Research Article

End-Completely-Regular and End-Inverse Lexicographic Products of Graphs

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A graph $X$ is said to be End-completely-regular (resp., End-inverse) if its endomorphism monoid $\text{End}(X)$ is completely regular (resp., inverse). In this paper, we will show that if $X[Y]$ is End-completely-regular (resp., End-inverse), then both $X$ and $Y$ are End-completely-regular (resp., End-inverse). We give several approaches to construct new End-completely-regular graphs by means of the lexicographic products of two graphs with certain conditions. In particular, we determine the End-completely-regular and End-inverse lexicographic products of bipartite graphs.

1. Introduction and Preliminary Concepts

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years, much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is to develop further relationships between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. The bipartite endomorphism monoids have been obtained. The aim of this research is to develop further relationship between graph theory and algebraic theory of semigroups and to apply the semigroups to graph theory. The bipartite endomorphism monoids have been obtained. The aim of this research is to develop further relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. The bipartite endomorphism monoids have been obtained. The aim of this research is to develop further relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. The bipartite endomorphism monoids have been obtained.

The graphs $X$ considered in this paper are undirected finite simple graphs. The vertex set of $X$ is denoted by $V(X)$ and the edge set of $X$ is denoted by $E(X)$. If two vertices $x_1$ and $x_2$ are adjacent in $X$, the edge connecting $x_1$ and $x_2$ is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. A subgraph $H$ is called an induced subgraph of $X$ if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph $X$ is called bipartite if $X$ has no odd cycle. It is known that if a graph $X$ is a bipartite graph, then its vertex set can be partitioned into two disjoint nonempty subsets such that no edge joins two vertices in the same set.

Let $X$ and $Y$ be two graphs. The join of $X$ and $Y$, denoted by $X + Y$, is a graph such that $V(X + Y) = V(X) \cup V(Y)$ and $E(X + Y) = E(X) \cup E(Y) \cup \{(x, y) \mid x \in V(X), y \in V(Y)\}$. The lexicographic product of $X$ and $Y$, denoted by $X[Y]$, is a graph with vertex set $V(X[Y]) = V(X) \times V(Y)$, and with edge set $E(X[Y]) = \{(x, y, (x_1, y_1)) \mid \{x, x_1\} \in E(X), \text{ or } x = x_1 \text{ and } \{y, y_1\} \in E(Y)\}$. Denote $Y_x = \{(x, y) \mid y \in V(Y)\}$ for any $x \in V(X)$.

Let $X$ and $Y$ be graphs. A mapping $f$ from $V(X)$ to $V(Y)$ is called a homomorphism (from $X$ to $Y$) if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism $f$ is called an isomorphism if $f$ is bijective and $f^{-1}$ is a homomorphism. A homomorphism (resp., isomorphism) $f$ from $X$ to itself is called an endomorphism (resp., automorphism) of $X$ (see [9]). The sets of all endomorphisms and automorphisms...
of $X$ are denoted by $\text{End}(X)$ and $\text{Aut}(X)$, respectively. A graph $X$ is said to be unreactive if $\text{End}(X) = \text{Aut}(X)$. For any $f \in \text{End}(X)$, it is easy to see that $f \in \text{Aut}(X)$ if and only if $f$ is injective.

A retraction of a graph $X$ is a homomorphism $f$ from $X$ to a subgraph $Y$ of $X$ such that the restriction $f|_Y$ of $f$ to $V(Y)$ is the identity mapping on $V(Y)$. In this case, $Y$ is called a retract of $X$. It is known that the idempotents of $\text{End}(X)$ are retracts of $X$. Denote by $\text{Idpt}(X)$ the set of all idempotents of $\text{End}(X)$. Let $f$ be an endomorphism of a graph $X$. A subgraph of $X$ is called the endomorphic image of $X$ under $f$, denoted by $I_f$, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $[c, d] \in E(X)$. By $\rho_f$ we denote the equivalence relation on $V(X)$ induced by $f$; that is, for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to $\rho_f$.

An element $a$ of a semigroup $S$ is called regular if there exists $x \in S$ such that $axa = a$. An element $a$ of a semigroup $S$ is called completely regular if $a = axa$ and $asa = ax$ hold for some $x \in S$. A semigroup $S$ is called regular (resp., completely regular) if all its elements are regular (resp., completely regular). An inverse semigroup is a regular semigroup in which the idempotents commute. A graph $X$ is said to be End-regular (resp., End-completely-regular, End-inverse) if its endomorphism monoid $\text{End}(X)$ is regular (resp., completely regular, inverse). Clearly, End-completely-regular graphs as well as End-inverse graphs are End-regular.

For undefined notation and terminology in this paper, the reader is referred to [9–14]. We list some known results which will be used in the sequel.

**Lemma 1** (see [2]). If $X[Y]$ is End-regular, then both $X$ and $Y$ are End-regular.

**Lemma 2** (see [15]). Let $G$ be a graph and let $f \in \text{End}(G)$. Then $f$ is completely regular if and only if $f|_{I_f} \in \text{Aut}(I_f)$.

**Lemma 3** (see [15]). Let $X$ be a bipartite graph. Then $X$ is End-completely-regular if and only if $X$ is one of $K_1, K_2, P_2, 2K_1, 2K_2$, and $K_1 \cup K_2$.

**Lemma 4** (see [4]). Let $X$ and $Y$ be two bipartite graphs. Then $X + Y$ is End-completely-regular if and only if any of them is End-completely-regular and the other is $K_1$ or $K_2$.

**Lemma 5** (see [16]). Let $G$ be a graph and $f \in \text{End}(G)$. Then $f$ is completely regular if and only if there exists $g \in \text{Idpt}(G)$ such that $\rho_g = \rho_f$ and $I_g = I_f$.

**Lemma 6** (see [4]). Let $X$ be a bipartite graph. Then $X$ is End-inverse if and only if $X = K_1$ or $X = K_2$.

**Lemma 7** (see [17]). Let $X$ and $Y$ be two graphs. Then $\text{End}(X[Y]) = \text{End}(X)[\text{End}(Y)]$ if and only if for any $f \in \text{End}(X[Y])$ and $x \in V(X)$, there exists $x' \in V(X)$ such that $f(V(x)) \subseteq Y_{x'}$.

### 2. Main Results

In this section, we will characterize the End-completely-regular and End-inverse lexicographic products of two graphs. We first show that if $X[Y]$ is End-completely-regular, then both $X$ and $Y$ are End-completely-regular.

**Theorem 8.** Let $X$ and $Y$ be two graphs. If $X[Y]$ is End-completely-regular, then both $X$ and $Y$ are End-completely-regular.

**Proof.** By Lemma 2, to show that $X$ is End-completely-regular, it is only necessary to verify that $f|_{I_f}$ is an automorphism of $I_f$ for each $f \in \text{End}(X)$. Define a mapping $F$ from $V(X[Y])$ to itself by $F((x, y)) = (f(x), y) \forall (x, y) \in V(X[Y])$. (1)

Then $F \in \text{End}(X[Y])$. Since $X[Y]$ is End-completely-regular, by Lemma 2, $F|_{I_f}$ is an automorphism of $I_f$. It is easy to see that $I_f = I_f[Y]$. For any distinct $x_1, x_2 \in V(I_f)$ and $y \in V(Y)$, $F((x_1, y)) = (f(x_1), y)$ and $F((x_2, y)) = (f(x_2), y)$ hold. Since $F|_{I_f}$ is an automorphism of $I_f$, $(f(x_1), y) \neq (f(x_2), y)$. Hence $f(x_1) \neq f(x_2)$ and so $f|_{I_f}$ is an automorphism of $I_f$.

Let $g \in \text{End}(Y)$. Define a mapping $G$ from $V(X[Y])$ to itself by $G((x, y)) = (x, g(y)) \forall (x, y) \in V(X[Y])$. (2)

Then $G \in \text{End}(X[Y])$. Since $X[Y]$ is End-completely-regular, by Lemma 2, $G|_{I_f}$ is an automorphism of $I_f$. It is easy to see that $I_f = X[I_f]$. For any $x \in V(X)$ and $y_1, y_2 \in V(I_f)$, $G((x, y_1)) = (x, g(y_1))$ and $G((x, y_2)) = (x, g(y_2))$. Since $G|_{I_f}$ is an automorphism of $I_f$, $(x, g(y_1)) \neq (x, g(y_2))$. Since $G|_{I_f}$ is an automorphism of $I_f$, $(x, g(y_1)) \neq (x, g(y_2))$, we get that $g(y_1) \neq g(y_2)$ and so $g|_{I_f}$ is an automorphism of $I_f$, as required.

The following example shows that $X$ and $Y$ being End-completely-regular does not yield that $X[Y]$ is End-completely-regular.

**Example 9.** Let $X$ and $Y$ be two bipartite graphs with $V(X) = \{x_1, x_2\}, V(Y) = \{y_1, y_2\}$, $E(X) = \{\{x_1, x_2\}\}$, and $E(Y) = \emptyset$. By Lemma 3, $X$ and $Y$ are End-completely-regular. It is easy to see that $X[Y] \cong C_4$. Also by Lemma 3, this is not End-completely-regular.

In the following, we give some sufficient conditions for $X[Y]$ to be End-completely-regular. To this aim, we need the following result due to Fan [17].

**Lemma 10** (see [17]). Let $X$ and $Y$ be two $K_3$-free connected graphs. If girth($X$) or girth($Y$) is odd, then $\text{End}(X[Y]) = \text{End}(X)[\text{End}(Y)]$, where $\text{End}(X)[\text{End}(Y)]$ is the wreath product of the monoids $\text{End}(X)$ and $\text{End}(Y)$.

Let $X$ and $Y$ be two $K_3$-free connected graphs such that girth($X$) or girth($Y$) is odd. In [2], Fan proved that if both of $X$ and $Y$ are End-regular and one of them is unreactive, then $X[Y]$ is End-regular. Here we prove that if $X$ is an End-completely-regular graph and $Y$ is an unreactive graph, then $X[Y]$ is End-completely-regular.
Theorem 11. Let $X$ and $Y$ be two $K_3$-free connected graphs with girth($X$) or girth($Y$) being odd, and assume that

1. $X$ is End-completely-regular,
2. $Y$ is unretractive.

Then $X[Y]$ is End-completely-regular.

Proof. Let $X$ and $Y$ be two graphs satisfying the assumptions. To show that $X[Y]$ is End-completely-regular, we prove that for any $F \in \text{End}(X[Y])$, there exists an idempotent endomorphism $G \in \text{End}(X[Y])$ such that $\rho_F = \rho_G$ and $I_G = I_F$.

Let $F \in \text{End}(X)[\text{End}(Y)]$. Since $\text{End}(X[Y]) = \text{End}(X)[\text{End}(Y)]$, $F = (s, f)$ for some $s \in \text{End}(X)$ and $f \in \text{End}(Y)^X$. Thus, for any $u \in V(X)$, there exists $f_u = f(u) \in \text{End}(X)$. Let $X$ and $Y$ be $K_3$-free connected graphs with girth($X$) or girth($Y$) being odd, by Lemma 7, for any $u \in V(X)$, $F(Y_u) \subseteq Y_u$ for some $v \in V(X)$. Note that $Y$ is unretractive. Then $F(Y_u) = Y_v$. Since $X$ is End-completely-regular and $s \in \text{End}(X)$, by Lemma 5, there exists $t \in \text{Idpt}(X)$ such that $\rho_t \rho_s = \rho_s$ and $I_t = I_s$. Clearly, $I_t$ is an induced subgraph of $X$. Hence $I_t = I_s[Y]$ is an induced subgraph of $X[Y]$.

Since $X$ is End-completely-regular, $s|_I$ is an automorphism of $I$. Thus for any $u \in I$, there is a unique vertex $u_1 \in I$ such that $u = (s|_I)(u_1)$. Define a mapping $G$ from $V(X[Y])$ to itself in the following way. If $(x, y) \in V(I_F)$, then $G((x, y)) = (x, y)$; if $(x, y) \notin V(I_F)$, then $F((x, y)) = (u, v)$ for some $u \in V(I_F)$. Now let $G((x, y)) = (u_1, v_1)$, where $(u_1, v_1)$ is the only vertex in $V(I_F)$ such that $(u_1, v_1) = (u, v)$. Then it is easy to see that $G$ is well-defined. Let $(x, y) \in V(X[Y])$. If $(x, y) \in V(I_F)$, then $t(x) = x$. Hence $G((x, y)) = (x, y) \in Y_{t(x)}$. If $(x, y) \notin V(I_F)$, then $t(x) = u_1$. Hence $G((x, y)) = (u_1, v_1) \in Y_{t(x)}$. Therefore, $G((x, y)) = (u_1, v_1) \in Y_{t(x)}$ for any $(x, y) \in V(X[Y])$.

Case 1. Assume that $(x_1, y_1)$, $(x_2, y_2) \in V(X[Y])$ be such that $[x_1, y_1], [x_2, y_2] \in E(X[Y])$. If $(x_1, y_1), (x_2, y_2) \in V(I_F)$, then $G((x_1, y_1)), G((x_2, y_2)) \in [x_1, y_1], [x_2, y_2] \in E(X[Y])$. If $(x_1, y_1) \notin V(I_F)$ and $(x_2, y_2) \notin V(I_F)$, then $x_1 \neq x_2$ and $x_1, x_2 \in E(X)$. Thus $G((x_1, y_1)) \in Y_{t(x_1)}$ and $G((x_2, y_2)) \in Y_{t(x_2)}$. Since $t \in \text{Idpt}(X)$ and $[x_1, x_2] \in E(X)$, $t([x_1, x_2]) \in E(X)$. Hence $G([x_1, y_1]), G([x_2, y_2]) \in E(X[Y])$. If $(x_1, y_1) \notin V(I_F)$ and $(x_2, y_2) \notin V(I_F)$, there are two cases.

Case 2. Assume that $x_1 = x_2$ and $y_1 \neq y_2 \in E(Y)$. Then we have $(x_1, y_1), (x_2, y_2) \in Y_{x_1}$ and $G((x_1, y_1)), G((x_2, y_2)) \in Y_{x_1}$. Since $G|_{Y_{x_1}}$ is an isomorphism from $Y_{x_1}$ to $Y_{x_1}$, $G(x_1, y_1)) = G((x_2, y_2)) \in E(X[Y])$. Therefore, $G \in \text{End}(X[Y])$.

If $(x, y) \in V(I_F)$, then $\text{G}^2((x, y)) = \text{G}((x, y)) = (x, y)$. If $(x, y) \notin V(I_F)$, then $\text{G}((x, y)) \notin V(I_F)$. Thus $\text{G}^2((x, y)) = \text{G}((x, y)) \in \text{End}(X[Y])$. Clearly, $I_G = I_F$.

Suppose $[x_1, y_1], p \in \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ for some $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in V(X[Y])$. In fact, it is easy to prove that $\rho_t \rho_s \subseteq \rho_s$. Let $(x_1, y_1)p(x_2, y_2)$. Then, by the definition of $G$, we have $G(x_1, y_1) = G((x_2, y_2)) = (u_1, v_1)$ and $G(x_2, y_2) = (u_2, v_2) \in V(I_F)$, and $G((u_1, v_1), (u_2, v_2)) = (u_1, v_1) \in F(x_1, y_1)$. So $(x_1, y_1)p(x_2, y_2)$ and thus $\rho_F \subseteq \rho_G$. Hence $\rho_F = \rho_G$.

Next we start to seek the conditions for bipartite graphs $X$ and $Y$ under which $X[Y]$ is End-completely-regular.

Lemma 12. Let $X$ be a graph and $R$ be a retract of $X$. If $R[Y]$ is not End-completely-regular, then $X[Y]$ is not End-completely-regular.

Proof. Let $R$ be a retract of $X$. Then there exists $f \in \text{Idpt}(X)$ such that $I_f = R$. Let $g \in \text{End}(R[Y])$. Since $R[Y]$ is not End-completely-regular, there exists $g \in \text{End}(R[Y])$ such that $g$ is not completely regular. By Lemma 2, $g|_R$ is not an automorphism of $I_g$. Thus there exist $x_1, x_2 \in I_g$ with $x_1 \neq x_2$ such that $g(x_1) = g(x_2)$. Define a mapping $F$ from $V(X[Y])$ to itself by

$$F((x, y)) = (f(x), y) \quad \forall (x, y) \in V(X[Y]).$$

Then $F \in \text{End}(X[Y])$ and $I_F = I_{g|_R}$. Now it is easy to see that $gF \in \text{End}(X[Y])$ and $I_{gF} = I_{g|_R}$. It follows from $(gF)(x_1) = (gF)(x_2)$ that $(gF)|_{I_{g|_R}}$ is not an automorphism of $I_{g|_R}$. Hence $X[Y]$ is not End-completely-regular.

Lemma 13. Let $X$ and $Y$ be two graphs. If at least one of $X$ and $Y$ is not End-completely-regular, then $X \cup Y$ is not End-completely-regular (where $X \cup Y$ is the disjoint union of $X$ and $Y$).

Proof. Without loss of generality, we may suppose that $X$ is not End-completely-regular. By Lemma 2, there exists $f \in \text{End}(X)$ such that $f|_{I_f}$ is not an automorphism of $I_f$. Define a mapping $F$ from $V(X \cup Y)$ to itself by

$$F(x) = \begin{cases} f(x), & x \in V(X), \\ x, & x \in V(Y). \end{cases}$$

Then $F \in \text{End}(X \cup Y)$. Now it is easy to see that $I_F = I_f \cup Y$ and $F(x) = f(x)$ for any $x \in V(X)$. Since $F|_{I_f}$ is not an automorphism of $I_f$, $F|_{I_f}$ is not an automorphism of $I_f$. Hence $X \cup Y$ is not End-completely-regular.

Theorem 14. Let $X$ and $Y$ be two bipartite graphs. Then $X[Y]$ is End-completely-regular if and only if

1. $X = K_1$ and $Y$ is End-completely-regular or
2. $X$ is End-completely-regular and $Y = K_1$ or $K_2$. 

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Proof

Sufficiency. Since $K_1[Y] = Y$ and $X[K_1] = X$, we have immediately that $K_1[Y](X[K_1])$ is End-completely-regular if and only if $Y(X)$ is End-completely-regular. If $X = Y = K_2$, then $X[Y] = K_4$. Thus $\text{End}(X[Y])$ is a group. Since any group is a completely regular semigroup, $X[Y]$ is End-completely-regular. If $X = 2K_2$ and $Y = K_2$, then $X[Y] = 2K_2$. By Lemma 3, it is End-completely-regular. In the following, we show that $X[Y]$ is End-completely-regular for the following cases (see Figure 1).

Case 1. $X = P_4$ and $Y = K_2$. Let $f \in \text{End}(X[Y])$. If $[x]_{P_4} = \{x_1\}$, then $[x]_{P_4} = \{x_2\}$. Otherwise, $f(x_1) = f(z_1) = f(z_2)$. Without loss of generality, we can suppose $f(x_1) = f(z_1)$. Since $z_1$ is adjacent to every vertex of $\{z_2, y_1, y_2\}$ and $\{x_1, x_2\} \in E(X[Y])$, $f(z_1)$ is adjacent to every vertex of $\{f(z_2), f(y_1), f(y_2), f(x_1)\}$. Note that there is no vertex in $X[Y]$ adjacent to 4 vertices. This is a contradiction. Hence $f \in \text{Aut}(X[Y])$ and so $f$ is completely regular. If $[x]_{P_4} \neq \{x_1\}$, then $f(x_1) = f(z_1)$ or $f(x_1) = f(z_2)$. Without loss of generality, we can suppose $f(x_1) = f(z_1)$. Then $f(x_1) = f(z_1)$. Otherwise, a similar argument as above will show that $f(z_1)$ is adjacent to every vertex of $\{f(z_2), f(y_1), f(y_2), f(z_1)\}$, which yield a contradiction. Thus $I_f \equiv K_4$. Since any endomorphism $f$ maps a clique to a clique of the same size, $f(t_4) = t_4$. By Lemma 2, $f$ is completely regular. Hence $P_2[K_2]$ is End-completely-regular.

Case 2. $X = K_1 \cup K_2$ and $Y = K_2$. Let $f \in \text{End}(X[Y])$. If $[c_1]_{P_1} = \{c_1\}$, then $[c_2]_{P_1} = \{c_2\}$. Otherwise, $f(c_2) \in \{f(a_1), f(a_2), f(b_1), f(b_2)\}$. Without loss of generality, we can suppose $f(c_2) = f(a_1)$. Since any endomorphism $f$ maps a clique to a clique of the same size and there is only one clique of size 4 in $Y[Y]$, $\{f(a_1), f(a_2), f(b_1), f(b_2)\} = \{a_1, a_2, b_1, b_2\}$. Note that $c_1, c_2 \in E(X[Y])$. Then $\{f(c_1), f(c_2)\} \in E(X[Y])$. Thus $f(c_1) \in \{f(a_1), f(a_2), f(b_1), f(b_2)\}$, which is a contradiction. Clearly, $[x]_{P_1} = [x]$ for any $x \in \{a_1, a_2, b_1, b_2\}$. Hence $f \in \text{Aut}(X[Y])$ and so $f$ is completely regular. If $[c_1]_{P_1} \neq \{c_1\}$, then $f(c_1) = f(t)$ for some $t \in \{a_1, a_2, b_1, b_2\}$. Without loss of generality, we can suppose $f(c_1) = f(a_1)$. Then $f(c_2) \in \{f(a_1), f(b_1), f(b_2)\}$. Thus $I_f \equiv K_4$. Hence $f(t_4) = t_4$ and $f$ is completely regular. Consequently, $(K_1 \cup K_2)[K_2]$ is End-completely-regular.

Case 3. $X = 2K_2$ and $Y = K_2$. Let $f \in \text{End}(X[Y])$. If $[x]_{P_4} = [x]$ for any $x \in V(X[Y])$, then $f \in \text{Aut}(X[Y])$ and so $f$ is completely regular. If $f(x) = f(y)$ for some $x, y \in V(X[Y])$ with $x \neq y$, without loss of generality, we can suppose $f(a_1) = f(c_1)$. Since $b_1, b_2, c_1, c_2$ is a clique of size 4 in $X[Y]$, $f(b_1), f(b_2), f(c_1), f(c_2)$ is also a clique of size 4 in $X[Y]$. Note that $a_2, d_1, d_2$ are adjacent to $a_1$. Then $f(a_1), f(d_1), f(d_2)$ are adjacent to $f(a_1) = f(c_1)$. Thus $f(a_1), f(d_1), f(d_2) \in \{f(b_1), f(b_2), f(c_1), f(c_2)\}$ and $I_f \equiv K_4$. Hence $f(t_4) = t_4$ and $f$ is completely regular. Consequently, $(2K_2)[K_2]$ is End-completely-regular.

Necessity. We only need to show that $X[Y]$ is not End-completely-regular in the following cases.

Case 1. $(X = K_2)$. Then $X[Y] = Y + Y$. By Lemma 4, $K_2[Y]$ is not End-completely-regular for the corresponding $Y$.

Case 2. $(X = P_2)$. Then $K_2$ is a retract of $X$. Since $K_2[Y]$ is not End-completely-regular for $Y = P_2, 2K_1, 2K_2, K_1 \cup K_2$, by Lemma 1, $P_2[Y]$ is not End-completely-regular for the corresponding $Y$.

Case 3. $(X = 2K_1)$. Then $X[Y] = 2Y$. If $Y$ is bipartite, then $X[Y]$ is also bipartite. By Lemma 3, $(2K_1)[Y]$ is not End-completely-regular for the corresponding $Y$.

Case 4. $(X = 2K_2)$. Then $X[Y] = 2(Y + Y)$. Since $Y + Y$ is not End-completely-regular for $Y = P_2, 2K_1, 2K_2, K_1 \cup K_2$, by Lemma 13, $(2K_2)[Y]$ is not End-completely-regular for the corresponding $Y$.

Case 5. $(X = K_1 \cup K_2)$. Then $X[Y] = Y \cup (Y + Y)$. Since $Y + Y$ is not End-completely-regular for $Y = P_2, 2K_1, 2K_2, K_1 \cup K_2$, by Lemma 13, $(K_1 \cup K_2)[Y]$ is not End-completely-regular for the corresponding $Y$.

Next we start to seek the conditions for a lexicographic product of bipartite graphs $X$ and $Y$ under which $X[Y]$ is End-inverse.

Theorem 15. Let $X$ and $Y$ be two graphs. If $X[Y]$ is End-inverse, then both $X$ and $Y$ are End-inverse.

Proof. Since $X[Y]$ is End-inverse, $X[Y]$ is End-regular. By Lemma 1, both $X$ and $Y$ are End-regular. To show that $X$ is End-inverse, we only need to prove that the idempotents of $\text{End}(X)$ commute.

Let $f_1$ and $f_2$ be two idempotents in $\text{End}(X)$. Define two mappings $g_1$ and $g_2$ from $V(X[Y])$ to itself by

$$g_1((x, y)) = (f_1(x), y) \quad \forall (x, y) \in V(X[Y]),$$

$$g_2((x, y)) = (f_2(x), y) \quad \forall (x, y) \in V(X[Y]).$$

Then $g_1$ and $g_2$ are two idempotents of $\text{End}(X[Y])$ and so $g_2g_1 = g_1g_2$, since $X[Y]$ is End-inverse. For any $(x, y) \in V(X[Y])$, we have

$$g_1g_2((x, y)) = g_1((f_2(x), y)) = ((f_1f_2)(x), y) = (g_2g_1)((x, y)) = ((f_2f_1)(x), y).$$

Clearly, $f_1f_2 = f_2f_1$. Hence $X$ is End-inverse.
Similarly, let \( f_3 \) and \( f_4 \) be two idempotents in \( \text{End}(Y) \). Define two mappings \( g_3 \) and \( g_4 \) from \( V(X[Y]) \) to itself by

\[
\begin{align*}
g_3((x, y)) &= (x, f_3(y)) \quad \forall (x, y) \in V(X[Y]), \\
g_4((x, y)) &= (x, f_4(y)) \quad \forall (x, y) \in V(X[Y]).
\end{align*}
\]

(7)

Then \( g_3 \) and \( g_4 \) are two idempotents of \( \text{End}(X[Y]) \) and so \( g_3g_4 = g_4g_3 \), since \( X[Y] \) is End-inverse. For any \((x, y) \in V(X[Y])\), we have

\[
(g_3g_4)((x, y)) = (x, (f_3f_4)(y)) = (g_4g_3)((x, y)) = (x, (f_4f_3)(y)).
\]

(8)

Clearly, \( f_3f_4 = f_4f_3 \). Hence \( Y \) is End-inverse, as required.

The next theorem characterizes the End-inverse lexicographic products of bipartite graphs.

**Theorem 16.** Let \( X \) and \( Y \) be two bipartite graphs. Then \( X[Y] \) is End-inverse if and only if \( X[Y] \) is one of \( K_1[K_1], K_1[K_2], K_2[K_1], \) and \( K_2[K_2] \).

**Proof**

**Necessity.** This follows directly from Lemma 6 and Theorem 15.

**Sufficiency.** It is easy to see that \( K_1[K_1] = K_1, K_1[K_2] = K_2[K_1] = K_2, \) and \( K_2[K_2] = K_4 \) are End-inverse, since they are unretractive.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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