Research Article

Fixed Points of Contractive Mappings in b -Metric-Like Spaces

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We discuss topological structure of b -metric-like spaces and demonstrate a fundamental lemma for the convergence of sequences. As an application we prove certain fixed point results in the setup of such spaces for different types of contractive mappings. Finally, some periodic point results in b -metric-like spaces are obtained. Two examples are presented in order to verify the effectiveness and applicability of our main results.

1. Introduction

There are a lot of generalizations of the concept of metric space. The concepts of b -metric space and partial metric space were introduced by Czerwik [1] and Matthews [2], respectively. Combining these two notions, Shukla [3] introduced another generalization which is called a partial b -metric space.

On the other hand, Amini-Harandi [4] introduced a new extension of the concept of partial metric space, called a metric-like space. The concept of b -metric-like space which generalizes the notions of partial metric space, metric-like space, and b -metric space was introduced by Alghamdi et al. in [5]. They established the existence and uniqueness of fixed points in a b -metric-like space as well as in a partially ordered b -metric-like space. In addition, as an application, they derived some new fixed point and coupled fixed point results in partial metric spaces, metric-like spaces, and b -metric spaces (see also [6–14]).

The aim of this paper is to examine more closely the topological structure of these spaces. In this context, we demonstrate a fundamental lemma for the convergence of sequences in b -metric-like spaces and by using it we prove some fixed point results in the setup of such spaces. Finally, some periodic point results in b -metric-like spaces

are obtained. Two examples are presented in order to verify the effectiveness and applicability of our main results.

2. Preliminaries

2.1. Definitions and Basic Properties of Certain Types of Spaces. To clarify the issue, we first recall definitions of b -metric, partial metric, partial b -metric, and metric-like spaces.

Definition 1 (see [1]). Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is a b -metric if, for all $x, y, z \in X$, the following conditions are satisfied:

- (b₁) $d(x, y) = 0$ if and only if $x = y$,
- (b₂) $d(x, y) = d(y, x)$,
- (b₃) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b -metric space with coefficient s .

Definition 2 (see [2]). A partial metric on a nonempty set X is a mapping $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$

- (p₁) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
- (p₂) $p(x, x) \leq p(x, y)$,
- (p₃) $p(x, y) = p(y, x)$,
- (p₄) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X .

Definition 3 (see [3]). A partial b -metric on a nonempty set X is a mapping $p_b : X \times X \rightarrow \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $x, y, z \in X$

- (p_{b1}) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$,
- (p_{b2}) $p_b(x, x) \leq p_b(x, y)$,
- (p_{b3}) $p_b(x, y) = p_b(y, x)$,
- (p_{b4}) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

A partial b -metric space is a pair (X, p_b) such that X is a nonempty set and p_b is a partial b -metric on X . The number s is called the coefficient of (X, p_b) .

Definition 4 (see [4]). A metric-like on a nonempty set X is a mapping $\sigma : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$

- $(\sigma 1)$ $\sigma(x, y) = 0$ implies $x = y$,
- $(\sigma 2)$ $\sigma(x, y) = \sigma(y, x)$,
- $(\sigma 3)$ $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

Every partial metric space is a metric-like space. Below we give some other examples of metric-like spaces.

Example 5 (see [15]). Let $X = [0, 1]$. Then the mapping $\sigma_1 : X \times X \rightarrow \mathbb{R}^+$ defined by $\sigma_1(x, y) = x + y - xy$ is a metric-like on X .

Example 6 (see [15]). Let $X = \mathbb{R}$; then the mappings $\sigma_i : X \times X \rightarrow \mathbb{R}^+$ ($i \in \{2, 3, 4\}$), defined by

$$\begin{aligned}\sigma_2(x, y) &= |x| + |y| + a, \\ \sigma_3(x, y) &= |x - b| + |y - b|, \\ \sigma_4(x, y) &= x^2 + y^2,\end{aligned}\quad (1)$$

are metric-like on X , where $a \geq 0$ and $b \in \mathbb{R}$.

Definition 7 (see [5]). Let X be a nonempty set and $s \geq 1$ a given real number. A function $\sigma_b : X \times X \rightarrow \mathbb{R}^+$ is b -metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

- $(\sigma_b 1)$ $\sigma_b(x, y) = 0$ implies $x = y$,
- $(\sigma_b 2)$ $\sigma_b(x, y) = \sigma_b(y, x)$,
- $(\sigma_b 3)$ $\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.

A b -metric-like space is a pair (X, σ_b) such that X is a nonempty set and σ_b is b -metric-like on X . The number s is called the coefficient of (X, σ_b) .

In a b -metric-like space (X, σ_b) if $x, y \in X$ and $\sigma_b(x, y) = 0$, then $x = y$, but the converse may not be true and $\sigma_b(x, x)$ may be positive for $x \in X$. It is clear that every partial b -metric space is a b -metric-like space with the same coefficient

s and every b -metric space is also a b -metric-like space with the same coefficient s . However, the converses of these facts need not hold.

Example 8. Let $X = \mathbb{R}^+$, $p > 1$ a constant, and $\sigma_b : X \times X \rightarrow \mathbb{R}^+$ be defined by

$$\sigma_b(x, y) = (x + y)^p, \quad \forall x, y \in X. \quad (2)$$

Then (X, σ_b) is a b -metric-like space with coefficient $s = 2^{p-1}$, but it is not a partial b -metric space. Indeed, for any $0 < y < x$ we have $\sigma_b(x, x) = (x + x)^p > (x + y)^p = \sigma_b(x, y)$, so (p_{b2}) of Definition 3 is not satisfied.

The following propositions help us to construct some more examples of b -metric-like spaces.

Proposition 9. Let (X, σ) be a metric-like space and $\sigma_b(x, y) = [\sigma(x, y)]^p$, where $p > 1$ is a real number. Then σ_b is b -metric-like with coefficient $s = 2^{p-1}$.

Proof. The proof follows from the fact that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, where $a, b \in \mathbb{R}^+$. $\square$

From the above proposition and Examples 5 and 6 we have the following examples of b -metric-like spaces.

Example 10. Let $X = [0, 1]$. Then the mapping $\sigma_{b1} : X \times X \rightarrow \mathbb{R}^+$ defined by $\sigma_{b1}(x, y) = (x + y - xy)^p$, where $p > 1$ is a real number, is b -metric-like on X with coefficient $s = 2^{p-1}$.

Example 11. Let $X = \mathbb{R}$. Then the mappings $\sigma_{bi} : X \times X \rightarrow \mathbb{R}^+$ ($i \in \{2, 3, 4\}$), defined by

$$\begin{aligned}\sigma_{b2}(x, y) &= (|x| + |y| + a)^p, \\ \sigma_{b3}(x, y) &= (|x - b| + |y - b|)^p, \\ \sigma_{b4}(x, y) &= (x^2 + y^2)^p,\end{aligned}\quad (3)$$

are b -metric-like on X , where $p > 1$, $a \geq 0$, and $b \in \mathbb{R}$.

Proposition 12. Let X be a nonempty set such that d and p_b are b -metric and partial b -metric, respectively, $s > 1$, and σ is a metric-like on X . Then the mappings $\sigma_{bi} : X \times X \rightarrow \mathbb{R}^+$ ($i \in \{5, 6, 7\}$), defined by

$$\begin{aligned}\sigma_{b5}(x, y) &= p_b(x, y) + d(x, y), \\ \sigma_{b6}(x, y) &= \sigma(x, y) + p_b(x, y), \\ \sigma_{b7}(x, y) &= \sigma(x, y) + d(x, y),\end{aligned}\quad (4)$$

for all $x, y \in X$ are b -metric-like on X .

Proof. Let (X, p_b) be a partial b -metric space and (X, d) a b -metric space with $s > 1$. Then conditions $(\sigma_b 1)$, $(\sigma_b 2)$, and $(\sigma_b 3)$ are obvious for the function σ_{b5} . For instance, if

$x, y, z \in X$ are arbitrary then, as p_b is partial b -metric and d is b -metric on X , we have

$$\begin{aligned}\sigma_{b5}(x, y) &= p_b(x, y) + d(x, y) \\ &\leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z) \\ &\quad + s[d(x, z) + d(z, y)] \\ &\leq s[p_b(x, z) + d(x, z) + p_b(z, y) + d(z, y)] \\ &= s[\sigma_{b5}(x, z) + \sigma_{b5}(z, y)].\end{aligned}\quad (5)$$

Therefore, $(\sigma_b 3)$ is satisfied and so (X, σ_{b5}) is a b -metric-like space. Similarly, one can show that (X, σ_{b6}) and (X, σ_{b7}) are b -metric-like spaces. $\square$

From the above proposition and Examples 5 and 6 we have the following examples.

Example 13. Let $X = [0, 1]$. Then the mapping $\sigma_{b7} : X \times X \rightarrow \mathbb{R}^+$ defined by $\sigma_{b7}(x, y) = x + y - xy + |x - y|^p$, where $p > 1$ is a real number, is b -metric-like on X with coefficient $s = 2^{p-1}$.

Example 14. Let $X = \mathbb{R}$. Then the mappings $\sigma_{bi} : X \times X \rightarrow \mathbb{R}^+$ ($i \in \{8, 9, 10\}$), defined by

$$\begin{aligned}\sigma_{b8}(x, y) &= |x| + |y| + a + |x - y|^p, \\ \sigma_{b9}(x, y) &= |x - b| + |y - b| + |x - y|^p, \\ \sigma_{b10}(x, y) &= x^2 + y^2 + |x - y|^p,\end{aligned}\quad (6)$$

are b -metric-like on X with coefficient $s = 2^{p-1}$, where $p > 1$, $a \geq 0$, and $b \in \mathbb{R}$.

Each b -metric-like σ_b on X generates a topology τ_{σ_b} on X whose base is the family of all open σ_b -balls $\{B_{\sigma_b}(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_{\sigma_b}(x, \varepsilon) = \{y \in X : |\sigma_b(x, y) - \sigma_b(x, x)| < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Now, we define the concepts of Cauchy sequence and convergent sequence in a b -metric-like space.

Definition 15 (see [5]). Let (X, σ_b) be a b -metric-like space with coefficient s , and let $\{x_n\}$ be any sequence in X and $x \in X$. Then

- (i) the sequence $\{x_n\}$ is said to be convergent to x with respect to τ_{σ_b} , if $\lim_{n \rightarrow \infty} \sigma_b(x_n, x) = \sigma_b(x, x)$;
- (ii) the sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, σ_b) if $\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m)$ exists and is finite;
- (iii) (X, σ_b) is said to be a complete b -metric-like space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = \lim_{n \rightarrow \infty} \sigma_b(x_n, x) = \sigma_b(x, x). \quad (7)$$

It is clear that the limit of a sequence in a b -metric-like space is usually not unique (since already partial metric spaces share this property).

We start our work by proving the following crucial lemma.

Lemma 16. Let (X, σ_b) be a b -metric-like space with coefficient $s > 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y , respectively. Then one has

$$\begin{aligned}\frac{1}{s^2} \sigma_b(x, y) - \frac{1}{s} \sigma_b(x, x) - \sigma_b(y, y) \\ \leq \liminf_{n \rightarrow \infty} \sigma_b(x_n, y_n) \leq \limsup_{n \rightarrow \infty} \sigma_b(x_n, y_n) \\ \leq s \sigma_b(x, x) + s^2 \sigma_b(y, y) + s^2 \sigma_b(x, y).\end{aligned}\quad (8)$$

In particular, if $\sigma_b(x, y) = 0$, then one has $\lim_{n \rightarrow \infty} \sigma_b(x_n, y_n) = 0$.

Moreover, for each $z \in X$ one has

$$\begin{aligned}\frac{1}{s} \sigma_b(x, z) - \sigma_b(x, x) \\ \leq \liminf_{n \rightarrow \infty} \sigma_b(x_n, z) \leq \limsup_{n \rightarrow \infty} \sigma_b(x_n, z) \\ \leq s \sigma_b(x, z) + s \sigma_b(x, x).\end{aligned}\quad (9)$$

In particular, if $\sigma_b(x, x) = 0$, then

$$\begin{aligned}\frac{1}{s} \sigma_b(x, z) \leq \liminf_{n \rightarrow \infty} \sigma_b(x_n, z) \\ \leq \limsup_{n \rightarrow \infty} \sigma_b(x_n, z) \leq s \sigma_b(x, z).\end{aligned}\quad (10)$$

Proof. Using the triangle inequality in a b -metric-like space it is easy to see that

$$\begin{aligned}\sigma_b(x, y) \leq s \sigma_b(x, x_n) + s^2 \sigma_b(x_n, y_n) + s^2 \sigma_b(y_n, y), \\ \sigma_b(x_n, y_n) \leq \sigma_b(x_n, x) + s^2 \sigma_b(x, y) + s^2 \sigma_b(y, y_n).\end{aligned}\quad (11)$$

Taking the lower limit as $n \rightarrow \infty$ in the first inequality and the upper limit as $n \rightarrow \infty$ in the second inequality we obtain the first desired result. If $\sigma_b(x, y) = 0$, then by the triangle inequality we get $\sigma_b(x, x) = 0$ and $\sigma_b(y, y) = 0$. Therefore, we have $\lim_{n \rightarrow \infty} \sigma_b(x_n, y_n) = 0$. Similarly, using again the triangle inequality the other assertions follow. $\square$

2.2. Contraction Conditions and Fixed Point Results. It is well known that a self-map f on a metric space (X, d) is said to be a Banach contraction mapping, if there exists a number $k \in [0, 1)$ such that

$$d(fx, fy) \leq kd(x, y) \quad (12)$$

for all $x, y \in X$. A mapping $f : X \rightarrow X$ is called a quasi-contraction if for some constant $\alpha \in [0, 1)$ and for every $x, y \in X$

$$\begin{aligned}d(fx, fy) \leq \alpha \max\{d(x, y), d(x, fx), d(y, fy), \\ d(x, fy), d(y, fx)\}.\end{aligned}\quad (13)$$

This concept was introduced and studied by Ćirić in 1974 [16]. A result of Ćirić shows that every quasicontraction in a complete metric space f has a unique fixed point.

The existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [17] and then by Nieto and Rodríguez-López [18].

In this paper, we establish some fixed point theorems for quasicontractive type mappings in a partially ordered complete b -metric-like space. We investigate also the so-called P -property for mappings in such spaces.

3. Main Results

3.1. Fixed Points of Quasicontraction-Type Mappings. Throughout this paper, let $(X, \leq)$ be a partially ordered set, and let (X, σ_b) be a b -metric-like space (we will say, for short, that $(X, \leq, \sigma_b)$ is a partially ordered b -metric-like space). Further, let $F(f) = \{x \in X : fx = x\}$ be the fixed point set of f , $(LF)_f = \{x \in X : x \leq fx\}$ the lower fixed point set of f , and

$$M(x, y) = \max \{ \sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, fy), \sigma_b(x, fy), \sigma_b(y, fx) \}. \quad (14)$$

In this section, we obtain some fixed point results for quasicontractions defined on a partially ordered complete b -metric-like space.

We will also make use of the following notion.

Definition 17. An ordered b -metric-like space $(X, \leq, \sigma_b)$ is said to have the sequential limit comparison property if for every nondecreasing sequence (nonincreasing sequence) $\{x_n\}_{n \in \mathbb{N}}$ in X , $x_n \rightarrow x$ implies that $x_n \leq x$ ($x \leq x_n$), for all $n \in \mathbb{N}$.

Theorem 18. Let $(X, \leq, \sigma_b)$ be a complete partially ordered b -metric-like space. If $f : X \rightarrow X$ is a nondecreasing map such that, for all elements $x, y \in X$ with $x \leq y$,

$$\sigma_b(fx, fy) \leq \alpha M(x, y), \quad (15)$$

where $\alpha \in [0, 1/2s^2]$, then $F(f) \neq \emptyset$ provided that there exists an $x_0 \in (LF)_f$ and one of the following two conditions is satisfied:

- (a) f is a continuous self-map on X ,
- (b) $(X, \leq, \sigma_b)$ has the sequential limit comparison property.

Moreover, f has a unique fixed point if and only if each pair of fixed points of f is comparable.

Proof. Since $x_0 \in (LF)_f$ and f is nondecreasing, therefore $f^n x_0 \leq f^{n+1} x_0$ for each $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in X with $x_n = f^n x_0$ and so $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there exists a positive integer n such that $x_n = x_{n+1}$, then $f^n x_0 = f^{n+1} x_0 = ff^n x_0$ which implies that $f^n x_0$ is a fixed point of f . Assume that $x_n \neq x_{n+1}$ for every positive integer n .

Since $x_{n-1} \leq x_n$, therefore by replacing x by x_{n-1} and y by x_n in (15), we have

$$\begin{aligned} \sigma_b(x_n, x_{n+1}) &= \sigma_b(fx_{n-1}, fx_n) \\ &\leq \alpha \max \{ \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, fx_{n-1}), \sigma_b(x_n, fx_n), \\ &\quad \sigma_b(x_{n-1}, fx_n), \sigma_b(x_n, fx_{n-1}) \} \\ &= \alpha \max \{ \sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), \\ &\quad \sigma_b(x_{n-1}, x_{n+1}), \sigma_b(x_n, x_n) \} \\ &\leq \alpha \max \{ \sigma_b(x_{n-1}, x_n), \sigma_b(x_n, x_{n+1}), s\sigma_b(x_{n-1}, x_n) \\ &\quad + s\sigma_b(x_n, x_{n+1}), 2s\sigma_b(x_{n-1}, x_n) \}. \end{aligned} \quad (16)$$

If, for some n , $\sigma_b(x_n, x_{n+1}) > \sigma_b(x_{n-1}, x_n) > 0$, then according to the above inequality $\sigma_b(x_n, x_{n+1}) \leq 2s\alpha\sigma_b(x_n, x_{n+1}) < \sigma_b(x_n, x_{n+1})$ which is a contradiction.

Hence, for all n , $\sigma_b(x_n, x_{n+1}) \leq \sigma_b(x_{n-1}, x_n)$ and therefore

$$\sigma_b(x_{n+1}, x_n) \leq h\sigma_b(x_n, x_{n-1}), \quad (17)$$

where $h = 2\alpha s$. Obviously, $0 \leq h < 1$. Repeating the above process, we get

$$\sigma_b(x_{n+1}, x_n) \leq h\sigma_b(x_n, x_{n-1}) \leq \dots \leq h^n \sigma_b(x_1, x_0), \quad (18)$$

for all $n \geq 1$, and so, for $m > n$, we have

$$\begin{aligned} \sigma_b(x_n, x_m) &\leq s\sigma_b(x_n, x_{n+1}) + s^2\sigma_b(x_{n+1}, x_{n+2}) \\ &\quad + \dots + s^{m-n}\sigma_b(x_{m-1}, x_m) \\ &\leq sh^n\sigma_b(x_0, x_1) + s^2h^{n+1}\sigma_b(x_0, x_1) \\ &\quad + \dots + s^{m-n}h^{m-1}\sigma_b(x_0, x_1) \\ &= sh^n(1 + sh + \dots + s^{m-n-1}h^{m-n-1})\sigma_b(x_0, x_1) \\ &\leq \frac{sh^n}{1 - sh}\sigma_b(x_0, x_1). \end{aligned} \quad (19)$$

Since, by assumption, $h < 1/s$, it follows that $\lim_{m,n \rightarrow \infty} \sigma_b(x_n, x_m) = 0$. Since X is complete, there exists an element $u \in X$ such that

$$0 = \lim_{n,m \rightarrow \infty} \sigma_b(x_n, x_m) = \lim_{n \rightarrow \infty} \sigma_b(x_n, u) = \sigma_b(u, u). \quad (20)$$

If f is a continuous self-map on X , then $fu = u$. Indeed, by the triangle inequality we have

$$\begin{aligned} \sigma_b(fu, u) &\leq s(\sigma_b(fu, x_{n+1}) + \sigma_b(x_{n+1}, u)) \\ &= s(\sigma_b(fu, fx_n) + \sigma_b(x_{n+1}, u)). \end{aligned} \quad (21)$$

Taking the limit as $n \rightarrow \infty$ in the above inequality, the desired result is obtained.

If condition (b) is fulfilled then $x_n = f^n x_0 \leq u$ for all $n \in \mathbb{N}$, and it follows that

$$\begin{aligned} \sigma_b(fx_n, fu) &\leq \alpha \max \{ \sigma_b(x_n, u), \sigma_b(x_n, fx_n), \sigma_b(u, fu), \\ &\quad \sigma_b(x_n, fu), \sigma_b(u, fx_n) \} \\ &= \alpha \max \{ \sigma_b(x_n, u), \sigma_b(x_n, x_{n+1}), \sigma_b(u, fu), \\ &\quad \sigma_b(x_n, fu), \sigma_b(u, x_{n+1}) \}. \end{aligned} \quad (22)$$

On taking the upper limit as $n \rightarrow \infty$, (using Lemma 16, as $\sigma_b(u, u) = 0$) it follows that

$$\frac{1}{s} \sigma_b(u, fu) \leq \alpha s \sigma_b(u, fu), \quad (23)$$

and hence $\sigma_b(u, fu) \leq \alpha s^2 \sigma_b(u, fu)$, or equivalently, $fu = u$.

Suppose that the fixed points of f are comparable. Let w be another fixed point of f such that $w \neq u$. Assume, for example, that $u \leq w$. Using (15), we obtain that

$$\begin{aligned} \sigma_b(u, w) &= \sigma_b(fu, fw) \\ &\leq \alpha \max \{ \sigma_b(u, w), \sigma_b(u, fu), \sigma_b(w, fw), \\ &\quad \sigma_b(u, fw), \sigma_b(w, fu) \} \\ &= \alpha \max \{ \sigma_b(u, w), \sigma_b(u, u), \sigma_b(w, w), \\ &\quad \sigma_b(u, w), \sigma_b(w, u) \} \\ &\leq 2s\alpha \sigma_b(u, w), \end{aligned} \quad (24)$$

which further implies that $u = w$. The converse is trivial. $\square$

Example 19. Let $X = [0, 1]$ be endowed with the usual order. Define $\sigma_b : X \times X \rightarrow \mathbb{R}^+$ by

$$\sigma_b(x, y) = x^2 + y^2 + |x - y|^2 \quad (25)$$

for all $x, y \in X$. Then (X, σ_b) is a complete b -metric-like space with coefficient $s = 2$ (see Example 14).

Let $f : X \rightarrow X$ be defined by $fx = \ln(1 + x/4)$. It is easy to see that f is a nondecreasing and continuous self-map on X , and $0 \in (LF)_f$. Using the Mean Value Theorem for any $x, y \in X$ with $x \leq y$ and that $fx \leq x/4$, we have

$$\begin{aligned} \sigma_b(fx, fy) &= f^2 x + f^2 y + |fy - fx|^2 \\ &= \left(\ln \left(1 + \frac{x}{4} \right) \right)^2 + \left(\ln \left(1 + \frac{y}{4} \right) \right)^2 \\ &\quad + \left| \ln \left(1 + \frac{y}{4} \right) - \ln \left(1 + \frac{x}{4} \right) \right|^2 \\ &\leq \left(\frac{x}{4} \right)^2 + \left(\frac{y}{4} \right)^2 + \left| \frac{y}{4} - \frac{x}{4} \right|^2 \\ &= \frac{1}{16} (x^2 + y^2 + |x - y|^2) \\ &= \frac{1}{16} \sigma_b(x, y) \leq \frac{1}{16} M(x, y). \end{aligned} \quad (26)$$

Thus (15) is satisfied with $\alpha = 1/16 \in [0, 1/8)$. Thus all conditions of Theorem 18 are satisfied. Moreover, 0 is the unique fixed point of f .

Theorem 20. Let $(X, \leq, \sigma_b)$ be a partially ordered complete b -metric-like space with coefficient $s > 1$. If a nondecreasing map $f : X \rightarrow X$ satisfies

$$s\sigma_b(fx, fy) \leq M(x, y) - \varphi(m(x, y)), \quad (27)$$

for all $x, y \in X$ with $x \leq y$, where

$$M(x, y) = \frac{\sigma_b(x, fy) + \sigma_b(y, fx)}{4s}, \quad (28)$$

$$m(x, y) = \min \{ \sigma_b(x, fy), \sigma_b(y, fx) \},$$

and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) > 0$ for all $t \in (0, \infty)$ and $\varphi(0) = 0$, then $F(f) \neq \emptyset$ provided that there exists an $x_0 \in (LF)_f$ and one of the following two conditions is satisfied:

- (a) f is a continuous self-map on X .
- (b) $(X, \leq, \sigma_b)$ has the sequential limit comparison property.

Moreover, f has a unique fixed point if and only if the fixed points of f are comparable.

Proof. Define the sequence $\{x_n\}$ as given in the proof of Theorem 18. If there exists a positive integer n such that $x_n = x_{n+1}$, then x_n is a fixed point of f . Assume that $x_n \neq x_{n+1}$ for every positive integer n . Since $x_{n-1} \leq x_n$, therefore by replacing x by x_{n-1} and y by x_n in (27), we obtain

$$\begin{aligned} s\sigma_b(x_n, x_{n+1}) &= s\sigma_b(fx_{n-1}, fx_n) \\ &\leq M(x_{n-1}, x_n) - \varphi(m(x_{n-1}, x_n)) \\ &\leq \frac{\sigma_b(x_{n-1}, fx_n) + \sigma_b(x_n, fx_{n-1})}{4s} - \varphi(m(x_{n-1}, x_n)) \\ &= \frac{\sigma_b(x_{n-1}, x_{n+1}) + \sigma_b(x_n, x_n)}{4s} - \varphi(m(x_{n-1}, x_n)) \\ &\leq \frac{s\sigma_b(x_{n-1}, x_n) + s\sigma_b(x_n, x_{n+1}) + 2s\sigma_b(x_n, x_{n+1})}{4s} \\ &\quad - \varphi(m(x_{n-1}, x_n)) \\ &\leq \frac{s\sigma_b(x_{n-1}, x_n) + s\sigma_b(x_n, x_{n+1}) + 2s\sigma_b(x_n, x_{n+1})}{4s}. \end{aligned} \quad (29)$$

Therefore, for all $n \geq 0$, $\sigma_b(x_n, x_{n+1}) \leq \sigma_b(x_n, x_{n-1})$ and $\{\sigma_b(x_n, x_{n+1})\}$ is nonincreasing and bounded from below. Hence, there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \sigma_b(x_{n+1}, x_n) = r$.

From the above argument we have

$$\begin{aligned} s\sigma_b(x_{n+1}, x_n) &\leq \frac{\sigma_b(x_{n-1}, x_{n+1}) + \sigma_b(x_n, x_n)}{4s} \\ &\leq \frac{s\sigma_b(x_{n-1}, x_n) + s\sigma_b(x_n, x_{n+1}) + 2s\sigma_b(x_n, x_{n+1})}{4s}. \end{aligned} \quad (30)$$

If $n \rightarrow \infty$, we get

$$sr \leq \lim_{n \rightarrow \infty} \frac{\sigma_b(x_{n-1}, x_{n+1}) + \sigma_b(x_n, x_n)}{4s} \leq r, \quad (31)$$

which is, because of $s > 1$, possible only if $r = 0$. So we have

$$\lim_{n \rightarrow \infty} \sigma_b(x_{n+1}, x_n) = 0. \quad (32)$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If not, then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of the sequence $\{x_n\}$ where n_k is the smallest index for which $n_k > m_k > k$ with

$$\sigma_b(x_{m_k}, x_{n_k}) \geq \varepsilon. \quad (33)$$

Then,

$$\sigma_b(x_{m_k}, x_{n_k-1}) < \varepsilon. \quad (34)$$

From (33) and (34), we obtain

$$\begin{aligned} \varepsilon &\leq \sigma_b(x_{m_k}, x_{n_k}) \\ &\leq s\sigma_b(x_{m_k}, x_{n_k-1}) + s\sigma_b(x_{n_k-1}, x_{n_k}) \\ &< s\varepsilon + s\sigma_b(x_{n_k-1}, x_{n_k}). \end{aligned} \quad (35)$$

Taking the upper and lower limits as $k \rightarrow \infty$, from (32) we conclude that

$$\varepsilon \leq \liminf_{k \rightarrow \infty} \sigma_b(x_{m_k}, x_{n_k}) \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k}, x_{n_k}) \leq s\varepsilon. \quad (36)$$

Note that

$$\sigma_b(x_{m_k+1}, x_{n_k-1}) \leq s\sigma_b(x_{m_k+1}, x_{m_k}) + s\sigma_b(x_{m_k}, x_{n_k-1}). \quad (37)$$

Using (34) and (32), we get

$$\limsup_{k \rightarrow \infty} \sigma_b(x_{m_k+1}, x_{n_k-1}) \leq s\varepsilon. \quad (38)$$

On the other hand,

$$\begin{aligned} \sigma_b(x_{m_k}, x_{n_k}) &\leq s\sigma_b(x_{m_k}, x_{m_k+1}) + s^2\sigma_b(x_{m_k+1}, x_{n_k-1}) \\ &\quad + s^2\sigma_b(x_{n_k-1}, x_{n_k}). \end{aligned} \quad (39)$$

Using (36) and (32), we get

$$\frac{\varepsilon}{s^2} \leq \liminf_{k \rightarrow \infty} \sigma_b(x_{m_k+1}, x_{n_k-1}). \quad (40)$$

From (38) and (40), we have

$$\begin{aligned} \frac{\varepsilon}{s^2} &\leq \liminf_{k \rightarrow \infty} \sigma_b(x_{m_k+1}, x_{n_k-1}) \\ &\leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k+1}, x_{n_k-1}) \leq s\varepsilon. \end{aligned} \quad (41)$$

Also,

$$\varepsilon \leq \sigma_b(x_{m_k}, x_{n_k}) \leq s\sigma_b(x_{m_k}, x_{m_k+1}) + s\sigma_b(x_{m_k+1}, x_{n_k}). \quad (42)$$

Taking the upper limit as $k \rightarrow \infty$, we conclude that

$$\varepsilon \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k}, x_{n_k}) \leq \limsup_{k \rightarrow \infty} s\sigma_b(x_{m_k+1}, x_{n_k}), \quad (43)$$

or equivalently,

$$\frac{\varepsilon}{s} \leq \limsup_{k \rightarrow \infty} \sigma_b(x_{m_k+1}, x_{n_k}). \quad (44)$$

As $\{x_n\}$ is nondecreasing and $m_k < n_k$, from (27) we have

$$\begin{aligned} s\sigma_b(x_{m_k+1}, x_{n_k}) &= s\sigma_b(fx_{m_k}, fx_{n_k-1}) \\ &\leq M(x_{m_k}, x_{n_k-1}) - \varphi(m(x_{m_k}, x_{n_k-1})), \end{aligned} \quad (45)$$

where

$$\begin{aligned} M(x_{m_k}, x_{n_k-1}) &= \frac{\sigma_b(x_{m_k}, fx_{n_k-1}) + \sigma_b(x_{n_k-1}, fx_{m_k})}{4s} \\ &= \frac{\sigma_b(x_{m_k}, x_{n_k}) + \sigma_b(x_{n_k-1}, x_{m_k+1})}{4s}, \end{aligned}$$

$$m(x_{m_k}, x_{n_k-1}) = \min \{ \sigma_b(x_{m_k}, x_{n_k}), \sigma_b(x_{n_k-1}, x_{m_k+1}) \}, \quad (46)$$

wherefrom on taking the upper limit as $k \rightarrow \infty$, from (36) and (41), we have

$$\limsup_{k \rightarrow \infty} M(x_{m_k}, x_{n_k-1}) \leq \frac{\varepsilon}{2}, \quad (47)$$

$$\liminf_{k \rightarrow \infty} m(x_{m_k}, x_{n_k-1}) \geq \min \left\{ \varepsilon, \frac{\varepsilon}{s^2} \right\} = \frac{\varepsilon}{s^2}.$$

Thus, from (44) and (45), we have

$$\varepsilon \leq \frac{\varepsilon}{2} - \varphi\left(\frac{\varepsilon}{s^2}\right), \quad (48)$$

which gives $\varepsilon = 0$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X . Since X is complete, there exists an element $u \in X$ such that

$$0 = \lim_{n, m \rightarrow \infty} \sigma_b(x_n, x_m) = \lim_{n \rightarrow \infty} \sigma_b(x_n, u) = \sigma_b(u, u). \quad (49)$$

If f is a continuous self-map on X , then $fu = u$. If assumption (b) is satisfied then $x_n = f^n x_0 \leq u$ for all $n \in \mathbb{N}$; it follows that

$$s\sigma_b(fx_n, fu) \leq M(x_n, u) - \varphi(m(x_n, u)), \quad (50)$$

where

$$M(x_n, u) = \frac{\sigma_b(x_n, fu) + \sigma_b(u, x_{n+1})}{4s}, \quad (51)$$

$$m(x_n, u) = \min\{\sigma_b(x_n, fu), \sigma_b(u, x_{n+1})\}.$$

On taking the upper limit as $n \rightarrow \infty$ in (50), from Lemma 16 we obtain

$$s \cdot \frac{1}{s} \sigma_b(u, fu) \leq \frac{s\sigma_b(u, fu)}{4s} - \varphi(0), \quad (52)$$

and hence $\sigma_b(u, fu) = 0$, implying that $u = fu$.

Now, suppose that the fixed points of f are comparable. Let w be another fixed point of f . We will show that $w = u$. If not, then without loss of generality, we assume that $u \leq w$. Using (27), we obtain

$$\sigma_b(u, w) = \sigma_b(fu, fw) \leq M(u, w) - \varphi(m(u, w)), \quad (53)$$

where

$$\begin{aligned} M(u, w) &= \frac{\sigma_b(u, fw) + \sigma_b(w, fu)}{4s} \\ &= \frac{\sigma_b(u, w) + \sigma_b(w, u)}{4s} = \frac{\sigma_b(u, w)}{2s}. \end{aligned} \quad (54)$$

Thus,

$$\sigma_b(u, w) = \sigma_b(fu, fw) \leq \frac{\sigma_b(u, w)}{2s} - \varphi(\sigma_b(u, w)), \quad (55)$$

which implies that $\sigma_b(u, w) = 0$, which yields that $u = w$. The converse is trivial. $\square$

Example 21. Let $X = [0, \infty)$ be endowed with the usual order $\leq$. Define $\sigma_b : X \times X \rightarrow \mathbb{R}^+$ by

$$\sigma_b(x, y) = (x^2 + y^2)^2 \quad (56)$$

for all $x, y \in X$. Then $(X, \leq, \sigma_b)$ is a complete ordered b -metric-like space with coefficient $s = 2$ (see Example 11).

Let $f : X \rightarrow X$ be defined by $fx = (1/4)\sqrt{\ln((x^2/3) + 1)}$. It is easy to see that f is a nondecreasing and continuous self-map on X , and $0 \in (LF)_f$. Using the Mean Value Theorem for the function $f^2x = (1/16)\ln(x^2/3 + 1)$ for any $x, y \in X$ with $x \leq y$, we have

$$\begin{aligned} f^2y - f^2x &= \frac{1}{16} \left(\ln\left(\frac{y^2}{3} + 1\right) - \ln\left(\frac{x^2}{3} + 1\right) \right) \\ &\leq \frac{1}{48} (y^2 - x^2) \leq y^2 - x^2, \end{aligned} \quad (57)$$

which implies that

$$f^2y + x^2 \leq f^2x + y^2. \quad (58)$$

Hence

$$m(x, y) = \min\{\sigma_b(fy, x), \sigma_b(fx, y)\} = \sigma_b(fy, x). \quad (59)$$

Now we define $\varphi : [0, \infty) \rightarrow [0, \infty)$ by $\varphi(t) = t/8$, and, for any $x, y \in X$ with $x \leq y$, we have

$$\begin{aligned} s\sigma_b(fx, fy) &\leq M(x, y) - \varphi(m(x, y)) \\ &\iff 2\sigma_b(fx, fy) \leq \frac{\sigma_b(fy, x) + \sigma_b(fx, y)}{8} \\ &\quad - \frac{\sigma_b(fy, x)}{8} \\ &\iff 16\sigma_b(fx, fy) \leq \sigma_b(fx, y) \\ &\iff 16(f^2x + f^2y)^2 \leq (f^2x + y^2)^2 \\ &\iff 4(f^2x + f^2y) \leq f^2x + y^2 \\ &\iff 3f^2x + 4f^2y \leq y^2 \\ &\iff 7f^2y \leq y^2 \\ &\iff f^2y \leq \frac{y^2}{7} \\ &\iff \frac{1}{16} \ln\left(\frac{y^2}{3} + 1\right) \leq \frac{y^2}{48}, \end{aligned} \quad (60)$$

since $f^2x = (1/16)\ln(x^2/3 + 1)$ is an increasing function and $f^2x \leq x^2/48$.

Thus, all conditions of Theorem 20 are satisfied. Moreover, 0 is the unique fixed point of f .

3.2. Coincidence Points of Four Mappings under Generalized Weakly Contractive Conditions

Definition 22. Let f and g be two self-maps on partially ordered set X . A pair (f, g) is said to be

- (i) weakly increasing if $fx \leq gfx$ and $gx \leq fgx$, for all $x \in X$ [19, 20],
- (ii) partially weakly increasing if $fx \leq gfx$, for all $x \in X$ [21].

Let X be a nonempty set and $f : X \rightarrow X$ a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

Definition 23. Let $(X, \leq)$ be a partially ordered set and $f, g, h : X \rightarrow X$ mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. The ordered pair (f, g) is said to be

- (a) weakly increasing with respect to h if and only if, for all $x \in X$, $fx \leq gy$ for all $y \in h^{-1}(fx)$, and $gx \leq fy$ for all $y \in h^{-1}(gx)$ [22],
- (b) partially weakly increasing with respect to h if $fx \leq gy$, for all $y \in h^{-1}(fx)$ [23].

Remark 24. In the above definition (i) if $f = g$, we say that f is weakly increasing (partially weakly increasing) with respect to h and (ii) if $h = I$ (the identity mapping on X), then the above definition reduces to the weakly increasing (partially weakly increasing) mapping (see [22, 24]).

The study of unique common fixed points of mappings satisfying weakly contractive conditions has been at the center of vigorous research activity. Motivated by the work in [21, 23–30], we prove some coincidence point results for nonlinear generalized (ψ, φ) -weakly contractive mappings in partially ordered b -metric-like spaces.

Recall [31] that an altering distance function is a mapping $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfies that

- (ψ_1) $\psi(t)$ is increasing and continuous,
- (ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

Let $(X, \leq, \sigma_b)$ be an ordered b -metric-like space and $f, g, R, S : X \rightarrow X$ four self-mappings. Throughout this subsection, unless otherwise stated, for all $x, y \in X$, let

$$M(x, y) = \max \left\{ \sigma_b(Sx, Ry), \sigma_b(Sx, fx), \sigma_b(Ry, gy), \frac{\sigma_b(Sx, gy) + \sigma_b(Ry, fx)}{4s} \right\}. \quad (61)$$

Suppose that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$; let x_0 be an arbitrary point of X . Choose $x_1 \in X$ such that $fx_0 = Rx_1$ and $x_2 \in X$ such that $gx_1 = Sx_2$. Continuing in this way, construct a sequence $\{z_n\}$ defined by $z_{2n+1} = Rx_{2n+1} = fx_{2n}$ and $z_{2n+2} = Sx_{2n+2} = gx_{2n+1}$, for all $n \geq 0$. The sequence $\{z_n\}$ in X is said to be a Jungck-type iterative sequence with initial guess x_0 .

Theorem 25. Let $(X, \leq, \sigma_b)$ be a partially ordered b -metric-like space with sequential limit comparison property, $f, g, R, S : X \rightarrow X$ four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$, and RX and SX σ_b -complete subsets of X . Suppose that, for comparable elements $Sx, Ry \in X$, one has

$$\psi(s^2 \sigma_b(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \quad (62)$$

where $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ are altering distance functions. Then, the pairs (f, S) and (g, R) have a coincidence point z^* in X provided that the pairs (f, S) and (g, R) are weakly compatible, and the pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S , respectively.

Proof. Let $\{z_n\}$ be a Jungck-type iterative sequence with initial guess x_0 in X ; that is, $z_{2n+1} = Rx_{2n+1} = fx_{2n}$, $z_{2n+2} = Sx_{2n+2} = gx_{2n+1}$ for all $n \geq 0$.

As $x_1 \in R^{-1}(fx_0)$ and $x_2 \in S^{-1}(gx_1)$, and the pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S , so we have

$$Rx_1 = fx_0 \leq gx_1 = Sx_2. \quad (63)$$

Continuing this process, we obtain $Rx_{2n+1} \leq Sx_{2n+2}$, for $n \geq 0$. We will complete the proof in three steps.

Step I. We will prove that $\lim_{k \rightarrow \infty} \sigma_b(z_k, z_{k+1}) = 0$.

Define $\sigma_{b_k} = \sigma_b(z_k, z_{k+1})$. Suppose $\sigma_{b_{k_0}} = 0$, for some k_0 . Then, $z_{k_0} = z_{k_0+1}$. If $k_0 = 2n$, then $z_{2n} = z_{2n+1}$ gives $z_{2n+1} = z_{2n+2}$. Indeed,

$$\begin{aligned} \psi(\sigma_b(z_{2n+1}, z_{2n+2})) &= \psi(\sigma_b(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(s^2 \sigma_b(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) \\ &\quad - \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \quad (64)$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &= \max \left\{ \sigma_b(Sx_{2n}, Rx_{2n+1}), \sigma_b(Sx_{2n}, fx_{2n}), \right. \\ &\quad \sigma_b(Rx_{2n+1}, gx_{2n+1}), \\ &\quad \left. \frac{\sigma_b(Sx_{2n}, gx_{2n+1}) + \sigma_b(Rx_{2n+1}, fx_{2n})}{4s} \right\} \\ &= \max \left\{ \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n+1}, z_{2n+2}), \right. \\ &\quad \left. \frac{\sigma_b(z_{2n}, z_{2n+2}) + \sigma_b(z_{2n+1}, z_{2n+1})}{4s} \right\} \\ &\leq \max \left\{ \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n+1}, z_{2n+2}), \right. \\ &\quad (s\sigma_b(z_{2n}, z_{2n+1}) + s\sigma_b(z_{2n+1}, z_{2n+2}) \\ &\quad \left. + 2s\sigma_b(z_{2n+1}, z_{2n+2})) \times (4s)^{-1} \right\} \\ &= \max \left\{ 0, 0, \sigma_b(z_{2n+1}, z_{2n+2}), \frac{0 + 3s\sigma_b(z_{2n+1}, z_{2n+2})}{4s} \right\} \\ &= \max \left\{ \sigma_b(z_{2n+1}, z_{2n+2}), \frac{3\sigma_b(z_{2n+1}, z_{2n+2})}{4} \right\} \\ &= \sigma_b(z_{2n+1}, z_{2n+2}). \end{aligned} \quad (65)$$

Thus,

$$\begin{aligned} \psi(\sigma_b(z_{2n+1}, z_{2n+2})) &\leq \psi(\sigma_b(z_{2n+1}, z_{2n+2})) \\ &\quad - \varphi(M(z_{2n}, z_{2n+1})), \end{aligned} \quad (66)$$

which implies that $\varphi(M(z_{2n}, z_{2n+1})) = 0$; that is, $z_{2n+1} = z_{2n+2}$. Similarly, if $k_0 = 2n+1$, then $z_{2n+1} = z_{2n+2}$ gives $z_{2n+2} = z_{2n+3}$. Consequently, the sequence $\{z_k\}$ becomes constant for $k \geq k_0$ and hence $\lim_{k \rightarrow \infty} \sigma_b(z_k, z_{k+1}) = 0$.

Suppose that

$$\sigma_{b_k} = \sigma_b(z_k, z_{k+1}) > 0, \quad (67)$$

for each k . We now claim that the following inequality holds:

$$\sigma_b(z_{k+1}, z_{k+2}) \leq \sigma_b(z_k, z_{k+1}) = M(x_k, x_{k+1}) \quad (68)$$

for each $k = 1, 2, \dots$

Let $k = 2n$, and for an $n \geq 0$, $\sigma_b(z_{2n+1}, z_{2n+2}) > \sigma_b(z_{2n}, z_{2n+1}) > 0$. Then, as $Sx_{2n} \leq Rx_{2n+1}$, using (62) we obtain that

$$\begin{aligned} & \psi(\sigma_b(z_{2n+1}, z_{2n+2})) \\ & \leq \psi(s^2 \sigma_b(z_{2n+1}, z_{2n+2})) \\ & = \psi(s^2 \sigma_b(fx_{2n}, gx_{2n+1})) \\ & \leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \quad (69)$$

where

$$\begin{aligned} & M(x_{2n}, x_{2n+1}) \\ & = \max \left\{ \sigma_b(Sx_{2n}, Rx_{2n+1}), \sigma_b(Sx_{2n}, fx_{2n}), \right. \\ & \quad \sigma_b(Rx_{2n+1}, gx_{2n+1}), \\ & \quad \left. \frac{\sigma_b(Sx_{2n}, gx_{2n+1}) + \sigma_b(Rx_{2n+1}, fx_{2n})}{4s} \right\} \\ & = \max \left\{ \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n}, z_{2n+1}), \right. \\ & \quad \sigma_b(z_{2n+1}, z_{2n+2}), \\ & \quad \left. \frac{\sigma_b(z_{2n}, z_{2n+2}) + \sigma_b(z_{2n+1}, z_{2n+1})}{4s} \right\} \quad (70) \\ & \leq \max \left\{ \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n+1}, z_{2n+2}), \right. \\ & \quad (s\sigma_b(z_{2n}, z_{2n+1}) + s\sigma_b(z_{2n+1}, z_{2n+2}) \\ & \quad + 2s\sigma_b(z_{2n+1}, z_{2n+2})) \times (4s)^{-1} \left. \right\} \\ & = \max \left\{ \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n+1}, z_{2n+2}), \right. \\ & \quad \left. \frac{s\sigma_b(z_{2n}, z_{2n+1}) + 3s\sigma_b(z_{2n+1}, z_{2n+2})}{4s} \right\} \\ & = \max \{ \sigma_b(z_{2n}, z_{2n+1}), \sigma_b(z_{2n+1}, z_{2n+2}) \}. \end{aligned}$$

If, for some n , $\sigma_b(z_{2n+1}, z_{2n+2}) > \sigma_b(z_{2n}, z_{2n+1}) > 0$, (69) implies that

$$\begin{aligned} & \psi(\sigma_b(z_{2n+1}, z_{2n+2})) \leq \psi(\sigma_b(z_{2n+1}, z_{2n+2})) \\ & \quad - \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \quad (71)$$

which is possible only if $M(x_{2n}, x_{2n+1}) = 0$; that is, $\sigma_b(z_{2n}, z_{2n+1}) = 0$, a contradiction to (67). Hence, $\sigma_b(z_{2n+1}, z_{2n+2}) \leq \sigma_b(z_{2n}, z_{2n+1})$ and

$$M(x_{2n}, x_{2n+1}) = \sigma_b(z_{2n}, z_{2n+1}). \quad (72)$$

Therefore, (68) is proved for $k = 2n$.

Similarly, it can be shown that

$$\sigma_b(z_{2n+2}, z_{2n+3}) \leq \sigma_b(z_{2n+1}, z_{2n+2}) = M(x_{2n+1}, x_{2n+2}). \quad (73)$$

Hence, $\{\sigma_b(z_k, z_{k+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Therefore, there is an $r \geq 0$ such that

$$\lim_{k \rightarrow \infty} \sigma_b(z_k, z_{k+1}) = r. \quad (74)$$

Taking the limit as $k \rightarrow \infty$ in (68), we obtain

$$\lim_{k \rightarrow \infty} M(x_k, x_{k+1}) = r. \quad (75)$$

Taking the limit as $n \rightarrow \infty$ in (69), using (74), (75), and the continuity of ψ and φ , we have $\psi(r) \leq \psi(r) - \varphi(r)$. Therefore $\varphi(r) = 0$. Hence,

$$\lim_{k \rightarrow \infty} \sigma_b(z_k, z_{k+1}) = 0, \quad (76)$$

from our assumptions about φ .

Step II. We now show that $\{z_n\}$ is a σ_b -Cauchy sequence in X . Because of (76), it is sufficient to show that $\{z_{2n}\}$ is σ_b -Cauchy.

We assume on the contrary that there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{2m(k)}\}$ and $\{z_{2n(k)}\}$ of $\{z_{2n}\}$ such that $n(k) > m(k) \geq k$ and

$$\sigma_b(z_{2m(k)}, z_{2n(k)}) \geq \varepsilon, \quad (77)$$

and $n(k)$ is the smallest number such that the above statement holds; that is,

$$\sigma_b(z_{2m(k)}, z_{2n(k)-2}) < \varepsilon. \quad (78)$$

From triangle inequality, we have

$$\begin{aligned} \sigma_b(z_{2m(k)}, z_{2n(k)}) & \leq s\sigma_b(z_{2m(k)}, z_{2m(k)+1}) \\ & \quad + s\sigma_b(z_{2m(k)+1}, z_{2n(k)}). \end{aligned} \quad (79)$$

Taking the upper limit as $k \rightarrow \infty$ in (79), from (77) and (76) we obtain that

$$\frac{\varepsilon}{s} \leq \limsup_{k \rightarrow \infty} \sigma_b(z_{2m(k)+1}, z_{2n(k)}). \quad (80)$$

Using triangle inequality, we have

$$\begin{aligned} \sigma_b(z_{2m(k)}, z_{2n(k)-1}) & \leq s\sigma_b(z_{2m(k)}, z_{2n(k)-2}) \\ & \quad + s\sigma_b(z_{2n(k)-2}, z_{2n(k)-1}). \end{aligned} \quad (81)$$

Taking the upper limit as $k \rightarrow \infty$ in (81), from (78) and (76) we have

$$\limsup_{k \rightarrow \infty} \sigma_b(z_{2m(k)}, z_{2n(k)-1}) \leq s\varepsilon. \quad (82)$$

Also,

$$\sigma_b(z_{2m(k)}, z_{2n(k)}) \leq s\sigma_b(z_{2m(k)}, z_{2n(k)-2}) + s\sigma_b(z_{2n(k)-2}, z_{2n(k)}). \quad (83)$$

Taking the upper limit as $k \rightarrow \infty$ in (83) and using (76) and (78) we have

$$\limsup_{k \rightarrow \infty} \sigma_b(z_{2m(k)}, z_{2n(k)}) \leq s\varepsilon. \quad (84)$$

Consider

$$\sigma_b(z_{2m(k)+1}, z_{2n(k)-1}) \leq s\sigma_b(z_{2m(k)+1}, z_{2m(k)}) + s\sigma_b(z_{2m(k)}, z_{2n(k)-1}). \quad (85)$$

Taking the limit as $k \rightarrow \infty$ and using (76) and (82), we have

$$\limsup_{k \rightarrow \infty} \sigma_b(z_{2m(k)+1}, z_{2n(k)-1}) \leq s^2\varepsilon. \quad (86)$$

As $Sx_{2m(k)} \preceq Rx_{2n(k)-1}$, so from (62), we have

$$\begin{aligned} \psi(\sigma_b(z_{2m(k)+1}, z_{2n(k)})) &= \psi(\sigma_b(fx_{2m(k)}, gx_{2n(k)-1})) \\ &\leq \psi(M(x_{2m(k)}, x_{2n(k)-1})) \\ &\quad - \varphi(M(x_{2m(k)}, x_{2n(k)-1})), \end{aligned} \quad (87)$$

where

$$\begin{aligned} &M(x_{2m(k)}, x_{2n(k)-1}) \\ &= \max \left\{ \sigma_b(Sx_{2m(k)}, Rx_{2n(k)-1}), \sigma_b(Sx_{2m(k)}, fx_{2m(k)}), \right. \\ &\quad \sigma_b(Rx_{2n(k)-1}, gx_{2n(k)-1}), \\ &\quad \left. \frac{\sigma_b(Sx_{2m(k)}, gx_{2n(k)-1}) + \sigma_b(Rx_{2n(k)-1}, fx_{2m(k)})}{4s} \right\} \\ &= \max \left\{ \sigma_b(z_{2m(k)}, z_{2n(k)-1}), \sigma_b(z_{2m(k)}, z_{2m(k)+1}), \right. \\ &\quad \sigma_b(z_{2n(k)-1}, z_{2n(k)}), \\ &\quad \left. \frac{\sigma_b(z_{2m(k)}, z_{2n(k)}) + \sigma_b(z_{2n(k)-1}, z_{2m(k)+1})}{4s} \right\}. \end{aligned} \quad (88)$$

Taking the upper limit in the above and using (76), (82), (84), and (86), we get

$$\limsup_{k \rightarrow \infty} M(x_{2m(k)}, x_{2n(k)-1}) \leq s\varepsilon. \quad (89)$$

Now, taking the upper limit as $k \rightarrow \infty$ in (87) and using (80) and (89) we have

$$\begin{aligned} \psi(s\varepsilon) &= \psi\left(s^2 \frac{\varepsilon}{s}\right) \leq \psi\left(s^2 \limsup_{k \rightarrow \infty} \sigma_b(z_{2m(k)+1}, z_{2n(k)})\right) \\ &\leq \psi\left(\limsup_{k \rightarrow \infty} M(x_{2m(k)}, x_{2n(k)-1})\right) \\ &\quad - \varphi\left(\liminf_{k \rightarrow \infty} M(x_{2m(k)}, x_{2n(k)-1})\right) \\ &\leq \psi(s\varepsilon) - \varphi\left(\liminf_{k \rightarrow \infty} M(x_{2m(k)}, x_{2n(k)-1})\right), \end{aligned} \quad (90)$$

which implies that $\varphi(\liminf_{k \rightarrow \infty} M(x_{2m(k)}, x_{2n(k)-1})) = 0$. Hence,

$$\liminf_{k \rightarrow \infty} \sigma_b(x_{2m(k)}, x_{2n(k)}) = 0, \quad (91)$$

which is in contradiction with (77). Hence, $\{z_n\}$ is a σ_b -Cauchy sequence.

Step III. We will show that f, g, R , and S have a coincidence point.

Since $\{z_n\}$ is a σ_b -Cauchy sequence and $R(X)$ and $S(X)$ are σ_b -complete σ_b -metric spaces, there exists $z^* \in R(X) \cap S(X)$ such that

$$\lim_{k \rightarrow \infty} \sigma_b(z_k, z^*) = \sigma_b(z^*, z^*) = \lim_{m, n \rightarrow \infty} \sigma_b(z_n, z_m) = 0. \quad (92)$$

There exists $u \in X$ such that $z^* = Ru$ and

$$\lim_{n \rightarrow \infty} \sigma_b(z_{2n+1}, Ru) = \lim_{n \rightarrow \infty} \sigma_b(Rx_{2n+1}, Ru) = \sigma_b(Ru, Ru). \quad (93)$$

Similarly, there exists $v \in X$ such that $z^* = Sv$ and

$$\lim_{n \rightarrow \infty} \sigma_b(z_{2n+2}, Sv) = \lim_{n \rightarrow \infty} \sigma_b(Sx_{2n+2}, Sv) = \sigma_b(Sv, Sv). \quad (94)$$

Now, we prove that z^* is a coincidence point of g and R . For this purpose, we show that $Ru = gu$. Since $Sx_{2n+2} \rightarrow z^* = Ru$ as $n \rightarrow \infty$, so $Sx_{2n+2} \preceq Ru$.

Therefore, from (62), we have

$$\begin{aligned} \psi\left(s^2 \sigma_b(fx_{2n+2}, gu)\right) &\leq \psi(M(x_{2n+2}, u)) \\ &\quad - \varphi(M(x_{2n+2}, u)), \end{aligned} \quad (95)$$

where

$$\begin{aligned} M(x_{2n+2}, u) &= \max \left\{ \sigma_b(Sx_{2n+2}, Ru), \sigma_b(Sx_{2n+2}, fx_{2n+2}), \right. \\ &\quad \sigma_b(Ru, gu), \\ &\quad \left. \frac{\sigma_b(Sx_{2n+2}, gu) + \sigma_b(Ru, fx_{2n+2})}{4s} \right\}. \end{aligned} \quad (96)$$

Taking the limit as $n \rightarrow \infty$ in (95), as $\sigma_b(z^*, z^*) = 0$, and using Lemma 16 we obtain that

$$\begin{aligned}
 & \psi(\sigma_b(z^*, gu)) \\
 &= \psi\left(s^2 \frac{1}{s} \sigma_b(z^*, gu)\right) \\
 &\leq \psi\left(\max\left\{\sigma_b(z^*, Ru), 0, \sigma_b(Ru, gu), \frac{\sigma_b(z^*, gu) + \sigma_b(z^*, Ru)}{4s}\right\}\right) \\
 &\quad - \varphi\left(\max\left\{\sigma_b(z^*, Ru), 0, \sigma_b(Ru, gu), \frac{\sigma_b(z^*, gu) + \sigma_b(z^*, Ru)}{4s}\right\}\right) \\
 &= \psi(\sigma_b(z^*, gu)) - \varphi(\sigma_b(z^*, gu)) \\
 &\leq \psi(\sigma_b(z^*, gu)) - \varphi(\sigma_b(z^*, gu))
 \end{aligned} \tag{97}$$

which implies that $Ru = z^* = gu$.

As g and R are weakly compatible, we have $gz^* = gRu = Rgu = Rz^*$. Thus z^* is a coincidence point of g and R .

Similarly it can be shown that z^* is a coincidence point of the pair (f, S) $\square$

Theorem 26. Let $(X, \leq, \sigma_b)$ be a partially ordered σ_b -complete σ_b -metric-like space. Let $f, g : X \rightarrow X$ be two mappings. Suppose that for every two comparable elements $x, y \in X$ one has

$$\psi(2s\sigma_b(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{98}$$

where $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ are altering distance functions and

$$\begin{aligned}
 M(x, y) = \max\left\{\sigma_b(x, y), \sigma_b(x, fx), \sigma_b(y, gy), \frac{\sigma_b(x, gy) + \sigma_b(y, fx)}{4s}\right\}.
 \end{aligned} \tag{99}$$

Let f and g be continuous and let the pair (f, g) be weakly increasing. Then, f and g have a common fixed point z^* in X .

Proof. Let $x_0 \in X$. Let $\{x_n\}$ in X be constructed such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$, for all nonnegative integers n . As f and g are weakly increasing, we have

$$\begin{aligned}
 x_1 &= fx_0 \leq gfx_0 = x_2 = gx_1 \leq fgx_1 = x_3 \leq \dots \leq x_{2n+1} \\
 &= fx_{2n} \leq gfx_{2n} = x_{2n+2} \leq \dots
 \end{aligned} \tag{100}$$

Following the proof of the above theorem there exists $z^* \in X$ such that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sigma_b(x_{2n+1}, z^*) &= \lim_{n \rightarrow \infty} \sigma_b(fx_{2n}, z^*) = 0, \\
 \lim_{n \rightarrow \infty} \sigma_b(x_{2n+2}, z^*) &= \lim_{n \rightarrow \infty} \sigma_b(gx_{2n+1}, z^*) = 0.
 \end{aligned} \tag{101}$$

Using the triangular inequality, we get

$$\begin{aligned}
 \sigma_b(z^*, fz^*) &\leq s\sigma_b(z^*, fx_{2n}) + s\sigma_b(fx_{2n}, fz^*), \\
 \sigma_b(z^*, gz^*) &\leq s\sigma_b(z^*, gx_{2n+1}) + s\sigma_b(gx_{2n+1}, gz^*).
 \end{aligned} \tag{102}$$

Letting $n \rightarrow \infty$ and using continuity of f and g , we get

$$\begin{aligned}
 \sigma_b(z^*, fz^*) &\leq s \lim_{n \rightarrow \infty} \sigma_b(z^*, fx_{2n}) + s \lim_{n \rightarrow \infty} \sigma_b(fx_{2n}, fz^*) \\
 &= s\sigma_b(fz^*, fz^*) \leq 2s\sigma_b(fz^*, gz^*), \\
 \sigma_b(z^*, gz^*) &\leq s \lim_{n \rightarrow \infty} \sigma_b(z^*, gx_{2n+1}) + s \lim_{n \rightarrow \infty} \sigma_b(gx_{2n+1}, gz^*) \\
 &= s\sigma_b(gz^*, gz^*) \leq 2s\sigma_b(fz^*, gz^*).
 \end{aligned} \tag{103}$$

Therefore,

$$\begin{aligned}
 &\max\{\sigma_b(z^*, fz^*), \sigma_b(z^*, gz^*)\} \\
 &\leq \max\{s\sigma_b(fz^*, fz^*), s\sigma_b(gz^*, gz^*)\} \\
 &\leq 2s\sigma_b(gz^*, fz^*).
 \end{aligned} \tag{104}$$

From (98), we have

$$\psi(2s\sigma_b(fz^*, gz^*)) \leq \psi(M(z^*, z^*)) - \varphi(M(z^*, z^*)), \tag{105}$$

where

$$\begin{aligned}
 M(z^*, z^*) &= \max\left\{\sigma_b(z^*, z^*), \sigma_b(z^*, fz^*), \sigma_b(z^*, gz^*), \frac{\sigma_b(z^*, gz^*) + \sigma_b(z^*, fz^*)}{4s}\right\} \\
 &= \max\{\sigma_b(z^*, fz^*), \sigma_b(z^*, gz^*)\}.
 \end{aligned} \tag{106}$$

Hence, by (104) and (105), we get that $\varphi(M(z^*, z^*)) = 0$. Thus, we have $fz^* = gz^* = z^*$. $\square$

3.3. Common Fixed Points of Dominated Maps on Closed Balls. Motivated by [32, Theorem 2.1] we have the following result.

Theorem 27. Let $(X, \leq, \sigma_b)$ be a complete ordered b -metric-like space, $S, T : X \rightarrow X$ dominated maps, and x_0 an arbitrary point in X . Suppose that for $k \in [0, 1/s)$ and for $S \neq T$ one has

$$\sigma_b(Sx, Ty) \leq k\sigma_b(x, y) \tag{107}$$

for all comparable elements x, y in $\overline{B(x_0, rs)}$,

$$\sigma_b(x_0, Sx_0) \leq (1 - ks)r. \tag{108}$$

Let, for each nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, rs)}$, $\{x_n\} \rightarrow u$ imply that $u \leq x_n$. Then there exists $x^* \in \overline{B(x_0, rs)}$ such that $\sigma_b(x^*, x^*) = 0$ and $x^* = Sx^* = Tx^*$. Also, if for any two points x, y in $\overline{B(x_0, rs)}$ there exists a point $z \in \overline{B(x_0, rs)}$ such that $z \leq x$ and $z \leq y$, that is, every pair of elements has a lower bound, then x^* is a unique common fixed point in $\overline{B(x_0, rs)}$.

Proof. Choose a point x_1 in X such that $x_1 = Sx_0$. As $Sx_0 \leq x_0$, so $x_1 \leq x_0$ and let $x_2 = Tx_1$. Now $Tx_1 \leq x_1$ gives $x_2 \leq x_1$, and continuing this process, we construct a sequence x_n of points in X such that

$$x_{2i+1} = Sx_{2i}, \quad x_{2i+2} = Tx_{2i+1}, \quad x_{2i+1} = Sx_{2i} \leq x_{2i} \\ \text{where } i = 0, 1, 2, \dots \quad (109)$$

First we show that $x_n \in \overline{B(x_0, rs)}$ for all $n \in N$. Using inequality (108), we have

$$\sigma_b(x_0, x_1) \leq (1 - ks)r \leq rs. \quad (110)$$

It follows that $x_1 \in \overline{B(x_0, rs)}$.

Let $x_2, \dots, x_j \in \overline{B(x_0, rs)}$ for some $j \in N$. If $j = 2i + 1$, then $x_{2i+1} \leq x_{2i}$, where $i = 0, 1, 2, \dots, (j-1)/2$ so, using inequality (107), we obtain

$$\sigma_b(x_{2i+1}, x_{2i+2}) \\ = \sigma_b(Sx_{2i}, Tx_{2i+1}) \leq k[\sigma_b(x_{2i}, x_{2i+1})] \\ \leq k^2[\sigma_b(x_{2i-1}, x_{2i})] \leq \dots \leq k^{2i+1}\sigma_b(x_0, x_1). \quad (111)$$

If $j = 2i+2$, then as $x_1, x_2, \dots, x_j \in \overline{B(x_0, rs)}$ and $x_{2i+2} \leq x_{2i+1}$ ($i = 0, 1, 2, \dots, (j-2)/2$) we obtain

$$\sigma_b(x_{2i+2}, x_{2i+3}) \leq k^{2(i+1)}\sigma_b(x_0, x_1). \quad (112)$$

Thus from inequalities (111) and (112), we have

$$\sigma_b(x_j, x_{j+1}) \leq k^j\sigma_b(x_0, x_1). \quad (113)$$

Now

$$\sigma_b(x_0, x_{j+1}) \\ \leq s\sigma_b(x_0, x_1) + s^2\sigma_b(x_1, x_2) + \dots + s^j\sigma_b(x_{j-1}, x_j) \\ + s^j\sigma_b(x_j, x_{j+1}) \\ \leq s\sigma_b(x_0, x_1) + ks^2\sigma_b(x_0, x_1) + \dots + k^{j-1}s^j\sigma_b(x_0, x_1) \\ + k^js^j\sigma_b(x_0, x_1) \quad (\text{by (113)}) \\ \leq s\sigma_b(x_0, x_1)[1 + ks + \dots + k^{j-1}s^{j-1} + k^js^j] \\ \leq s \frac{(1 - k^{j+1}s^{j+1})}{1 - ks} \sigma_b(x_0, x_1) \\ \leq s \frac{(1 - k^{j+1}s^{j+1})}{1 - ks} (1 - ks)r \quad (\text{by (108)}) \\ \leq s(1 - k^{j+1}s^{j+1})r \leq rs. \quad (114)$$

Thus, $x_{j+1} \in \overline{B(x_0, rs)}$. Hence $x_n \in \overline{B(x_0, rs)}$ for all $n \in N$. This implies that

$$\sigma_b(x_n, x_{n+1}) \leq k^n\sigma_b(x_0, x_1), \quad \forall n \in N. \quad (115)$$

It follows that

$$\sigma_b(x_n, x_{n+i}) \\ \leq s\sigma_b(x_n, x_{n+1}) + s^2\sigma_b(x_{n+1}, x_{n+2}) \\ + \dots + s^{i-1}\sigma_b(x_{n+i-2}, x_{n+i-1}) + s^{i-1}\sigma_b(x_{n+i-1}, x_{n+i}) \\ \leq sk^n\sigma_b(x_0, x_1) + s^2k^{n+1}\sigma_b(x_0, x_1) \\ + \dots + s^{i-1}k^{n+i-2}\sigma_b(x_0, x_1) + s^{i-1}k^{n+i-1}\sigma_b(x_0, x_1) \\ (\text{by (115)}) \\ \leq sk^n\sigma_b(x_0, x_1)[1 + ks + \dots + k^{i-2}s^{i-2} + k^{i-1}s^{i-2}] \\ \leq s \frac{k^n(1 - k^is^i)}{1 - ks} \sigma_b(x_0, x_1) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (116)$$

Notice that the sequence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, rs)}, \sigma_b)$. Therefore there exists a point $x^* \in \overline{B(x_0, rs)}$ with $\lim_{n \rightarrow \infty} x_n = x^*$. Also,

$$\lim_{n \rightarrow \infty} \sigma_b(x_n, x^*) = 0. \quad (117)$$

Now,

$$\sigma_b(x^*, Sx^*) \leq s\sigma_b(x^*, x_{2n+2}) + s\sigma_b(x_{2n+2}, Sx^*). \quad (118)$$

On taking the limit as $n \rightarrow \infty$ and using the fact that $x^* \leq x_n$ when $x_n \rightarrow x^*$, we have

$$\sigma_b(x^*, Sx^*) \leq s \lim_{n \rightarrow \infty} [\sigma_b(x^*, x_{2n+2}) + k\sigma_b(x_{2n+1}, x^*)]. \quad (119)$$

By (117) we obtain

$$\sigma_b(x^*, Sx^*) \leq 0, \quad (120)$$

and hence $x^* = Sx^*$. Similarly, by using

$$\sigma_b(x^*, Tx^*) \leq s\sigma_b(x^*, x_{2n+1}) + s\sigma_b(x_{2n+1}, Tx^*), \quad (121)$$

we can show that $x^* = Tx^*$. Hence S and T have a common fixed point in $\overline{B(x_0, rs)}$. Now,

$$\sigma_b(x^*, x^*) = \sigma_b(Sx^*, Tx^*) \leq k\sigma_b(x^*, x^*). \quad (122)$$

This implies that

$$\sigma_b(x^*, x^*) = 0. \quad (123)$$

For the uniqueness, assume that y is another fixed point of T and S in $\overline{B(x_0, rs)}$. If x^* and y are comparable then

$$\sigma_b(x^*, y) = \sigma_b(Sx^*, Ty) \leq k\sigma_b(x^*, y). \quad (124)$$

This shows that $x^* = y$. Now if x^* and y are not comparable then there exists a point $z_0 \in \overline{B(x_0, rs)}$ such that $z_0 \leq x^*$ and $z_0 \leq y$. Choose a point z_1 in X such that $z_1 = Tz_0$.

As $Tz_0 \leq z_0$, so $z_1 \leq z_0$; let $z_2 = Sz_1$. Now $Sz_1 \leq z_1$ gives $z_2 \leq z_1$, and continuing this process choose z_n in X such that

$$z_{2i+1} = Tz_{2i}, \quad z_{2i+2} = Sz_{2i+1}, \quad z_{2i+1} = Tz_{2i} \leq z_{2i} \\ \text{where } i = 0, 1, 2, \dots \quad (125)$$

It follows that $z_{n+1} \leq z_n \leq \dots \leq z_0 \leq x^*$. As $z_0 \leq x^*$ and $z_0 \leq y$, it follows that $z_n \leq Tx^*$ and $z_n \leq Ty$ for all $n \in \mathbb{N}$. We will prove that $z_n \in \overline{B(x_0, rs)}$ for all $n \in \mathbb{N}$ by using mathematical induction. For $n = 1$,

$$\sigma_b(x_0, z_1) \leq s\sigma_b(x_0, x_1) + s\sigma_b(x_1, z_1) \\ \leq s(1 - ks)r + ks\sigma_b(x_0, z_0) \quad (126) \\ \leq s(1 - ks)r + ks^2r = sr.$$

It follows that $z_1 \in \overline{B(x_0, rs)}$. Let $z_2, z_3, \dots, z_j \in \overline{B(x_0, rs)}$ for some $j \in \mathbb{N}$. Note that if j is odd then

$$\sigma_b(x_{j+1}, z_{j+1}) = \sigma_b(Tx_j, Sz_j) \leq k\sigma_b(x_j, z_j) \\ \leq \dots \leq k^{j+1}\sigma_b(x_0, z_0), \quad (127)$$

and if j is even then

$$\sigma_b(x_{j+1}, z_{j+1}) = \sigma_b(Sx_j, Tz_j) \leq k\sigma_b(x_j, z_j) \\ \leq \dots \leq k^{j+1}\sigma_b(x_0, z_0). \quad (128)$$

Now,

$$\sigma_b(x_0, z_{j+1}) \\ \leq s\sigma_b(x_0, x_1) + s^2\sigma_b(x_1, x_2) + \dots + s^j\sigma_b(x_j, x_{j+1}) \\ + s^j\sigma_b(x_{j+1}, z_{j+1}) \\ \leq s\sigma_b(x_0, x_1) + ks^2\sigma_b(x_0, x_1) + \dots + k^j s^j\sigma_b(x_0, x_1) \\ + k^{j+1} s^j\sigma_b(x_0, z_0) \\ \leq s\sigma_b(x_0, x_1) [1 + ks + \dots + k^j s^{j-1}] + k^{j+1} s^{j+1} r \\ \leq s\sigma_b(x_0, x_1) [1 + ks + \dots + k^j s^j] + k^{j+1} s^{j+1} r \\ \leq s(1 - ks)r \frac{(1 - k^{j+1} s^{j+1})}{1 - ks} + k^{j+1} s^{j+1} r \\ = sr - k^{j+1} s^{j+2} r + k^{j+1} s^{j+1} r \\ \leq sr - k^{j+1} s^{j+2} r + k^{j+1} s^{j+2} r = sr, \quad (129)$$

and this implies that

$$\sigma_b(x_0, z_{j+1}) \leq sr. \quad (130)$$

Thus $z_{j+1} \in \overline{B(x_0, rs)}$. Hence $z_n \in \overline{B(x_0, rs)}$ for all $n \in \mathbb{N}$. As $z_0 \leq x^*$ and $z_0 \leq y$, it follows that $z_n \leq T^n x^*$, $z_n \leq S^n x^*$,

$z_n \leq S^n y$, and $z_n \leq T^n y$ for all $n \in \mathbb{N}$ as $S^n x^* = T^n x^* = x^*$ and $S^n y = T^n y = y$ for all $n \in \mathbb{N}$. If n is odd then

$$\sigma_b(x^*, y) \\ = \sigma_b(T^n x^*, T^n y) \\ \leq s\sigma_b(T^n x^*, Sz_n) + s\sigma_b(Sz_n, T^n y) \\ \leq ks\sigma_b(T^{n-1} x^*, z_n) + ks\sigma_b(z_n, T^{n-1} y) \\ = ks\sigma_b(S^{n-1} x^*, Tz_{n-1}) + ks\sigma_b(Tz_{n-1}, S^{n-1} y) \\ \leq k^2 s\sigma_b(S^{n-2} x^*, z_{n-1}) + k^2 s\sigma_b(z_{n-1}, S^{n-2} y) \\ \vdots \\ \leq k^{n+1} s\sigma_b(x^*, z_0) + k^{n+1} s\sigma_b(z_0, y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (131)$$

Hence, $x^* = y$. Similarly, we can show that $x^* = y$ if n is even.

Thus, x^* is a unique common fixed point of T and S in $\overline{B(x_0, rs)}$. $\square$

3.4. Periodic Point Results. Clearly, a fixed point of f is also a fixed point of f^n for every $n \in \mathbb{N}$; that is, $F(f) \subset F(f^n)$. However, the converse is false. For example, the mapping $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $fx = 1/2 - x$, has a unique fixed point $1/4$ but every $x \in \mathbb{R}$ is a fixed point of f^2 . If $F(f) = F(f^n)$ for every $n \in \mathbb{N}$, then f is said to have property P . For more details, we refer to [33, 34] and the references mentioned therein.

Recently, the study of periodic points for contraction mappings has been considered by many authors; for instance, every quasicontraction $f: X \rightarrow X$ with the constant $\alpha \in [0, 1/2)$, where X is a cone metric space, has the property P [35, Theorem 3.1] and if (X, d) is a cone metric space and T -Hardy-Rogers contraction $f: X \rightarrow X$ satisfies some appropriate conditions, then f has property P [36, Corollary 3.3].

Definition 28 (see [21]). Let $(X, \leq)$ be a partially ordered set. A mapping f is called dominating on X if $x \leq fx$ for each x in X .

Example 29 (see [21]). Let $X = [0, \infty)$ be endowed with the usual ordering. Let $f: X \rightarrow X$ be defined by $fx = \sqrt[n]{x}$ for $x \in [0, 1)$ and $fx = x^n$ for $x \in [1, \infty)$, for any $n \in \mathbb{N}$. Then, for all $x \in X$, $x \leq fx$; that is, f is a dominating map.

We have the following results.

Theorem 30. Let $(X, \leq, \sigma_b)$ be a partially ordered complete b -metric-like space. Let $f: X \rightarrow X$ be a nondecreasing mapping such that for all $x \in X$ with $x \leq fx$ one has

$$\sigma_b(fx, f^2x) \leq \lambda \sigma_b(x, fx), \quad (132)$$

where $\lambda \in [0, 1)$. Then f has the property P provided that $F(f)$ is nonempty and f is dominating on $F(f^n)$.

Proof. Let $u \in F(f^n)$ for some $n > 1$. We will show that $u = fu$. Since f is dominating on $F(f^n)$, therefore $u \leq fu$ which implies that $f^{n-1}u \leq f^n u$ as f is nondecreasing. Using (132), we obtain that

$$\begin{aligned}\sigma_b(u, fu) &= \sigma_b(ff^{n-1}u, f^2f^{n-1}u) \\ &\leq \lambda\sigma_b(f^{n-1}u, f^n u) = \lambda\sigma_b(ff^{n-2}u, f^2f^{n-2}u).\end{aligned}\quad (133)$$

Repeating the above process, we get

$$\sigma_b(u, fu) \leq \lambda^n \sigma_b(u, fu), \quad (134)$$

which, on taking the limit as $n \rightarrow \infty$, implies that $\sigma_b(u, fu) = 0$, implying that $u = fu$. $\square$

Theorem 31. Let X and f be as in Theorem 18. If f is dominating on X , then f satisfies property P .

Proof. From Theorem 18, $F(f) \neq \emptyset$. We will prove that (132) is satisfied for all $x \leq fx$.

If $\sigma_b(fx, f^2x) = 0$, then it is easy to see that (132) is satisfied. Now we suppose that $\sigma_b(fx, f^2x) > 0$. Since f is dominating, we have $x \leq fx$. Also, $fx \leq f^2x$ as f is nondecreasing. Using (15), we have

$$\begin{aligned}\sigma_b(fx, f^2x) &= \sigma_b(fx, ffx) \\ &\leq \alpha \max\{\sigma_b(x, fx), \sigma_b(x, fx), \sigma_b(fx, f^2x), \\ &\quad \sigma_b(x, f^2x), \sigma_b(fx, fx)\} \\ &= \alpha \max\{\sigma_b(x, fx), \sigma_b(fx, f^2x), \sigma_b(x, f^2x), \\ &\quad \sigma_b(fx, fx)\} \\ &\leq \alpha \max\{\sigma_b(x, fx), \sigma_b(fx, f^2x), s\sigma_b(x, fx) \\ &\quad + s\sigma_b(fx, f^2x), 2s\sigma_b(x, fx)\}.\end{aligned}\quad (135)$$

We will show that $\sigma_b(fx, f^2x) \leq \sigma_b(x, fx)$. On the contrary, if $\sigma_b(x, fx) < \sigma_b(fx, f^2x)$, then from the above inequality we have

$$\sigma_b(fx, f^2x) < 2s\alpha\sigma_b(fx, f^2x), \quad (136)$$

which implies that $1 < 2s\alpha < 2s \cdot (1/2s^2) < 1$, a contradiction. So we have

$$\sigma_b(fx, f^2x) \leq \lambda\sigma_b(x, fx), \quad (137)$$

where $\lambda = 2s\alpha$. Obviously, $\lambda \in [0, 1)$. By Theorem 30, f has property P . $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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