Research Article

The Exponential Diophantine Equation $2^x + b^y = c^z$

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Let $b$ and $c$ be fixed coprime odd positive integers with $\min\{b, c\} > 1$. In this paper, a classification of all positive integer solutions $(x, y, z)$ of the equation $2^x + b^y = c^z$ is given. Further, by an elementary approach, we prove that if $c = b + 2$, then the equation has only the positive integer solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$ and $(2^r - 1, r + 2, 2, 2)$, where $r$ is a positive integer with $r \geq 2$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let $a, b, c$ be fixed coprime positive integers with $\min\{x, y, z\} > 1$. In recent years, the solutions $(x, y, z)$ of the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N}$$

have been investigated in many papers (see [1–3] and its references). In this paper we deal with (1) for the case that $a = 2$. Then (1) can be rewritten as

$$2^x + b^y = c^z, \quad x, y, z \in \mathbb{N},$$

where $b$ and $c$ are fixed coprime odd positive integers with $\min\{b, c\} > 1$. We will give a classification of all solutions $(x, y, z)$ of (2) as follows.

**Theorem 1.** Every solution $(x, y, z)$ of (2) satisfies one of the following types:

(i) $(b, c, x, y, z) = (7, 3, 5, 2, 4)$;
(ii) $(b, c, x, y, z) = (2^r - 1, 2^r + 1, r + 2, 2, 2)$, where $r$ is a positive integer with $r \geq 2$;
(iii) $(b, c, x, y, z) = (5, 3, 1, 2, 3)$;
(iv) $(b, c, x, y, z) = (11, 5, 2, 2, 3)$;
(v) $2 \mid y$ and $z = 1$;
(vi) $(b, c, x, y, z) = (17, 71, 7, 3, 2)$;
(vii) $x = 1, y > 1, 2 \nmid y$ and $2 \mid z$;
(viii) $x > 1, y = 1, 2 \mid z$ and $2^x < b^{50/13}$;
(ix) $2 \nmid yz$.

Recently, Miyazaki and Togbé [4] showed that if $b \geq 5$ and $c = b + 2$, then (2) has only the solution $(x, y, z) = (1, 1, 1)$, except for $(b, x, y, z) = (89, 13, 1, 2)$ and $(2^r - 1, r + 2, 2, 2)$, where $r$ is a positive integer with $r \geq 2$.

2. Preliminaries

**Lemma 3** (see [5, Formula 1.76]). For any positive integer $n$ and any complex numbers $\alpha$ and $\beta$, one has

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} \binom{n}{i} \alpha^i \beta^{n-2i} (-\alpha \beta)^i,$$

where $[n/2]$ is the integer part of $n/2$;

$$\binom{n}{i} = \frac{(n-i-1)!n!}{(n-2i)!i!}, \quad i = 0, \ldots, \left[\frac{n}{2}\right]$$

are positive integers.
Lemma 4 (see [6]). Let $A$ and $B$ be coprime odd positive integers with $\min\{A, B\} > 1$. If the equation
\[ Au^2 - Bv^2 = 2, \quad u, v \in \mathbb{N} \]  
has solutions $(u, v)$, then it has a unique solution $(u_1, v_1)$ such that $u_1 \sqrt{A} + v_1 \sqrt{B} \leq u \sqrt{A} + v \sqrt{B}$, where $(u, v)$ is the unique solution of $(5)$. The solution $(u_1, v_1)$ is the least solution of $(5)$. Every solution $(u, v)$ of $(5)$ can be expressed as
\[ \frac{u \sqrt{A} + v \sqrt{B}}{\sqrt{2}} = \left( \frac{u_1 \sqrt{A} + v_1 \sqrt{B}}{\sqrt{2}} \right)^n, \quad n \in \mathbb{N}, \; 2 \nmid n. \]  
Furthermore, by (6), we have $u_1 \mid u$ and $v_1 \mid v$.

Lemma 5. Equation $(5)$ has no solutions $(u, v)$ such that $u > u_1$, $v > v_1$, and every prime divisor of $u|u_1$ and $v|v_1$ divides $A$ and $B$, respectively.

Proof. We now assume that $(u, v)$ is a solution of $(5)$ satisfying the hypothesis. Since $u > u_1$, by Lemma 4, the $(u, v)$ is all solutions of $(5)$. Let
\[ \alpha = \frac{u_1 \sqrt{A} + v_1 \sqrt{B}}{\sqrt{2}}, \quad \beta = \frac{u_1 \sqrt{A} - v_1 \sqrt{B}}{\sqrt{2}}. \]  
We get
\[ \frac{u_n}{u_1} = \frac{\alpha^n - (-\beta)^n}{\alpha - (-\beta)}, \quad \frac{v_n}{v_1} = \frac{\alpha^n - (\beta)^n}{\alpha - (\beta)}, \]  
where $n$ is odd. Numbers $\alpha$ and $\beta$ are such that $(\alpha - \beta)$ satisfy $x^2 - 2\sqrt{2}v_1^2 x + 1 = 0$ and $(\alpha, \beta)$ satisfy $x^2 - 2Au_1^2 x + 1 = 0$. Thus, $(u_n/u_1)_{n \geq 1}$ and $(v_n/v_1)_{n \geq 1}$ are the odd indexed subsequences of the two Lehmer sequence of roots $(\alpha, -\beta)$ and $(\alpha, \beta)$. Their discriminants are $(\alpha + \beta)^2 = 2Au_1^2$ and $(\alpha - \beta)^2 = 2Bv_1^2$, respectively. Saying that all prime factors of $u_n/u_1$ divide $A$ implies that all primes of the $n$th term of a Lehmer sequence divide its discriminant. The same is true for $v_n/v_1$. Hence, $u_n/u_1$ and $v_n/v_1$ are terms of a Lehmer sequence of real roots lacking primitive divisors. By Table 2 in [7], this is possible only for $n = 3, 5$. Even more, in the present case,
\[ \frac{(\alpha^2 - (\beta^2)^n)}{\alpha^2 - \beta^2} = \frac{u_n v_n}{u_1 v_1}. \]  
is the $n$th term of the Lucas sequence of positive real roots $(\alpha^2, \beta^2)$ whose all prime factors divide its discriminant $(\alpha^2 - \beta^2)^2 = 4Au_1^2v_1^2$, and by Table 1 in [7] this is possible for $n$ odd only if $n = 3$ or $n = 5$. Furthermore, when $n = 5$, we must have $\alpha^2 = (1 + \sqrt{5})/2$, but this is not possible since $\alpha = \sqrt{(1 + \sqrt{5})/2}$ is not of the form $(u_1 \sqrt{A} + v_1 \sqrt{B})/\sqrt{2}$ for some positive integers $A > 1, B > 1, u_1$ and $v_1$. So, only $n = 3$ is possible. Now by some simple numerical computation for $u_3$ and $v_3$, we see that it is not possible that all prime factors of $u_3$ and all prime factors of $v_3$ divide $B$. Thus, Lemma 5 is proved.

Lemma 6 (see [8]). The equation
\[ X^2 + 7 = 2^{m+2}, \quad X, n \in \mathbb{N} \]  
has only the solutions $(X, n) = (1, 1), (3, 2), (5, 3), (11, 5), \text{ and } (181, 13)$.

Lemma 7 (see [9]). Let $D$ be an odd positive integer with $D > 1$. If $(X, n)$ is a solution of the equation
\[ X^2 - D = 2^n, \quad X, n \in \mathbb{N}, \]  
then $2^n < D^{50/13}$.

Lemma 8 (see [10, 11]). The equation
\[ X^2 + 2m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \quad \gcd (X, Y) = 1, \quad n \geq 3 \]  
has only the solutions $(X, Y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4), \text{ and } (11, 5, 2, 3)$.

Lemma 9 (see [12]). The equation
\[ X^2 - 2m = Y^n, \quad X, Y, m, n \in \mathbb{N}, \quad \gcd (X, Y) = 1, \quad Y > 1, \quad m > 1, \quad n \geq 3 \]  
has only the solution $(X, Y, m, n) = (71, 17, 7, 3)$.

Lemma 10 (see [13]). The equation
\[ X^m - Y^n = 1, \quad X, Y, m, n \in \mathbb{N}, \quad \min \{X, Y, m, n\} > 1 \]  
has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.

3. Proof of Theorem
Let $(x, y, z)$ be a solution of $(2)$. If $2 \mid y$ and $2 \mid z$, then we have $x \geq 3, c^{x/2} + b^{y/2} = 2^{x-1}$, and $c^{x/2} - b^{y/2} = 2$. It follows that
\[ c^{x/2} = 2^{x-1} + 1, \quad b^{y/2} = 2^{x-2} - 1. \]  
Applying Lemma 10 to (15), we can only obtain the solutions of types (i) and (ii).

If $2 \nmid y$ and $2 \nmid z$, then we have
\[ \left( b^{y/2} \right)^2 + 2^x = c^2, \quad 2 \nmid z. \]  
Applying Lemma 8 to (16), we can only get the solutions of types (iii), (iv), and (v).

Similarly, if $2 \nmid y$ and $2 \mid z$, using Lemmas 7 and 9, then we can only obtain the solutions of types (vi), (vii), and (viii). Finally, if $2 \mid yz$, then the solutions are of type (ix). Thus, the theorem is proved.

4. Proof of Corollary
Since $c = b + 2$, $(2)$ can be rewritten as
\[ 2^x + b^y = (b + 2)^x, \quad x, y, z \in \mathbb{N}. \]
Let \((x, y, z)\) be a solution of (17). By the theorem, (17) has only the solutions
\[
(b, x, y, z) = (2^r - 1, r + 2, 2, 2), \quad r \in \mathbb{N}, \quad r \geq 2 \tag{18}
\]
satisfying \(2 \mid y\) and \(2 \mid z\).

If \(x = 1, 2 \nmid y\) and \(2 \mid z\), then from (17) we get
\[
2 + b^y = 2 + ((b + 1) - 1)^y = 1 + (b + 1) \sum_{j=1}^{y} (-1)^{j-1} \left( \frac{y}{j} \right) (b + 1)^{y-j} = 1 + (b + 1) \sum_{j=1}^{\frac{z}{2}} \left( \frac{z}{j} \right) (b + 1)^{y-j} = ((b + 1) + 1)^z = (b + 2)^z,
\]
whence we obtain
\[
y \equiv z \pmod{(b + 1)} \tag{20}
\]
But, since \(2 \mid b + 1\) and \(2 \nmid y - z\), congruence (20) is impossible.

If \(x > 1, 2 \nmid y\) and \(2 \mid z\), by the theorem, then we have
\[
y = 1 \text{ and } 2^x < b^{50/13}. \quad \text{Hence, by (17), we get}
\]
\[
b^y < (b + 2)^z \leq (b + 2)^z = 2^x + b < b^{50/13} + b. \tag{21}
\]

Since \(b \geq 3\) and \(2 \mid z\), we see from (21) that \(z = 2\). Substituting it into (17), we have
\[
b^2 + 3b - 4(2^x - 1) = 0 \quad \text{and} \quad b = \frac{1}{2} \left( -3 + \sqrt{2 \cdot 2^x + 7} \right). \tag{22}
\]

By (22), we get
\[
b = \frac{1}{2} (X - 3), \quad x = n, \tag{23}
\]
where \((X, n)\) is a solution of (10). Since \(2 \mid b\) and \(b > 1\), by Lemma 6, we can only have \((X, n) = (181, 13)\) and
\[
(b, x, y, z) = (89, 13, 1, 2). \tag{24}
\]

By (23).

If \(2 \nmid yz\), then \(b^{y-1} \equiv (b + 2)^{z-1} \equiv 1 \pmod{8}\). Hence, by (17), we get
\[
2^x \equiv (b + 2)^2 - b^y \equiv (b + 2)^2 - b \equiv 2 \pmod{8} \quad \text{and} \quad x = 1. \text{ It implies that the equation}
\]
\[
(b + 2) u^2 - b v^2 = 2, \quad u, v \in \mathbb{N} \tag{25}
\]
has the solution
\[
(u, v) = \left( \left( b + 2 \right)^{\frac{y-1}{2}}, b^{\frac{y-1}{2}} \right). \tag{26}
\]

Notice that the least solution of (25) is \((u_1, v_1) = (1, 1)\); \(y\) and \(z\) satisfy either \(y = z = 1\) or \(\min(y, z) > 1\). Applying Lemma 5 to (26), we only obtain that \((u, v) = (1, 1)\) and
\[
(x, y, z) = (1, 1, 1). \tag{27}
\]

Thus, (17) has only the solutions (18), (24), and (27). The corollary is proved.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


