $\{x_n\}$ be a bounded sequence in H . Assume that

- (i) the weak w -limit set $w_w(x_n) \subset C$ where $w_w(x_n) = \{x: x_{n_i} \rightharpoonup x\}$;
- (ii) for each $z \in C$, $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists.

Then $\{x_n\}$ is weakly convergent to a point in C .

3. The Proposed Method and Some Properties

In this section, we suggest and analyze our method and we prove a strong convergence theorem for finding the common solutions of the split equilibrium problem (15)–(16) and the hierarchical fixed point problem (17).

Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2 , respectively. Let $A: H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $F_1: C \times C \rightarrow \mathbb{R}$

and $F_2: Q \times Q \rightarrow \mathbb{R}$ are the bifunctions satisfying Assumption 3 and F_2 is upper semicontinuous in first argument. Let $S, T: C \rightarrow C$ be a nonexpansive mapping such that $\Lambda \cap F(T) \neq \emptyset$. Let $F: C \rightarrow C$ be an k -Lipschitzian mapping and η -strongly monotone and let $U: C \rightarrow C$ be τ -Lipschitzian mapping. Now we introduce the proposed method as follows.

Algorithm 11. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ be generated by

$$\begin{aligned} u_n &= T_{r_n}^{F_1} (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n); \\ y_n &= \beta_n S x_n + (1 - \beta_n) u_n; \\ x_{n+1} &= P_C [\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0, \end{aligned} \quad (33)$$

where $\{r_n\} \subset (0, 2\zeta)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A , and A^* is the adjoint of A . Suppose that the parameters satisfy $0 < \mu < (2\eta/k^2)$, $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. And $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (b) $\lim_{n \rightarrow \infty} (\beta_n / \alpha_n) = 0$;
- (c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$;
- (d) $\liminf_{n \rightarrow \infty} r_n < \limsup_{n \rightarrow \infty} r_n < 2\zeta$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Remark 12. Our method can be viewed as extension and improvement for some well-known results as follows

- (i) The proposed method is an extension and improvement of the method of Wang and Xu [23] for finding the approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.
- (ii) If the Lipschitzian mapping $U = f$, $F = I$, $\rho = \mu = 1$, we obtain an extension and improvement of the method of Yao et al. [14] for finding the approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.
- (iii) The contractive mapping f with a coefficient $\alpha \in [0, 1)$ in other papers (see [14, 19, 22, 27]) is extended to the cases of the Lipschitzian mapping U with a coefficient constant $\gamma \in [0, \infty)$.

This shows that Algorithm 11 is quite general and unifying.

Lemma 13. Let $x^* \in \Lambda \cap F(T)$. Then $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ are bounded.

Proof. Let $x^* \in \Lambda \cap F(T)$; we have $x^* = T_{r_n}^{F_1}(x^*)$ and $Ax^* = T_{r_n}^{F_2}(Ax^*)$. Then

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}^{F_1} (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) - x^*\|^2 \\ &= \|T_{r_n}^{F_1} (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) - T_{r_n}^{F_1}(x^*)\|^2 \\ &\leq \|x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 + \gamma^2 \|A^* (T_{r_n}^{F_2} - I) A x_n\|^2 \\ &\quad + 2\gamma \langle x_n - x^*, A^* (T_{r_n}^{F_2} - I) A x_n \rangle \\ &= \|x_n - x^*\|^2 \\ &\quad + \gamma^2 \langle (T_{r_n}^{F_2} - I) A x_n, A A^* (T_{r_n}^{F_2} - I) A x_n \rangle \\ &\quad + 2\gamma \langle x_n - x^*, A^* (T_{r_n}^{F_2} - I) A x_n \rangle. \end{aligned} \quad (34)$$

From the definition of L , it follows that

$$\begin{aligned} &\gamma^2 \langle (T_{r_n}^{F_2} - I) A x_n, A A^* (T_{r_n}^{F_2} - I) A x_n \rangle \\ &\leq L\gamma^2 \langle (T_{r_n}^{F_2} - I) A x_n, (T_{r_n}^{F_2} - I) A x_n \rangle \\ &= L\gamma^2 \|(T_{r_n}^{F_2} - I) A x_n\|^2. \end{aligned} \quad (35)$$

It follows from (8) that

$$\begin{aligned} &2\gamma \langle x_n - x^*, A^* (T_{r_n}^{F_2} - I) A x_n \rangle \\ &= 2\gamma \langle A(x_n - x^*), (T_{r_n}^{F_2} - I) A x_n \rangle \\ &= 2\gamma \langle A(x_n - x^*) + (T_{r_n}^{F_2} - I) A x_n \\ &\quad - (T_{r_n}^{F_2} - I) A x_n, (T_{r_n}^{F_2} - I) A x_n \rangle \\ &= 2\gamma \langle (T_{r_n}^{F_2} A x_n - A x^*, (T_{r_n}^{F_2} - I) A x_n \rangle \\ &\quad - \|(T_{r_n}^{F_2} - I) A x_n\|^2 \\ &\leq 2\gamma \left(\frac{1}{2} \|(T_{r_n}^{F_2} - I) A x_n\|^2 - \|(T_{r_n}^{F_2} - I) A x_n\|^2 \right) \\ &= -\gamma \|(T_{r_n}^{F_2} - I) A x_n\|^2. \end{aligned} \quad (36)$$

Applying (36) and (35) to (34) and from the definition of γ , we get

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + \gamma(L\gamma - 1) \|(T_{r_n}^{F_2} - I) A x_n\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \quad (37)$$

Denote $V_n = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))$. Next, we prove that the sequence $\{x_n\}$ is bounded; without loss of generality we can assume that $\beta_n \leq \alpha_n$ for all $n \geq 1$. From (33), we have

$$\begin{aligned}
 & \|x_{n+1} - x^*\| \\
 &= \|P_C[V_n] - P_C[x^*]\| \\
 &\leq \|\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n)) - x^*\| \\
 &\leq \alpha_n \|\rho U(x_n) - \mu F(x^*)\| \\
 &\quad + \|(I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)T(x^*)\| \\
 &= \alpha_n \|\rho U(x_n) - \rho U(x^*) + (\rho U - \mu F)(x^*)\| \\
 &\quad + \|(I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)T(x^*)\| \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)(x^*)\| \\
 &\quad + (1 - \alpha_n \nu) \|y_n - x^*\| \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)(x^*)\| \\
 &\quad + (1 - \alpha_n \nu) \|\beta_n Sx_n + (1 - \beta_n)u_n - x^*\| \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)(x^*)\| \\
 &\quad + (1 - \alpha_n \nu) (\beta_n \|Sx_n - Sx^*\| + \beta_n \|Sx^* - x^*\| \\
 &\quad \quad + (1 - \beta_n) \|u_n - x^*\|) \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)(x^*)\| \\
 &\quad + (1 - \alpha_n \nu) (\beta_n \|Sx_n - Sx^*\| + \beta_n \|Sx^* - x^*\| \\
 &\quad \quad + (1 - \beta_n) \|x_n - x^*\|) \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)(x^*)\| \\
 &\quad + (1 - \alpha_n \nu) (\beta_n \|x_n - x^*\| + \beta_n \|Sx^* - x^*\| \\
 &\quad \quad + (1 - \beta_n) \|x_n - x^*\|) \\
 &= (1 - \alpha_n (\nu - \rho \tau)) \|x_n - x^*\| \\
 &\quad + \alpha_n \|(\rho U - \mu F)(x^*)\| \\
 &\quad + (1 - \alpha_n \nu) \beta_n \|Sx^* - x^*\| \\
 &\leq (1 - \alpha_n (\nu - \rho \tau)) \|x_n - x^*\| \\
 &\quad + \alpha_n \|(\rho U - \mu F)(x^*)\| + \beta_n \|Sx^* - x^*\| \\
 &\leq (1 - \alpha_n (\nu - \rho \tau)) \|x_n - x^*\| \\
 &\quad + \alpha_n (\|(\rho U - \mu F)(x^*)\| + \|Sx^* - x^*\|) \\
 &= (1 - \alpha_n (\nu - \rho \tau)) \|x_n - x^*\| \\
 &\quad + \frac{\alpha_n (\nu - \rho \tau)}{\nu - \rho \tau} (\|(\rho U - \mu F)(x^*)\| + \|Sx^* - x^*\|) \\
 &\leq \max \left\{ \|x_n - x^*\|, \frac{1}{\nu - \rho \tau} \right. \\
 &\quad \left. \times (\|(\rho U - \mu F)(x^*)\| + \|Sx^* - x^*\|) \right\}, \tag{38}
 \end{aligned}$$

where the third inequality follows from Lemma 8.

By induction on n , we obtain $\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, (1/(1 - \rho))(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\}$, for $n \geq 0$ and $x_0 \in C$. Hence $\{x_n\}$ is bounded and, consequently, we deduce that $\{u_n\}$, $\{y_n\}$, $\{S(x_n)\}$, $\{T(x_n)\}$, $\{F(T(y_n))\}$, and $\{U(x_n)\}$ are bounded. $\square$

Lemma 14. Let $x^* \in \Lambda \cap F(T)$ and $\{x_n\}$ the sequence generated by the Algorithm 11. Then one has

- (a) $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$;
 (b) the weak w -limit set $w_w(x_n) \subset F(T)$, $(w_w(x_n)) = \{x : x_{n_i} \rightharpoonup x\}$.

Proof. Since $u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)$ and $u_{n-1} = T_{r_{n-1}}^{F_1}(x_{n-1} + \gamma A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1})$ it follows from Lemma 5 that

$$\begin{aligned}
 & \|u_n - u_{n-1}\| \\
 &\leq \|x_n - x_{n-1}\| \\
 &\quad + \gamma \|A^*(T_{r_n}^{F_2} - I)Ax_n - A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1}\| \\
 &\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) \\
 &\quad \quad - (x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
 &\leq \|x_n - x_{n-1} - \gamma A^*(x_n - x_{n-1})\| \\
 &\quad + \gamma \|A\| \|T_{r_n}^{F_2}Ax_n - T_{r_{n-1}}^{F_2}Ax_{n-1}\| \\
 &\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) \\
 &\quad \quad - (x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
 &\leq (\|x_n - x_{n-1}\|^2 - 2\gamma \|A\| \|x_n - x_{n-1}\|)^2 \\
 &\quad + \gamma^2 \|A\|^4 \|x_n - x_{n-1}\|^2)^{1/2} \\
 &\quad + \gamma \|A\| \left(\|A(x_n - x_{n-1})\| \right. \\
 &\quad \quad \left. + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_2}Ax_n - Ax_n\| \right) \\
 &\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) \\
 &\quad \quad - (x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
 &\leq (1 - 2\gamma \|A\|^2 + \gamma^2 \|A\|^4)^{1/2} \|x_n - x_{n-1}\| \\
 &\quad + \gamma \|A\|^2 \|x_n - x_{n-1}\| \\
 &\quad + \gamma \|A\| \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_2}Ax_n - Ax_n\|
 \end{aligned}$$

$$\begin{aligned}
& + \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) \right. \\
& \quad \left. - (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) \right\| \\
& = (1 - \gamma \|A\|^2) \|x_n - x_{n-1}\| \\
& \quad + \gamma \|A\|^2 \|x_n - x_{n-1}\| \\
& \quad + \gamma \|A\| \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_2} A x_n - A x_n \right\| \\
& \quad + \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) \right. \\
& \quad \left. - (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) \right\| \\
& = \|x_n - x_{n-1}\| + \gamma \|A\| \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_2} A x_n - A x_n \right\| \\
& \quad + \left| 1 - \frac{r_{n-1}}{r_n} \right| \left\| T_{r_n}^{F_1} (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) \right. \\
& \quad \left. - (x_n + \gamma A^* (T_{r_n}^{F_2} - I) A x_n) \right\| \\
& = \|x_n - x_{n-1}\| + \left| \frac{r_n - r_{n-1}}{r_n} \right| (\gamma \|A\| \sigma_n + \chi_n), \tag{39}
\end{aligned}$$

where $\sigma_n := \|T_{r_n}^{F_2} A x_n - A x_n\|$ and $\chi_n := \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)A x_n) - (x_n + \gamma A^*(T_{r_n}^{F_2} - I)A x_n)\|$. Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$, for all positive integers n . Then we get

$$\|u_{n-1} - u_n\| \leq \|x_{n-1} - x_n\| + \frac{1}{\mu} |r_{n-1} - r_n| (\gamma \|A\| \sigma_n + \chi_n). \tag{40}$$

From (33) and the above inequality, we get

$$\begin{aligned}
& \|y_n - y_{n-1}\| \\
& = \|\beta_n S x_n + (1 - \beta_n) u_n \\
& \quad - (\beta_{n-1} S x_{n-1} + (1 - \beta_{n-1}) u_{n-1})\| \\
& = \|\beta_n (S x_n - S x_{n-1}) + (\beta_n - \beta_{n-1}) S x_{n-1} \\
& \quad + (1 - \beta_n) (u_n - u_{n-1}) + (\beta_{n-1} - \beta_n) u_{n-1}\| \\
& \leq \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \|u_n - u_{n-1}\| \\
& \quad + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|u_{n-1}\|) \\
& \leq \beta_n \|x_n - x_{n-1}\| + (1 - \beta_n) \\
& \quad \times \left\{ \|x_{n-1} - x_n\| + \frac{1}{\mu} |r_{n-1} - r_n| (\gamma \|A\| \sigma_n + \chi_n) \right\} \\
& \quad + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|u_{n-1}\|) \\
& \leq \|x_n - x_{n-1}\| + \frac{1}{\mu} |r_{n-1} - r_n| (\gamma \|A\| \sigma_n + \chi_n) \\
& \quad + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|u_{n-1}\|). \tag{41}
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
& = \|P_C[V_n] - P_C[V_{n-1}]\| \\
& \leq \|\alpha_n \rho (U(x_n) - U(x_{n-1})) + (\alpha_n - \alpha_{n-1}) \rho U(x_{n-1}) \\
& \quad + (I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_{n-1} \mu F) T(y_{n-1}) \\
& \quad + (I - \alpha_n \mu F)(T(y_{n-1})) - (I - \alpha_{n-1} \mu F)(T(y_{n-1}))\| \\
& \leq \alpha_n \rho \tau \|x_n - x_{n-1}\| + (1 - \alpha_n \nu) \|y_n - y_{n-1}\| \\
& \quad + |\alpha_n - \alpha_{n-1}| (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|), \tag{42}
\end{aligned}$$

where the second inequality follows from Lemma 8. From (41) and (42), we have

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
& \leq \alpha_n \rho \tau \|x_n - x_{n-1}\| + (1 - \alpha_n \nu) \\
& \quad \times \left\{ \|x_n - x_{n-1}\| + \frac{1}{\mu} |r_{n-1} - r_n| (\gamma \|A\| \sigma_n + \chi_n) \right. \\
& \quad \left. + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|u_{n-1}\|) \right\} \\
& \quad + |\alpha_n - \alpha_{n-1}| (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|) \\
& \leq (1 - (\nu - \rho \tau) \alpha_n) \|x_n - x_{n-1}\| \\
& \quad + \frac{1}{\mu} |r_{n-1} - r_n| (\gamma \|A\| \sigma_n + \chi_n) \\
& \quad + |\beta_n - \beta_{n-1}| (\|S x_{n-1}\| + \|u_{n-1}\|) \\
& \quad + |\alpha_n - \alpha_{n-1}| (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|) \\
& \leq (1 - (\nu - \rho \tau) \alpha_n) \|x_n - x_{n-1}\| \\
& \quad + M \left(\frac{1}{\mu} |r_{n-1} - r_n| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| \right), \tag{43}
\end{aligned}$$

where

$$\begin{aligned}
M = \max \left\{ \sup_{n \geq 1} (\gamma \|A\| \sigma_n + \chi_n), \right. \\
\sup_{n \geq 1} (\|S x_{n-1}\| + \|u_{n-1}\|), \\
\left. \sup_{n \geq 1} (\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|) \right\}. \tag{44}
\end{aligned}$$

It follows from conditions (a)–(d) of Algorithm 11 and Lemma 9 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{45}$$

Next, we show that $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$. Since $x^* \in \Lambda \cap F(T)$ by using (34) and (37), we obtain

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 &= \langle P_C[V_n] - x^*, x_{n+1} - x^* \rangle \\
 &= \langle P_C[V_n] - V_n, P_C[V_n] - x^* \rangle + \langle V_n - x^*, x_{n+1} - x^* \rangle \\
 &\leq \langle \alpha_n(\rho U(x_n) - \mu F(x^*) + (I - \alpha_n \mu F)(T(y_n))) \\
 &\quad - (I - \alpha_n \mu F)(T(x^*)), x_{n+1} - x^* \rangle \\
 &= \langle \alpha_n \rho(U(x_n) - U(x^*)), x_{n+1} - x^* \rangle \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \langle (I - \alpha_n \mu F)(T(y_n)) \\
 &\quad - (I - \alpha_n \mu F)(T(x^*)), x_{n+1} - x^* \rangle \\
 &\leq \alpha_n \rho \tau \|x_n - x^*\| \|x_{n+1} - x^*\| \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + (1 - \alpha_n \gamma) \|y_n - x^*\| \|x_{n+1} - x^*\| \\
 &\leq \frac{\alpha_n \rho \tau}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \gamma)}{2} (\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
 &\leq \frac{(1 - \alpha_n(\gamma - \rho \tau))}{2} \|x_{n+1} - x^*\|^2 \\
 &\quad + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \gamma)}{2} (\beta_n \|Sx_n - x^*\|^2 + (1 - \beta_n) \|u_n - x^*\|^2) \\
 &\leq \frac{(1 - \alpha_n(\gamma - \rho \tau))}{2} \|x_{n+1} - x^*\|^2 \\
 &\quad + \frac{\alpha_n \rho \tau}{2} \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \gamma) \beta_n}{2} \|Sx_n - x^*\|^2 + \frac{(1 - \alpha_n \gamma)(1 - \beta_n)}{2} \\
 &\quad \times \left\{ \|x_n - x^*\|^2 + \gamma(L\gamma - 1) \|(T_{r_n}^{F_2} - I)Ax_n\|^2 \right\}, \quad (46)
 \end{aligned}$$

where the last inequality follows from (37), which implies that

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 &\leq \frac{\alpha_n \rho \tau}{1 + \alpha_n(\gamma - \rho \tau)} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n \gamma) \beta_n}{1 + \alpha_n(\gamma - \rho \tau)} \|Sx_n - x^*\|^2 + \frac{(1 - \alpha_n \gamma)(1 - \beta_n)}{1 + \alpha_n(\gamma - \rho \tau)} \\
 &\quad \times \left\{ \|x_n - x^*\|^2 + \gamma(L\gamma - 1) \|(T_{r_n}^{F_2} - I)Ax_n\|^2 \right\} \\
 &\leq \frac{\alpha_n \rho \tau}{1 + \alpha_n(\gamma - \rho \tau)} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho \tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \|x_n - x^*\|^2 + \frac{(1 - \alpha_n \gamma) \beta_n}{1 + \alpha_n(\gamma - \rho \tau)} \|Sx_n - x^*\|^2 \\
 &\quad - \frac{(1 - \alpha_n \gamma)(1 - \beta_n) \gamma(1 - L\gamma)}{1 + \alpha_n(\gamma - \rho \tau)} \|(T_{r_n}^{F_2} - I)Ax_n\|^2. \quad (47)
 \end{aligned}$$

Then from the above inequality, we get

$$\begin{aligned}
 & \frac{(1 - \alpha_n \gamma)(1 - \beta_n) \gamma(1 - L\gamma)}{1 + \alpha_n(\gamma - \rho \tau)} \|(T_{r_n}^{F_2} - I)Ax_n\|^2 \\
 &\leq \frac{\alpha_n \rho \tau}{1 + \alpha_n(\gamma - \rho \tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho \tau)} \\
 &\quad \times \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \beta_n \|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
 &\leq \frac{\alpha_n \rho \tau}{1 + \alpha_n(\gamma - \rho \tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho \tau)} \\
 &\quad \times \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \beta_n \|Sx_n - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
 &\quad \times \|x_{n+1} - x_n\|. \quad (48)
 \end{aligned}$$

Since $\gamma(1 - L\gamma) > 0$, $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, $\alpha_n \rightarrow 0$, and $\beta_n \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} \|(T_{r_n}^{F_2} - I)Ax_n\| = 0. \quad (49)$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
 & \|u_n - x^*\|^2 \\
 &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}(x^*)\|^2 \\
 &\leq \langle u_n - x^*, x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^* \rangle \\
 &= \frac{1}{2} \left\{ \|u_n - x^*\|^2 + \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\|^2 \right. \\
 &\quad \left. - \|u_n - x^* - [x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*]\|^2 \right\} \\
 &= \frac{1}{2} \left\{ \|u_n - x^*\|^2 + \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\|^2 \right. \\
 &\quad \left. - \|u_n - x_n - \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 \right\} \\
 &\leq \frac{1}{2} \left\{ \|u_n - x^*\|^2 + \|x_n - x^*\|^2 \right. \\
 &\quad \left. - \|u_n - x_n - \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 \right\} \\
 &= \frac{1}{2} \left\{ \|u_n - x^*\|^2 + \|x_n - x^*\|^2 \right. \\
 &\quad - [\|u_n - x_n\|^2 + \gamma^2 \|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 \\
 &\quad \left. - 2\gamma \langle u_n - x_n, A^*(T_{r_n}^{F_2} - I)Ax_n \rangle] \right\}, \quad (50)
 \end{aligned}$$

where the last inequality follows from (34) and (37). Hence, we get

$$\begin{aligned}
 \|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\
 &\quad + 2\gamma \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\|. \quad (51)
 \end{aligned}$$

From (46) and the above inequality, we have

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 &\leq \frac{(1 - \alpha_n(\gamma - \rho\tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2} \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n\gamma)}{2} (\beta_n \|Sx_n - x^*\|^2 + (1 - \beta_n) \|u_n - x^*\|^2) \\
 &\leq \frac{(1 - \alpha_n(\gamma - \rho\tau))}{2} \|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2} \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle + \frac{(1 - \alpha_n\gamma)}{2} \\
 &\quad \times \{ \beta_n \|Sx_n - x^*\|^2 + (1 - \beta_n) \\
 &\quad \times (\|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\
 &\quad + 2\gamma \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\|) \}, \quad (52)
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 &\leq \frac{\alpha_n\rho\tau}{1 + \alpha_n(\gamma - \rho\tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho\tau)} \\
 &\quad \times \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n\gamma)\beta_n}{1 + \alpha_n(\gamma - \rho\tau)} \|Sx_n - x^*\|^2 + \frac{(1 - \alpha_n\gamma)(1 - \beta_n)}{1 + \alpha_n(\gamma - \rho\tau)} \\
 &\quad \times \{ \|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\
 &\quad + 2\gamma \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\| \} \\
 &\leq \frac{\alpha_n\rho\tau}{1 + \alpha_n(\gamma - \rho\tau)} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho\tau)} \\
 &\quad \times \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n\gamma)\beta_n}{1 + \alpha_n(\gamma - \rho\tau)} \|Sx_n - x^*\|^2 \\
 &\quad + \|x_n - x^*\|^2 + \frac{(1 - \alpha_n\gamma)(1 - \beta_n)}{1 + \alpha_n(\gamma - \rho\tau)} \\
 &\quad \times \{ -\|u_n - x_n\|^2 + 2\gamma \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\| \}. \quad (53)
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \frac{(1 - \alpha_n\gamma)(1 - \beta_n)}{1 + \alpha_n(\gamma - \rho\tau)} \|u_n - x_n\|^2 \\
 &\leq \frac{\alpha_n\rho\tau}{1 + \alpha_n(\gamma - \rho\tau)} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n\gamma)\beta_n}{1 + \alpha_n(\gamma - \rho\tau)} \|Sx_n - x^*\|^2 \\
 &\quad + \frac{2(1 - \alpha_n\gamma)(1 - \beta_n)\gamma}{1 + \alpha_n(\gamma - \rho\tau)} \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\| \\
 &\quad + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
 &\leq \frac{\alpha_n\rho\tau}{1 + \alpha_n(\gamma - \rho\tau)} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\gamma - \rho\tau)} \langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
 &\quad + \frac{(1 - \alpha_n\gamma)\beta_n}{1 + \alpha_n(\gamma - \rho\tau)} \|Sx_n - x^*\|^2 \\
 &\quad + \frac{2(1 - \alpha_n\gamma)(1 - \beta_n)\gamma}{1 + \alpha_n(\gamma - \rho\tau)} \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\| \\
 &\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\|. \quad (54)
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0$, and $\lim_{n \rightarrow \infty} \|(T_{r_n}^{F_2} - I)Ax_n\| = 0$, we obtain

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (55)$$

Now, let $z \in \Lambda \cap F(T)$; since $T(x_n) \in C$, we have

$$\begin{aligned} & \|x_n - T(x_n)\| \\ & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\| \\ & = \|x_n - x_{n+1}\| + \|P_C[V_n] - P_C[T(x_n)]\| \\ & \leq \|x_n - x_{n+1}\| \\ & \quad + \|\alpha_n(\rho U(x_n) - \mu F(T(y_n))) + T(y_n) - T(x_n)\| \\ & \leq \|x_n - x_{n+1}\| \\ & \quad + \alpha_n \|\rho U(x_n) - \mu F(T(y_n))\| + \|y_n - x_n\| \\ & \leq \|x_n - x_{n+1}\| + \alpha_n \|\rho U(x_n) - \mu F(T(y_n))\| \\ & \quad + \|\beta_n Sx_n + (1 - \beta_n)u_n - x_n\| \\ & \leq \|x_n - x_{n+1}\| + \alpha_n \|\rho U(x_n) - \mu F(T(y_n))\| \\ & \quad + \beta_n \|Sx_n - x_n\| + (1 - \beta_n)\|u_n - x_n\|. \end{aligned} \quad (56)$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0$, $\|\rho U(x_n) - \mu F(T(y_n))\|$, and $\|Sx_n - x_n\|$ are bounded and $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T(x_n)\| = 0. \quad (57)$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n \rightharpoonup x^* \in C$. It follows from Lemma 6 that $x^* \in F(T)$. Therefore $w_w(x_n) \subset F(T)$. $\square$

Theorem 15. *The sequence $\{x_n\}$ generated by Algorithm 11 converges strongly to z , which is the unique solution of the variational inequality*

$$\langle \rho U(z) - \mu F(z), x - z \rangle \leq 0, \quad \forall x \in \Lambda \cap F(T). \quad (58)$$

Proof. Since $\{x_n\}$ is bounded $x_n \rightharpoonup w$ and from Lemma 14, we have $w \in F(T)$. Next, we show that $w \in \text{EP}(F_1)$. Since $u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)$, we have

$$\begin{aligned} & F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\ & - \frac{1}{r_n} \langle y - u_n, \gamma A^*(T_{r_n}^{F_2} - I)Ax_n \rangle \geq 0, \quad \forall y \in C. \end{aligned} \quad (59)$$

It follows from monotonicity of F_1 that

$$\begin{aligned} & - \frac{1}{r_n} \langle y - u_n, \gamma A^*(T_{r_n}^{F_2} - I)Ax_n \rangle \\ & + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n), \quad \forall y \in C, \end{aligned} \quad (60)$$

$$\begin{aligned} & - \frac{1}{r_{n_k}} \langle y - u_{n_k}, \gamma A^*(T_{r_{n_k}}^{F_2} - I)Ax_{n_k} \rangle \\ & + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq F_1(y, u_{n_k}), \quad \forall y \in C. \end{aligned} \quad (61)$$

Since $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$, $\lim_{n \rightarrow \infty} \|(T_{r_n}^{F_2} - I)Ax_n\| = 0$, and $x_n \rightharpoonup w$, it is easy to observe that $u_{n_k} \rightarrow w$. It follows by Assumption 3(iv) that $F_1(y, w) \leq 0$, for all $y \in C$.

For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)w$; we have $y_t \in C$. Then, from Assumptions 3(i) and (iv)), we have

$$\begin{aligned} 0 & = F_1(y_t, y_t) \\ & \leq tF_1(y_t, y) + (1 - t)F_1(y_t, w) \\ & \leq tF_1(y_t, y). \end{aligned} \quad (62)$$

Therefore $F_1(y_t, y) \geq 0$. From Assumption 3(iii), we have $F_1(w, y) \geq 0$, which implies that $w \in \text{EP}(F_1)$.

Next, we show that $Aw \in \text{EP}(F_2)$. Since $\{x_n\}$ is bounded and $x_n \rightharpoonup w$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow w$ and since A is a bounded linear operator, $Ax_{n_k} \rightarrow Aw$. Now set $v_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{F_2}Ax_{n_k}$. It follows from (49) that $\lim_{k \rightarrow \infty} v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = T_{r_{n_k}}^{F_2}Ax_{n_k}$. Therefore from the definition of $T_{r_{n_k}}^{F_2}$, we have

$$\begin{aligned} & F_2(Ax_{n_k} - v_{n_k}, y) \\ & + \frac{1}{r_{n_k}} \langle y - (Ax_{n_k} - v_{n_k}), \rangle \\ & (Ax_{n_k} - v_{n_k}) - Ax_{n_k} \rangle \geq 0, \quad \forall y \in C. \end{aligned} \quad (63)$$

Since F_2 is upper semicontinuous in first argument, taking $\limsup$ to above inequality as $k \rightarrow \infty$ and using Assumption 3(iv), we obtain

$$F_2(Aw, y) \geq 0 \quad \forall y \in C, \quad (64)$$

which implies that $Aw \in \text{EP}(F_2)$ and hence $w \in \Lambda$.

Thus we have

$$w \in \Lambda \cap F(T). \quad (65)$$

Observe that the constants satisfy $0 \leq \rho\tau < \nu$ and

$$\begin{aligned}
 k &\geq \eta \\
 &\iff k^2 \geq \eta^2 \\
 &\iff 1 - 2\mu\eta + \mu^2 k^2 \geq 1 - 2\mu\eta + \mu^2 \eta^2 \\
 &\iff \sqrt{1 - \mu(2\eta - \mu k^2)} \geq 1 - \mu\eta \\
 &\iff \mu\eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \\
 &\iff \mu\eta \geq \nu.
 \end{aligned} \tag{66}$$

Therefore from Lemma 7, the operator $\mu F - \rho U$ is $\mu\eta - \rho\tau$ strongly monotone, and we get the uniqueness of the solution of the variational inequality (58) and denote it by $z \in \Lambda \cap F(T)$.

Next, we claim that $\limsup_{n \rightarrow \infty} \langle \rho U(z) - \mu F(z), x_n - z \rangle \leq 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \langle \rho U(z) - \mu F(z), x_n - z \rangle \\
 &= \limsup_{k \rightarrow \infty} \langle \rho U(z) - \mu F(z), x_{n_k} - z \rangle \\
 &= \langle \rho U(z) - \mu F(z), w - z \rangle \leq 0.
 \end{aligned} \tag{67}$$

Next, we show that $x_n \rightarrow z$. Consider

$$\begin{aligned}
 &\|x_{n+1} - z\|^2 \\
 &= \langle P_C[V_n] - z, x_{n+1} - z \rangle \\
 &= \langle P_C[V_n] - V_n, P_C[V_n] - z \rangle + \langle V_n - z, x_{n+1} - z \rangle \\
 &\leq \langle \alpha_n(\rho U(x_n) - \mu F(z)) + (I - \alpha_n \mu F)(T(y_n)) \\
 &\quad - (I - \alpha_n \mu F)(T(z)), x_{n+1} - z \rangle \\
 &\leq \langle \alpha_n \rho(U(x_n) - U(z)), x_{n+1} - z \rangle \\
 &\quad + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + \langle (I - \alpha_n \mu F)(T(y_n)) - (I - \alpha_n \mu F)(T(z)), x_{n+1} - z \rangle \\
 &\leq \alpha_n \rho\tau \|x_n - z\| \|x_{n+1} - z\| \\
 &\quad + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + (1 - \alpha_n \nu) \|y_n - z\| \|x_{n+1} - z\| \\
 &\leq \alpha_n \rho\tau \|x_n - z\| \|x_{n+1} - z\| \\
 &\quad + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + (1 - \alpha_n \nu) \{\beta_n \|Sx_n - Sz\| + \beta_n \|Sz - z\| \\
 &\quad + (1 - \beta_n) \|u_n - z\|\} \|x_{n+1} - z\| \\
 &\leq \alpha_n \rho\tau \|x_n - z\| \|x_{n+1} - z\|
 \end{aligned}$$

$$\begin{aligned}
 &+ \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\
 &+ (1 - \alpha_n \nu) \{\beta_n \|x_n - z\| + \beta_n \|Sz - z\| \\
 &\quad + (1 - \beta_n) \|x_n - z\|\} \|x_{n+1} - z\| \\
 &= (1 - \alpha_n(\nu - \rho\tau)) \|x_n - z\| \|x_{n+1} - z\| \\
 &\quad + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + (1 - \alpha_n \nu) \beta_n \|Sz - z\| \|x_{n+1} - z\| \\
 &\leq \frac{1 - \alpha_n(\nu - \rho\tau)}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
 &\quad + \alpha_n \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + (1 - \alpha_n \nu) \beta_n \|Sz - z\| \|x_{n+1} - z\|
 \end{aligned} \tag{68}$$

which implies that

$$\begin{aligned}
 &\|x_{n+1} - z\|^2 \\
 &\leq \frac{1 - \alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)} \|x_n - z\|^2 \\
 &\quad + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho\tau)} \langle \rho U(x_n) - \mu F(z), x_{n+1} - z \rangle \\
 &\quad + \frac{2(1 - \alpha_n \nu) \beta_n}{1 + \alpha_n(\nu - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \\
 &\leq (1 - \alpha_n(\nu - \rho\tau)) \|x_n - z\|^2 + \frac{2\alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)} \\
 &\quad \times \left\{ \frac{1}{\nu - \rho\tau} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle \right. \\
 &\quad \left. + \frac{(1 - \alpha_n \nu) \beta_n}{\alpha_n(\nu - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \right\}.
 \end{aligned} \tag{69}$$

Let $\gamma_n = \alpha_n(\nu - \rho\tau)$ and $\delta_n = (2\alpha_n(\nu - \rho\tau))/(1 + \alpha_n(\nu - \rho\tau))\{(1/(\nu - \rho\tau))\langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle + ((1 - \alpha_n \nu) \beta_n / \alpha_n(\nu - \rho\tau))\|Sz - z\| \|x_{n+1} - z\|\}$.

Since

$$\sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$\limsup_{n \rightarrow \infty} \left\{ \frac{1}{\nu - \rho\tau} \langle \rho U(z) - \mu F(z), x_{n+1} - z \rangle + \frac{(1 - \alpha_n \nu) \beta_n}{\alpha_n(\nu - \rho\tau)} \|Sz - z\| \|x_{n+1} - z\| \right\} \leq 0. \tag{70}$$

It follows that

$$\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0. \tag{71}$$

Thus all the conditions of Lemma 9 are satisfied. Hence we deduce that $x_n \rightarrow z$. This completes the proof. $\square$

Remark 16. In hierarchical fixed point problem (17), if $S = I - (\rho U - \mu F)$, then we can get the variational inequality (58). In (58), if $U = 0$, then we get the variational inequality $\langle F(z), x - z \rangle \geq 0$, for all $x \in \Lambda \cap F(T)$, which is just the variational inequality studied by Suzuki [27] extending the common set of solutions of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed point problem.

4. Conclusions

In this paper, we suggest and analyze an iterative method for finding the approximate element of the common set of solutions of (15)-(16) and (17) in real Hilbert space, which can be viewed as a refinement and improvement of some existing methods for solving a split equilibrium problem and a hierarchical fixed point problem. Some existing methods (e.g., [13, 14, 17–19, 21–23]) can be viewed as special cases of Algorithm 11. Therefore, the new algorithm is expected to be widely applicable.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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