Research Article

Weighted Multilinear Hardy Operators on Herz Type Spaces

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This paper focuses on the bounds of weighted multilinear Hardy operators on the product Herz spaces and the product Morrey-Herz spaces, respectively. We present a sufficient condition on the weight function that guarantees weighted multilinear Hardy operators to be bounded on the product Herz spaces. And the condition is necessary under certain assumptions. Finally, we extend the obtained results to the product Morrey-Herz spaces.

1. Introduction

Let f be a nonnegative integral function on $\mathbb{R}^+$. The Hardy operator H is defined by $Hf(x) = (1/x) \int_0^x f(t)dt$, $x > 0$. For the operator H , Hardy et al. [1] proved that the inequality

$$\|Hf\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p} \quad (1)$$

holds, where $1 < p < \infty$ and the constant $p/(p-1)$ is the best possible. Usually, we call (1) classical Hardy's inequality. With the development of analysis theory, many types of Hardy's inequalities have been discussed. For example, a quite number of papers dealt with the various generalizations, numerous variants, and applications of Hardy's inequalities in the past few years. On the detailed discussions of Hardy's inequalities, we choose to refer to [2, 3].

In 1984, Carton-Lebrun and Fosset [4] gave the definition of the weighted Hardy operator H_ω to be

$$H_\omega f(x) := \int_0^1 f(tx) \omega(t) dt, \quad x \in \mathbb{R}^n, \quad (2)$$

where $\omega : [0, 1] \rightarrow [0, \infty)$ is a measurable function and f is a complex-valued measurable function on $\mathbb{R}^n$. It is obvious that H_ω degenerates into the classical Hardy operator H when $\omega \equiv 1$, $n = 1$, and f is defined on $\mathbb{R}^+$. In addition, we call

the adjoint operator of H_ω the weighted Cesàro average $G_\omega f$. And the definition of $G_\omega f$ is

$$(G_\omega f)(x) = \int_0^1 f\left(\frac{x}{t}\right) t^{-n} \omega(t) dt. \quad (3)$$

When $\omega \equiv 1$ and $n = 1$, G_ω becomes the classical Cesàro operator G

$$Gf(x) = \begin{cases} \int_x^\infty \frac{f(y)}{y} dy, & x > 0, \\ -\int_{-\infty}^x \frac{f(y)}{y} dy, & x < 0. \end{cases} \quad (4)$$

It is easy to get that H_ω and G_ω satisfy

$$\int_{\mathbb{R}^n} (H_\omega f)(x) g(x) dx = \int_{\mathbb{R}^n} f(x) (G_\omega g)(x) dx, \quad (5)$$

when $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p < \infty$, $1/p + 1/q = 1$. This means that H_ω and G_ω satisfy the commutative rule $H_\omega G_\omega = G_\omega H_\omega$.

Under certain conditions on ω , Carton-Lebrun and Fosset [4] proved that H_ω maps $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$. They also pointed out that the operator H_ω commutes with the Hilbert transform when $n = 1$ and with certain Calderón-Zygmund singular integral operators including the Riesz

transform when $n \geq 2$. Refer to Xiao [5] for the further extension of the results above; see also [6, 7].

Since Herz space is a natural generalization of weighted Lebesgue spaces with power weights, researchers are also interested in studying the boundedness of H_ω on Herz spaces. To make the description more clear below, we review the definition of the Herz spaces now. In the following definitions, $\chi_k = \chi_{C_k}$, $C_k = B_k \setminus B_{k-1}$, and $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, for $k \in \mathbb{Z}$, $\widetilde{C}_k = C_k$, $\widetilde{C}_0 = B_0$, and $\widetilde{\chi}_k = \chi_{C_k}$, for $k \in \mathbb{N}$, and χ_E is the characteristic function of a set E .

Definition 1 (see [8]). Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$.

(1) The homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \quad (6)$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (7)$$

(2) The inhomogeneous Herz spaces $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \quad (8)$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}, \quad (9)$$

with usual modifications made when $p = \infty$ or $q = \infty$.

In [9], Wu gave the definition of Morrey-Herz spaces.

Definition 2. Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$, and $\lambda \geq 0$.

(1) The homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\}, \quad (10)$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (11)$$

(2) The inhomogeneous Morrey-Herz space $MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n) : \|f\|_{MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\}, \quad (12)$$

where

$$\|f\|_{MK_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} := \sup_{k_0 \in \mathbb{N}} 2^{-k_0\lambda} \left\{ \sum_{k=0}^{k_0} 2^{k\alpha p} \|f\widetilde{\chi}_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}, \quad (13)$$

with usual modifications made when $p = \infty$ or $q = \infty$.

From the above definitions, it is not difficult to note that $\dot{K}_p^{0,p}(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = K_p^{\alpha/p,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx)$ for all $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$. Moreover, the homogenous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, the homogenous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$, and the Morrey spaces $M^{q,\lambda}(\mathbb{R}^n)$ (see [10, 11]) satisfy $M\dot{K}_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, $M^{q,\lambda}(\mathbb{R}^n) \subset M\dot{K}_{q,q}^{0,\lambda}(\mathbb{R}^n)$. In [12], Liu and Fu discussed the boundedness of the weighted Hardy operator H_ω on the Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. Conditions on the weighted function ω were presented to guarantee that H_ω is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. They also estimated the corresponding operator norm. And in [13], Fu and Lu further extended the results of [12] to the Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$.

In the past few years, the properties of multilinear operators have also been extensively studied by researchers. There are two reasons for this. First, the multilinear operators are the generalization of the linear ones, and its study makes the research contents of analysis theory more rich. Second, the multilinear operators naturally appear in analysis. The study of multilinear operators is traced to the multilinear singular integral operator theory (see [14]). For more detailed studies on multilinear operators, the readers refer to [15–18] and the references therein. Recently, we have studied the boundedness of weighted multilinear Hardy operators $\mathcal{H}_\omega^m$ on the product of Lebesgue spaces and central Morrey spaces in [19]. Based on these results and inspired by the results of [12, 13], this paper further concerns the boundedness of $\mathcal{H}_\omega^m$ on the product Herz spaces and the product Morrey-Herz spaces. We first recall the definition of $\mathcal{H}_\omega^m$.

Definition 3. Let $m \in \mathbb{N}$, $m \geq 2$, and

$$\omega : \overbrace{[0, 1] \times [0, 1] \times \cdots \times [0, 1]}^{m \text{ times}} \rightarrow [0, \infty) \quad (14)$$

be an integrable function. The *weighted multilinear Hardy operator* $\mathcal{H}_\omega^m$ is defined by

$$\mathcal{H}_\omega^m(\vec{f})(x) := \int_{0 < t_1, t_2, \dots, t_m < 1} \left(\prod_{i=1}^m f_i(t_i x) \right) \omega(\vec{t}) d\vec{t}, \quad (15)$$

$$x \in \mathbb{R}^n,$$

where $\vec{f} := (f_1, \dots, f_m)$, $\omega(\vec{t}) := \omega(t_1, t_2, \dots, t_m)$, $d\vec{t} := dt_1 \cdots dt_m$, and f_i ($i = 1, \dots, m$) are complex-valued measurable functions on $\mathbb{R}^n$. When $m = 2$, $\mathcal{H}_\omega^m$ is referred to as bilinear.

In accordance with the case of H_ω , we also recall the definition of the weighted multilinear Cesàro operator $\mathcal{G}_\omega$ that is the adjoint operator of $\mathcal{H}_\omega^m$.

Definition 4. Let $m \in \mathbb{N}$, $m \geq 2$, and $\omega : [0, 1] \times [0, 1]^{m-1} \rightarrow [0, \infty)$ be an integrable function. The weighted multilinear Cesàro operator $\mathcal{G}_\omega^m$ is defined by

$$\mathcal{G}_\omega^m(\vec{f})(x) = \int_{0 < t_1, t_2, \dots, t_m < 1} \left(\prod_{i=1}^m f_i \left(\frac{x}{t_i} \right) (t_i)^{-n} \right) \omega(\vec{t}) d\vec{t},$$

$$x \in \mathbb{R}^n, \quad (16)$$

where f_i are measurable complex-valued functions on $\mathbb{R}^n$, $1 \leq i \leq m$.

Note that $\mathcal{H}_\omega^m$ and $\mathcal{G}_\omega^m$ do not satisfy the following commutative rule:

$$\int_{\mathbb{R}^n} (\mathcal{H}_\omega^m f)(x) g(x) dx = \int_{\mathbb{R}^n} f(x) (\mathcal{G}_\omega^m g)(x) dx. \quad (17)$$

This is different from the case of H_ω and G_ω .

The paper is organized as follows. In Section 2, we present the estimate of the boundedness of $\mathcal{H}_\omega^m$ on the product Herz spaces. In Section 3, we give the estimates of the boundedness of $\mathcal{H}_\omega^m$ on the product Morrey-Herz spaces.

2. Boundedness of $\mathcal{H}_\omega^m$ on the Product of Herz Spaces

We give the first main result of this paper.

Theorem 5. Let $\alpha, \alpha_1, \dots, \alpha_2 \in \mathbb{R}$, $1 < p, p_i, q, q_i < \infty$ and $\alpha_1 + \dots + \alpha_m = \alpha$, $1/q_1 + \dots + 1/q_m = 1/q$, $1/p_1 + \dots + 1/p_m = 1/p$, $i = 1, 2, \dots, m$. Then $\mathcal{H}_\omega^m$ is bounded from $\dot{K}_{q_1}^{\alpha_1, p_1} \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m}$ to $\dot{K}_q^{\alpha, p}$ if

$$\int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n/q_i)} \right) \omega(\vec{t}) d\vec{t} < \infty. \quad (18)$$

Conversely, if $\alpha_1 = \dots = \alpha_m = (1/m)\alpha$, $q_i = mq$, $p_i = mp$, $i = 1, 2, \dots, m$, and $\mathcal{H}_\omega^m$ is bounded from $\dot{K}_{q_1}^{\alpha_1, p_1} \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m}$ to $\dot{K}_q^{\alpha, p}$, then (18) holds. Moreover, in this case, one has

$$\|\mathcal{H}_\omega^m\|_{\dot{K}_{q_1}^{\alpha_1, p_1} \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m} \rightarrow \dot{K}_q^{\alpha, p}} \approx \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n/q_i)} \right) \omega(\vec{t}) d\vec{t}. \quad (19)$$

Similarly, we have the following result for the operator $\mathcal{G}_\omega^m$.

Theorem 6. Let $\alpha, \alpha_1, \dots, \alpha_2 \in \mathbb{R}$, $1 < p, p_i, q, q_i < \infty$, and $\alpha_1 + \dots + \alpha_m = \alpha$, $1/q_1 + \dots + 1/q_m = 1/q$, $1/p_1 + \dots + 1/p_m = 1/p$, $i = 1, 2, \dots, m$. If ω satisfies

$$\int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{\alpha_i - n(1-1/q_i)} \right) \omega(\vec{t}) d\vec{t} < \infty, \quad (20)$$

then $\mathcal{G}_\omega^m$ is bounded from $\dot{K}_{q_1}^{\alpha_1, p_1} \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m}$ to $\dot{K}_q^{\alpha, p}$.

Conversely, if $\alpha_1 = \dots = \alpha_m = (1/m)\alpha$, $q_i = mq$, $p_i = mp$, $i = 1, 2, \dots, m$, and $\mathcal{G}_\omega^m$ is bounded from $\dot{K}_{q_1}^{\alpha_1, p_1} \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m}$ to $\dot{K}_q^{\alpha, p}$, then (20) holds. Moreover, in this case, one has

$$\|\mathcal{G}_\omega^m\|_{\dot{K}_{q_1}^{\alpha_1, p_1} \times \dots \times \dot{K}_{q_m}^{\alpha_m, p_m} \rightarrow \dot{K}_q^{\alpha, p}} \approx \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{\alpha_i - n(1-1/q_i)} \right) \omega(\vec{t}) d\vec{t}. \quad (21)$$

Since the proofs of Theorems 5 and 6 are similar, we just give the proof of Theorem 5.

Proof of Theorem 5. In order to simplify the proof, we only consider the case of $m = 2$. Actually, a similar procedure works for all $m \in \mathbb{N}$.

Since $1/q_1 + 1/q_2 = 1/q$, by Hölder and Minkowski inequality, we have

$$\begin{aligned} & \|\mathcal{H}_\omega^2(f_1, f_2)\chi_k\|_{L^q(\mathbb{R}^n)} \\ &= \left(\int_{C_k} \left| \iint_0^1 f_1(t_1x) f(t_2x) \omega(t_1, t_2) dt_1 dt_2 \right|^q \right)^{1/q} \\ &\leq \iint_0^1 \left(\int_{C_k} |f_1(t_1x) f(t_2x)|^q \right)^{1/q} \omega(t_1, t_2) dt_1 dt_2 \quad (22) \\ &\leq \iint_0^1 \left(\int_{C_k} |f_1(t_1x)|^{q_1} dx \right)^{1/q_1} \\ &\quad \times \left(\int_{C_k} |f_2(t_2x)|^{q_2} dx \right)^{1/q_2} \omega(t_1, t_2) dt_1 dt_2. \end{aligned}$$

For the arbitrary $t_1, t_2 \in (0, 1)$, we can find $m, l \in \mathbb{Z}$, such that $2^{m-1} < t_1 \leq 2^m$ and $2^{l-1} < t_2 \leq 2^l$. By Minkowski inequality, we have

$$\begin{aligned} & \iint_0^1 \left(\int_{C_k} |f_1(t_1x)|^{q_1} dx \right)^{1/q_1} \\ &\quad \times \left(\int_{C_k} |f_2(t_2x)|^{q_2} dx \right)^{1/q_2} \omega(t_1, t_2) dt_1 dt_2 \\ &\leq \iint_0^1 \left(\int_{t_1 C_k} |f_1(x)|^{q_1} dx \right)^{1/q_1} \\ &\quad \times \left(\int_{t_2 C_k} |f_2(x)|^{q_2} dx \right)^{1/q_2} t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\ &\leq \iint_0^1 \left(\int_{2^{k+m-2} < |x| \leq 2^{k+m}} |f_1(x)|^{q_1} dx \right)^{1/q_1} \\ &\quad \times \left(\int_{2^{k+l-2} < |x| \leq 2^{k+l}} |f_2(x)|^{q_2} dx \right)^{1/q_2} \\ &\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \end{aligned}$$

$$\begin{aligned}
&\leq \iint_0^1 (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} + \|f_1 \chi_{k+m}\|_{L^{q_1}}) \\
&\quad \times (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}}) t_1^{-n/q_1} \\
&\quad \times t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{23}$$

For $1/p_1 + 1/p_2 = 1/p$ and $\alpha_1 + \alpha_2 = \alpha$, Hölder inequality and Minkowski inequality give

$$\begin{aligned}
&\|\mathcal{H}_\omega^2(f_1, f_2)\|_{\dot{K}_q^{\alpha, p}} \\
&= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\mathcal{H}_\omega^2(f_1, f_2) \chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \\
&\leq \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\iint_0^1 (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} + \|f_1 \chi_{k+m}\|_{L^{q_1}}) \right. \right. \\
&\quad \times (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}}) \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \Big)^p \Big)^{1/p} \\
&\leq \iint_0^1 \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} + \|f_1 \chi_{k+m}\|_{L^{q_1}})^p \right. \\
&\quad \times (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}})^p \Big)^{1/p} \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \iint_0^1 \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} \right. \\
&\quad \left. + \|f_1 \chi_{k+m}\|_{L^{q_1}})^{p_1} \right)^{1/p_1} \\
&\quad \times \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2} (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}})^{p_2} \right)^{1/p_2} \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \iint_0^1 \left\{ \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \|f_1 \chi_{k+m-1}\|_{L^{q_1}}^{p_1} \right)^{1/p_1} \right. \\
&\quad \left. + \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p_1} \|f_1 \chi_{k+m}\|_{L^{q_1}}^{p_1} \right)^{1/p_1} \right\} \\
&\quad \times \left\{ \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2} \|f_2 \chi_{k+l-1}\|_{L^{q_2}}^{p_2} \right)^{1/p_2} \right.
\end{aligned}$$

$$\begin{aligned}
&\quad \left. + \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p_2} \|f_2 \chi_{k+l}\|_{L^{q_2}}^{p_2} \right)^{1/p_2} \right\} \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \|f_1\|_{\dot{K}_{q_1}^{\alpha_1, p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2, p_2}} \\
&\quad \times \iint_0^1 (2^{-(m-1)\alpha_1} + 2^{-m\alpha_1}) (2^{-(l-1)\alpha_2} + 2^{-l\alpha_2}) \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \|f_1\|_{\dot{K}_{q_1}^{\alpha_1, p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2, p_2}} \\
&\quad \times \iint_0^1 t_1^{-(\alpha_1 + n/q_1)} t_2^{-(\alpha_2 + n/q_2)} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{24}$$

From the above inequality, we have that the first conclusion in Theorem 5 holds.

On the other hand, we suppose that $\mathcal{H}_\omega^2$ is bounded from $\dot{K}_{q_1}^{\alpha_1, p_1} \times \dot{K}_{q_2}^{\alpha_2, p_2}$ to $\dot{K}_q^{\alpha, p}$ and $\mathcal{H}_\omega^2$ has the operator norm $\|\mathcal{H}_\omega^2\|_{\dot{K}_{q_1}^{\alpha_1, p_1} \times \dot{K}_{q_2}^{\alpha_2, p_2} \rightarrow \dot{K}_q^{\alpha, p}}$. For any $0 < \varepsilon < 1$, let

$$\begin{aligned}
f_1(x) &= \begin{cases} 0, & |x| \leq 1, \\ |x|^{-\alpha_1 - n/q_1 - \varepsilon}, & |x| > 1, \end{cases} \\
f_2(x) &= \begin{cases} 0, & |x| \leq 1, \\ |x|^{-\alpha_2 - n/q_2 - \varepsilon}, & |x| > 1. \end{cases}
\end{aligned} \tag{25}$$

Obviously, $\|f_1 \chi_k\|_{L^{q_1}(\mathbb{R}^n)} = \|f_2 \chi_k\|_{L^{q_2}(\mathbb{R}^n)} = 0$, when $k = 0, -1, -2, \dots$. So, for any positive integer k , it is easy to get that

$$\begin{aligned}
\|f_1 \chi_k\|_{L^{q_1}(\mathbb{R}^n)} &= \left(\int_{2^{k-1} \leq |x| < 2^k} |x|^{(-\alpha_1 - n/q_1 - \varepsilon)q_1} dx \right)^{1/q_1} \\
&= 2^{-(\alpha_1 + \varepsilon)k} \left| \frac{S_n(2^{(\alpha_1 + \varepsilon)q_1} - 1)}{(\alpha_1 + \varepsilon)q_1} \right|^{1/q_1},
\end{aligned} \tag{26}$$

where $S_n = n\pi^{n/2}/\Gamma(1 + n/2)$. A simple computation gives

$$\begin{aligned}
&\|f_1\|_{\dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^n)} \\
&= 2^{-\varepsilon} \left(\frac{1}{1 - 2^{-\varepsilon p_1}} \right)^{1/p_1} \left| \frac{S_n(2^{(\alpha_1 + \varepsilon)q_1} - 1)}{(\alpha_1 + \varepsilon)q_1} \right|^{1/q_1}.
\end{aligned} \tag{27}$$

Similarly, we obtain

$$\begin{aligned}
&\|f_2\|_{\dot{K}_{q_2}^{\alpha_2, p_2}(\mathbb{R}^n)} \\
&= 2^{-\varepsilon} \left(\frac{1}{1 - 2^{-\varepsilon p_2}} \right)^{1/p_2} \left| \frac{S_n(2^{(\alpha_2 + \varepsilon)q_2} - 1)}{(\alpha_2 + \varepsilon)q_2} \right|^{1/q_2}.
\end{aligned} \tag{28}$$

When $|x| \leq 1$ and $t \in [0, 1]$, we have $|tx| \leq 1$. So, $\mathcal{H}_\omega^2(f_1, f_2)(x) = 0$ in this case based on (25). When $|x| > 1$, we get

$$\begin{aligned} \mathcal{H}_\omega^2(f_1, f_2)(x) &= \iint_{1/|x|}^1 f_1(t_1 x) f_2(t_2 x) \omega(t_1, t_2) dt_1 dt_2 \\ &= |x|^{-(\alpha+(n/q)+2\epsilon)} \iint_{1/|x|}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \\ &\quad \times \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (29)$$

Let $\delta = \epsilon^{-1} > 1$. It is easy to find a positive integer l such that $2^{l-1} \leq \delta < 2^l$. Thus, we have

$$\begin{aligned} &\|\mathcal{H}_\omega^2(f_1, f_2)\|_{\dot{K}_q^{\alpha,p}}^p \\ &= \sum_{k=1}^{\infty} 2^{k\alpha p} \|\mathcal{H}_\omega^2(f_1, f_2) \chi_k(x)\|_{L^q(\mathbb{R}^n)}^p \\ &= \sum_{k=1}^{\infty} 2^{k\alpha p} \left\{ \int_{|x|>1} (|x|^{-(\alpha+n/q+2\epsilon)} \chi_k(x) \right. \\ &\quad \times \iint_{1/|x|}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \\ &\quad \times \omega(t_1, t_2) dt_1 dt_2)^q dx \Big\}^{p/q} \\ &\geq \sum_{k=1}^{\infty} 2^{k\alpha p} \left(\int_{|x|>\delta} |x|^{-(\alpha+n/q+2\epsilon)q} \chi_k(x) dx \right)^{p/q} \\ &\quad \times \left(\iint_{1/\delta}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2 \right)^p \\ &\geq S_n^{p/q} \sum_{k=l+1}^{\infty} 2^{k\alpha p} \left(\int_{2^{k-1}}^{2^k} r^{-(\alpha+2\epsilon)q-1} dr \right)^{p/q} \\ &\quad \times \left(\iint_{1/\delta}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2 \right)^p \\ &= S_n^{p/q} \sum_{k=l+1}^{\infty} 2^{-2\epsilon p k} \left| \frac{(2^{(\alpha+2\epsilon)q} - 1)}{(\alpha + 2\epsilon)q} \right|^{p/q} \\ &\quad \times \left(\iint_{1/\delta}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2 \right)^p \\ &= S_n^{p/q} \frac{2^{-2\epsilon(l+1)p}}{1 - 2^{-2\epsilon p}} \left| \frac{(2^{(\alpha+2\epsilon)q} - 1)}{(\alpha + 2\epsilon)q} \right|^{p/q} \\ &\quad \times \left(\iint_{1/\delta}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2 \right)^p. \end{aligned} \quad (30)$$

Since $q_1 = q_2 = 2q$, $\alpha_1 = \alpha_2 = (1/2)\alpha$ and $1/p = 1/p_1 + 1/p_2$, $p_1 = p_2 = 2p$, we have

$$\begin{aligned} &\|\mathcal{H}_\omega^2(f_1, f_2)\|_{\dot{K}_q^{\alpha,p}} \\ &\geq S_n^{1/q} 2^{-2\epsilon(l+1)} \left(\frac{1}{1 - 2^{-2\epsilon p}} \right)^{1/p} \left| \frac{(2^{(\alpha+2\epsilon)q} - 1)}{(\alpha + 2\epsilon)q} \right|^{1/q} \\ &\quad \times \iint_{1/\delta}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2 \\ &\geq \epsilon^{2\epsilon} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}} \\ &\quad \times \iint_{\epsilon}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (31)$$

This gives

$$\begin{aligned} &\|\mathcal{H}_\omega^2(f_1, f_2)\|_{\dot{K}_q^{\alpha,p}} \\ &\geq \epsilon^{2\epsilon} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}} \\ &\quad \times \iint_{\epsilon}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (32)$$

Using the boundedness of $\mathcal{H}_\omega^2$ and its operator norm yields

$$\epsilon^{2\epsilon} \iint_{\epsilon}^1 t_1^{-\alpha_1-n/q_1-\epsilon} t_2^{-\alpha_2-n/q_2-\epsilon} \omega(t_1, t_2) dt_1 dt_2 \quad (33)$$

$$\leq \|\mathcal{H}_\omega^2\|_{\dot{K}_{q_1}^{\alpha_1,p_1} \times \dot{K}_{q_2}^{\alpha_2,p_2} \rightarrow \dot{K}_q^{\alpha,p}}.$$

Letting $\epsilon \rightarrow 0^+$ in (33) gives

$$\begin{aligned} &\iint_0^1 t_1^{-\alpha_1-n/q_1} t_2^{-\alpha_2-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\ &\leq \|\mathcal{H}_\omega^2\|_{\dot{K}_{q_1}^{\alpha_1,p_1} \times \dot{K}_{q_2}^{\alpha_2,p_2} \rightarrow \dot{K}_q^{\alpha,p}}. \end{aligned} \quad (34)$$

Thus, (18) holds. $\square$

Since $\dot{K}_p^{0,p}(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = K_p^{\alpha/p,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx)$ for all $0 < p \leq \infty$, we deduce the following corollaries from Theorems 5 and 6.

Corollary 7. Let $\alpha, \alpha_1, \dots, \alpha_2 \in \mathbb{R}$, $1 < p, p_i < \infty$ and $\alpha_1 + \dots + \alpha_m = \alpha$, $1/p_1 + \dots + 1/p_m = 1/p$, $i = 1, 2, \dots, m$. Then $\mathcal{H}_\omega^m$ is bounded from $L^{p_1}(|x|^{\alpha_1} dx) \times \dots \times L^{p_m}(|x|^{\alpha_m} dx)$ to $L^p(|x|^\alpha dx)$ if

$$\int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i+n)/p_i} \right) \omega(\vec{t}) d\vec{t} < \infty. \quad (35)$$

Conversely, if $\alpha_1 = \dots = \alpha_m = (1/m)\alpha$, $p_i = mp$, $i = 1, 2, \dots, m$, and $\mathcal{H}_\omega^m$ is bounded from $L^{p_1}(|x|^{\alpha_1} dx) \times \dots \times$

$L^{p_m}(|x|^{\alpha_m} dx)$ to $L^p(|x|^{\alpha} dx)$, then (35) holds. Moreover, in this case, one has

$$\begin{aligned} & \|\mathcal{H}_{\omega}^m\|_{L^{p_1}(|x|^{\alpha_1} dx) \times \dots \times L^{p_m}(|x|^{\alpha_m} dx) \rightarrow L^p(|x|^{\alpha} dx)} \\ & \simeq \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n)/p_i} \right) \omega(\vec{t}) d\vec{t}. \end{aligned} \quad (36)$$

Corollary 8. Let $\alpha, \alpha_1, \dots, \alpha_2 \in \mathbb{R}, 1 < p, p_i < \infty$ and $\alpha_1 + \dots + \alpha_m = \alpha, 1/p_1 + \dots + 1/p_m = 1/p, i = 1, 2, \dots, m$. If ω satisfies

$$\int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{(\alpha_i + n)/p_i - n} \right) \omega(\vec{t}) d\vec{t} < \infty, \quad (37)$$

then $\mathcal{G}_{\omega}^m$ is bounded from $L^{p_1}(|x|^{\alpha_1} dx) \times \dots \times L^{p_m}(|x|^{\alpha_m} dx)$ to $L^p(|x|^{\alpha} dx)$.

Conversely, if $\alpha_1 = \dots = \alpha_m = (1/m)\alpha, p_i = mp, i = 1, 2, \dots, m$ and $\mathcal{G}_{\omega}^m$ is bounded from $L^{p_1}(|x|^{\alpha_1} dx) \times \dots \times L^{p_m}(|x|^{\alpha_m} dx)$ to $L^p(|x|^{\alpha} dx)$, then (37) holds. Moreover, in this case, one has

$$\begin{aligned} & \|\mathcal{G}_{\omega}^m\|_{L^{p_1}(|x|^{\alpha_1} dx) \times \dots \times L^{p_m}(|x|^{\alpha_m} dx) \rightarrow L^p(|x|^{\alpha} dx)} \\ & \simeq \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{(\alpha_i + n)/p_i - n} \right) \omega(\vec{t}) d\vec{t}. \end{aligned} \quad (38)$$

3. Boundedness of $\mathcal{H}_{\omega}^m$ on the Product of Morrey-Herz Spaces

Lets give the second main result of this paper.

Theorem 9. Let $\alpha, \alpha_1, \dots, \alpha_2 \in \mathbb{R}, \lambda, \lambda_i > 0, 1 < p, p_i, q, q_i < \infty, i = 1, 2, \dots, m$, and $\alpha_1 + \dots + \alpha_m = \alpha, 1/q_1 + \dots + 1/q_m = 1/q, 1/p_1 + \dots + 1/p_m = 1/p, \lambda_1 + \dots + \lambda_m = \lambda$. Then $\mathcal{H}_{\omega}^m$ is bounded from $M\dot{K}_{p_1, q_1}^{\alpha_1, \lambda_1} \times \dots \times M\dot{K}_{p_m, q_m}^{\alpha_m, \lambda_m}$ to $M\dot{K}_{p, q}^{\alpha, \lambda}$ if ω satisfies

$$\int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n/q_i - \lambda_i)} \right) \omega(\vec{t}) d\vec{t} < \infty. \quad (39)$$

Conversely, when $\alpha_1 = \dots = \alpha_m = (1/m)\alpha, \lambda_1 = \dots = \lambda_m = (1/m)\lambda, q_i = mq, \text{ and } p_i = mp, i = 1, 2, \dots, m$, if $\mathcal{H}_{\omega}^m$ is bounded from $M\dot{K}_{p_1, q_1}^{\alpha_1, \lambda_1} \times \dots \times M\dot{K}_{p_m, q_m}^{\alpha_m, \lambda_m}$ to $M\dot{K}_{p, q}^{\alpha, \lambda}$, then (39) holds. And in this case, one has

$$\begin{aligned} & \|\mathcal{H}_{\omega}^m\|_{M\dot{K}_{p_1, q_1}^{\alpha_1, \lambda_1} \times \dots \times M\dot{K}_{p_m, q_m}^{\alpha_m, \lambda_m} \rightarrow M\dot{K}_{p, q}^{\alpha, \lambda}} \\ & \simeq \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n/q_i - \lambda_i)} \right) \omega(\vec{t}) d\vec{t}. \end{aligned} \quad (40)$$

For the operator $\mathcal{G}_{\omega}^m$, we have a corresponding result.

Theorem 10. Let $\alpha, \alpha_1, \dots, \alpha_2 \in \mathbb{R}, \lambda, \lambda_i > 0, 1 < p, p_i, q, q_i < \infty, i = 1, 2, \dots, m$, and $\alpha_1 + \dots + \alpha_m = \alpha, 1/q_1 + \dots + 1/q_m = 1/q, 1/p_1 + \dots + 1/p_m = 1/p, \lambda_1 + \dots + \lambda_m = \lambda$. Then $\mathcal{G}_{\omega}^m$ is bounded from $M\dot{K}_{p_1, q_1}^{\alpha_1, \lambda_1} \times \dots \times M\dot{K}_{p_m, q_m}^{\alpha_m, \lambda_m}$ to $M\dot{K}_{p, q}^{\alpha, \lambda}$ if ω satisfies

$$\int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{\alpha_i - \lambda_i - n(1-1/q_i)} \right) \omega(\vec{t}) d\vec{t} < \infty. \quad (41)$$

Conversely, when $\alpha_1 = \dots = \alpha_m = (1/m)\alpha, \lambda_1 = \dots = \lambda_m = (1/m)\lambda, q_i = mq, \text{ and } p_i = mp, i = 1, 2, \dots, m$, if $\mathcal{G}_{\omega}^m$ is bounded from $M\dot{K}_{p_1, q_1}^{\alpha_1, \lambda_1} \times \dots \times M\dot{K}_{p_m, q_m}^{\alpha_m, \lambda_m}$ to $M\dot{K}_{p, q}^{\alpha, \lambda}$, then (41) holds and

$$\begin{aligned} & \|\mathcal{G}_{\omega}^m\|_{M\dot{K}_{p_1, q_1}^{\alpha_1, \lambda_1} \times \dots \times M\dot{K}_{p_m, q_m}^{\alpha_m, \lambda_m} \rightarrow M\dot{K}_{p, q}^{\alpha, \lambda}} \\ & \simeq \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{\alpha_i - \lambda_i - n(1-1/q_i)} \right) \omega(\vec{t}) d\vec{t}. \end{aligned} \quad (42)$$

Since the proof of Theorem 10 is similar to the proof of Theorem 9, we give the proof of Theorem 9 only.

Proof of Theorem 9. By similarity, we only consider the case that $m = 2$. From the proof of Theorem 5, we know that

$$\begin{aligned} & \|\mathcal{H}_{\omega}^2(f_1, f_2) \chi_k\|_{L^q(\mathbb{R}^n)} \\ & \leq \iint_0^1 (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} + \|f_1 \chi_{k+m}\|_{L^{q_1}}) \\ & \quad \times (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}}) \\ & \quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (43)$$

For $1/p = 1/p_1 + 1/p_2$ and $\alpha = \alpha_1 + \alpha_2, \lambda = \lambda_1 + \lambda_2$, Hölder inequality and Minkowski inequality give

$$\begin{aligned} & \|\mathcal{H}_{\omega}^2(f_1, f_2)\|_{M\dot{K}_{p, q}^{\alpha, \lambda}} \\ & = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha p} \|\mathcal{H}_{\omega}^2(f_1, f_2) \chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \\ & \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha p} \right. \\ & \quad \times \left(\iint_0^1 (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} + \|f_1 \chi_{k+m}\|_{L^{q_1}}) \right. \\ & \quad \times (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}}) \\ & \quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \Big)^p \Big)^{1/p} \\ & \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \iint_0^1 \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha p} (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} \right. \\ & \quad \left. + \|f_1 \chi_{k+m}\|_{L^{q_1}})^p \right. \\ & \quad \times (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} + \|f_2 \chi_{k+l}\|_{L^{q_2}})^p \Big)^{1/p} \\ & \quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \iint_0^1 \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha_1 p_1} (\|f_1 \chi_{k+m-1}\|_{L^{q_1}} \right. \\
&\quad \left. + \|f_1 \chi_{k+m}\|_{L^{q_1}})^{p_1} \right)^{1/p_1} \\
&\quad \times \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha_2 p_2} (\|f_2 \chi_{k+l-1}\|_{L^{q_2}} \right. \\
&\quad \left. + \|f_2 \chi_{k+l}\|_{L^{q_2}})^{p_2} \right)^{1/p_2} \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda_1} \iint_0^1 \left\{ \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha_1 p_1} \|f_1 \chi_{k+m-1}\|_{L^{q_1}}^{p_1} \right)^{1/p_1} \right. \\
&\quad \left. + \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha_1 p_1} \|f_1 \chi_{k+m}\|_{L^{q_1}}^{p_1} \right)^{1/p_1} \right\} \\
&\quad \times \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda_2} \left\{ \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha_2 p_2} \|f_2 \chi_{k+l-1}\|_{L^{q_2}}^{p_2} \right)^{1/p_2} \right. \\
&\quad \left. + \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha_2 p_2} \right. \right. \\
&\quad \left. \left. \times \|f_2 \chi_{k+l}\|_{L^{q_2}}^{p_2} \right)^{1/p_2} \right\} \\
&\quad \times t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \|f_1\|_{\dot{M}K_{p_1, q_1}^{\alpha_1, \lambda_1}} \|f_2\|_{\dot{M}K_{p_2, q_2}^{\alpha_2, \lambda_2}} \\
&\quad \times \iint_0^1 \left(2^{-(m-1)\alpha_1} + 2^{-m\alpha_1} \right) \left(2^{-(l-1)\alpha_2} + 2^{-l\alpha_2} \right) \\
&\quad \times t_1^{-(n/q_1 - \lambda_1)} t_2^{-(n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2 \\
&\leq \|f_1\|_{\dot{M}K_{p_1, q_1}^{\alpha_1, \lambda_1}} \|f_2\|_{\dot{M}K_{p_2, q_2}^{\alpha_2, \lambda_2}} \\
&\quad \times \iint_0^1 t_1^{-(\alpha_1 + n/q_1 - \lambda_1)} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{44}$$

It means that $\mathcal{H}_\omega^m$ is bounded from $\dot{M}K_{p_1, q_1}^{\alpha_1, \lambda_1} \times \cdots \times \dot{M}K_{p_m, q_m}^{\alpha_m, \lambda_m}$ to $\dot{M}K_{p, q}^{\alpha, \lambda}$.

On the other hand, we define

$$\begin{aligned}
f_1(x) &= |x|^{-(\alpha_1 + n/q_1 - \lambda_1)}, \quad x \in \mathbb{R}^n, \\
f_2(x) &= |x|^{-(\alpha_2 + n/q_2 - \lambda_2)}, \quad x \in \mathbb{R}^n.
\end{aligned} \tag{45}$$

When $\alpha_1 \neq \lambda_1$ and $\alpha_2 \neq \lambda_2$, we get

$$\begin{aligned}
\|f_1 \chi_k\|_{L^{q_1}} &= \left(\int_{2^{k-1} \leq |x| < 2^k} |x|^{-(\alpha_1 + n/q_1 - \lambda_1)q_1} dx \right)^{1/q_1} \\
&= 2^{-(\alpha_1 - \lambda_1)k} \left| \frac{S_n (2^{(\alpha_1 - \lambda_1)q_1} - 1)}{(\alpha_1 - \lambda_1)q_1} \right|^{1/q_1},
\end{aligned} \tag{46}$$

where $S_n = n\pi^{n/2}/\Gamma(1 + n/2)$. Thus

$$\begin{aligned}
\|f_1\|_{\dot{M}K_{p_1, q_1}^{\alpha_1, \lambda_1}} \\
&= 2^{\lambda_1} \left(\frac{1}{2^{\lambda_1 p_1} - 1} \right)^{1/p_1} \left| \frac{S_n (2^{(\alpha_1 - \lambda_1)q_1} - 1)}{(\alpha_1 - \lambda_1)q_1} \right|^{1/q_1}.
\end{aligned} \tag{47}$$

It is similar to obtain

$$\begin{aligned}
\|f_2\|_{\dot{M}K_{p_2, q_2}^{\alpha_2, \lambda_2}} \\
&= 2^{\lambda_2} \left(\frac{1}{2^{\lambda_2 p_2} - 1} \right)^{1/p_2} \left| \frac{S_n (2^{(\alpha_2 - \lambda_2)q_2} - 1)}{(\alpha_2 - \lambda_2)q_2} \right|^{1/q_2}.
\end{aligned} \tag{48}$$

For $\alpha = \alpha_1 + \alpha_2$, $\lambda = \lambda_1 + \lambda_2$, and $1/q = 1/q_1 + 1/q_2$, we have

$$\begin{aligned}
&\mathcal{H}_\omega^2(f_1, f_2)(x) \\
&= |x|^{-(\alpha + n/q - \lambda)} \iint_0^1 t_1^{-(\alpha_1 + n/q_1 - \lambda_1)} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \\
&\quad \times \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{49}$$

By simple computation, we get

$$\begin{aligned}
&\|\mathcal{H}_\omega^2(f_1, f_2) \chi_k\|_{L^q(\mathbb{R}^n)}^p \\
&= \left(\int_{\mathbb{R}^n} |\mathcal{H}_\omega^2(f_1, f_2)(x)|^q \chi_k(x) dx \right)^{p/q} \\
&= \left(\int_{\mathbb{R}^n} |x|^{-(\alpha + n/q - \lambda)q} \chi_k(x) \right. \\
&\quad \times \left(\iint_0^1 t_1^{-(\alpha_1 + n/q_1 - \lambda_1)} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \right. \\
&\quad \left. \times \omega(t_1, t_2) dt_1 dt_2 \right)^q dx \Big)^{p/q} \\
&= \left(\int_{2^{k-1} \leq |x| < 2^k} |x|^{-(\alpha + n/q - \lambda)q} dx \right)^{p/q} \\
&\quad \times \left(\iint_0^1 t_1^{-(\alpha_1 + n/q_1 - \lambda_1)} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \right. \\
&\quad \left. \times \omega(t_1, t_2) dt_1 dt_2 \right)^p
\end{aligned}$$

$$\begin{aligned}
&= 2^{-(\alpha-\lambda)kp} \left| \frac{S_n (2^{(\alpha-\lambda)q} - 1)}{(\alpha-\lambda)q} \right|^{p/q} \\
&\quad \times \left(\iint_0^1 t_1^{-(\alpha_1+n/q_1-\lambda_1)} t_2^{-(\alpha_2+n/q_2-\lambda_2)} \right. \\
&\quad \left. \times \omega(t_1, t_2) dt_1 dt_2 \right)^p.
\end{aligned} \tag{50}$$

Since $\alpha_1 = \alpha_2 = (1/2)\alpha$, $p_1 = p_2 = 2p$, $q_1 = q_2 = 2q$, and $\lambda_1 = \lambda_2 = (1/2)\lambda$, we have

$$\begin{aligned}
&\| \mathcal{H}_\omega^2(f_1, f_2) \|_{\dot{MK}_{p,q}^{\alpha,\lambda}} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \| \mathcal{H}_\omega^2(f_1, f_2) \chi_k \|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} 2^{-(\alpha-\lambda)kp} \right)^{1/p} \\
&\quad \times \left| \frac{S_n (2^{(\alpha-\lambda)q} - 1)}{(\alpha-\lambda)q} \right|^{1/q} \\
&\quad \times \iint_0^1 t_1^{-(\alpha_1+n/q_1-\lambda_1)} t_2^{-(\alpha_2+n/q_2-\lambda_2)} \\
&\quad \times \omega(t_1, t_2) dt_1 dt_2 \\
&= 2^\lambda \left(\frac{1}{2^{\lambda p} - 1} \right)^{1/p} \left| \frac{S_n (2^{(\alpha-\lambda)q} - 1)}{(\alpha-\lambda)q} \right|^{1/q} \\
&\quad \times \iint_0^1 t_1^{-(\alpha_1+n/q_1-\lambda_1)} t_2^{-(\alpha_2+n/q_2-\lambda_2)} \\
&\quad \times \omega(t_1, t_2) dt_1 dt_2 \\
&= \|f_1\|_{\dot{MK}_{p_1,q_1}^{\alpha_1,\lambda_1}} \|f_2\|_{\dot{MK}_{p_2,q_2}^{\alpha_2,\lambda_2}} \\
&\quad \times \iint_0^1 t_1^{-(\alpha_1+n/q_1-\lambda_1)} t_2^{-(\alpha_2+n/q_2-\lambda_2)} \\
&\quad \times \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{51}$$

Thus, (39) holds in this case.

When $\alpha_1 = \lambda_1$ and $\alpha_2 = \lambda_2$, it is easy to obtain

$$\begin{aligned}
\|f_1 \chi_k\|_{L^{q_1}}^{q_1} &= \|f_2 \chi_k\|_{L^{q_2}}^{q_2} = \int_{2^{k-1} \leq |x| < 2^k} |x|^{-n} dx \\
&= S_n \ln 2.
\end{aligned} \tag{52}$$

Thus, we have

$$\begin{aligned}
\|f_1\|_{\dot{MK}_{p_1,q_1}^{\alpha_1,\lambda_1}} &= 2^{\lambda_1} \left(\frac{1}{2^{\lambda_1 p_1} - 1} \right)^{1/p_1} (S_n \ln 2)^{1/q_1}, \\
\|f_2\|_{\dot{MK}_{p_2,q_2}^{\alpha_2,\lambda_2}} &= 2^{\lambda_2} \left(\frac{1}{2^{\lambda_2 p_2} - 1} \right)^{1/p_2} (S_n \ln 2)^{1/q_2}.
\end{aligned} \tag{53}$$

So

$$\begin{aligned}
&\mathcal{H}_\omega^2(f_1, f_2)(x) \\
&= |x|^{-n/q} \iint_0^1 t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{54}$$

Then, we get

$$\begin{aligned}
&\| \mathcal{H}_\omega^2(f_1, f_2) \chi_k \|_{L^q(\mathbb{R}^n)} \\
&= (S_n \ln 2)^{1/q} \iint_0^1 t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{55}$$

Thus,

$$\begin{aligned}
&\| \mathcal{H}_\omega^2(f_1, f_2) \|_{\dot{MK}_{p,q}^{\alpha,\lambda}} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \| \mathcal{H}_\omega^2(f_1, f_2) \chi_k \|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} (S_n \ln 2)^{p/q} \right. \\
&\quad \times \left(\iint_0^1 t_1^{-n/q_1} t_2^{-n/q_2} \right. \\
&\quad \left. \left. \times \omega(t_1, t_2) dt_1 dt_2 \right)^p \right)^{1/p} \\
&= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \right)^{1/p} (S_n \ln 2)^{1/q} \\
&\quad \times \iint_0^1 t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{56}$$

Since $\alpha_1 = \lambda_1$ and $\alpha_2 = \lambda_2$, we get $\alpha = \lambda$. Thus

$$\begin{aligned}
&\| \mathcal{H}_\omega^2(f_1, f_2) \|_{\dot{MK}_{p,q}^{\alpha,\lambda}} \\
&= 2^\lambda \left(\frac{1}{2^{\lambda p} - 1} \right)^{1/p} (S_n \ln 2)^{1/q} \\
&\quad \times \iint_0^1 t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2 \\
&= \|f_1\|_{\dot{MK}_{p_1,q_1}^{\alpha_1,\lambda_1}} \|f_2\|_{\dot{MK}_{p_2,q_2}^{\alpha_2,\lambda_2}} \\
&\quad \times \iint_0^1 t_1^{-n/q_1} t_2^{-n/q_2} \omega(t_1, t_2) dt_1 dt_2.
\end{aligned} \tag{57}$$

This tells us that (39) also holds in this case.

Furthermore, when one of $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$ holds, we suppose $\alpha_1 = \lambda_1$ but $\alpha_2 \neq \lambda_2$ here. From the above computation, we have

$$\begin{aligned} \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}} &= 2^{\lambda_1} \left(\frac{1}{2^{\lambda_1 p_1} - 1} \right)^{1/p_1} (S_n \ln 2)^{1/q_1}, \\ \|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}} &= 2^{\lambda_2} \left(\frac{1}{2^{\lambda_2 p_2} - 1} \right)^{1/p_2} \left| \frac{S_n (2^{(\alpha_2 - \lambda_2) q_2} - 1)}{(\alpha_2 - \lambda_2) q_2} \right|^{1/q_2}. \end{aligned} \quad (58)$$

It gives

$$\begin{aligned} \mathcal{H}_\omega^2(f_1, f_2)(x) &= |x|^{-(\alpha_2 + (n/q) - \lambda_2)} \int_0^1 \int_0^1 t_1^{-n/q_1} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \\ &\quad \times \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (59)$$

So, we have

$$\begin{aligned} \|\mathcal{H}_\omega^2(f_1, f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}} &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha p} \|\mathcal{H}_\omega^2(f_1, f_2) \chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p} \\ &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k \alpha p} 2^{-(\alpha_2 - \lambda_2) k p} \right)^{1/p} \\ &\quad \times \left| \frac{S_n (2^{(\alpha_2 - \lambda_2) q} - 1)}{(\alpha_2 - \lambda_2) q} \right|^{1/q} \\ &\quad \times \int_0^1 \int_0^1 t_1^{-n/q_1} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2 \\ &= 2^{\lambda} \left(\frac{1}{2^{\lambda p} - 1} \right)^{1/p} \left| \frac{S_n (2^{(\alpha_2 - \lambda_2) q} - 1)}{(\alpha_2 - \lambda_2) q} \right|^{1/q} \\ &\quad \times \int_0^1 \int_0^1 t_1^{-n/q_1} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (60)$$

Since $\alpha_1 = \alpha_2 = (1/2)\alpha$, $p_1 = p_2 = 2p$, $q_1 = q_2 = 2q$, and $\lambda_1 = \lambda_2 = (1/2)\lambda$, using Taylor expansion yields

$$\begin{aligned} &\left| \frac{S_n (2^{(\alpha_2 - \lambda_2) q} - 1)}{(\alpha_2 - \lambda_2) q} \right|^{1/q} \\ &\geq (S_n \ln 2)^{1/q_1} \left| \frac{S_n (2^{(\alpha_2 - \lambda_2) q_2} - 1)}{(\alpha_2 - \lambda_2) q_2} \right|^{1/q_2}. \end{aligned} \quad (61)$$

Thus, we have

$$\begin{aligned} &\|\mathcal{H}_\omega^2(f_1, f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}} \\ &\geq \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}} \|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}} \\ &\quad \times \int_0^1 \int_0^1 t_1^{-n/q_1} t_2^{-(\alpha_2 + n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (62)$$

The above inequality gives

$$\begin{aligned} &\|\mathcal{H}_\omega^m\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1} \times \dots \times M\dot{K}_{p_m,q_m}^{\alpha_m,\lambda_m} \rightarrow M\dot{K}_{p,q}^{\alpha,\lambda}} \\ &\geq \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n/q_i - \lambda_i)} \right) \omega(\vec{t}) d\vec{t}. \end{aligned} \quad (63)$$

Since $\mathcal{H}_\omega^m$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1} \times \dots \times M\dot{K}_{p_m,q_m}^{\alpha_m,\lambda_m}$ to $M\dot{K}_{p,q}^{\alpha,\lambda}$, we know that (39) holds and

$$\begin{aligned} &\|\mathcal{H}_\omega^m\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1} \times \dots \times M\dot{K}_{p_m,q_m}^{\alpha_m,\lambda_m} \rightarrow M\dot{K}_{p,q}^{\alpha,\lambda}} \\ &\simeq \int_{0 < t_1, \dots, t_m < 1} \left(\prod_{i=1}^m t_i^{-(\alpha_i + n/q_i - \lambda_i)} \right) \omega(\vec{t}) d\vec{t}. \end{aligned} \quad (64)$$

The proof of Theorem 9 is complete. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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