Research Article

Extending an Almost Complete Pair of Matrices to a Complete Triple

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Motivated by the concept of complete pairs, which was introduced by Krein and Langer, we present the concepts of an almost complete pair of matrices and a complete triple of matrices. It is proved that an almost complete pair of matrices can be extended to a complete triple. An application of the result to differential equations is also given.

1. Introduction

Let

$$L(\lambda) = I\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0 \quad (1)$$

be a monic matrix polynomial, where $A_i \in M_n(\mathbb{C})$ for $i = 0, 1, \dots, l-1$. If there exist a matrix S and a polynomial $L_1(\lambda)$ of degree $l-1$ such that $L(\lambda) = L_1(\lambda)(I\lambda - S)$, then $I\lambda - S$ is said to be a monic right divisor of $L(\lambda)$. As is well known, $I\lambda - S$ is a monic right divisor of $L(\lambda)$ if and only if S is a root of $L(\lambda)$; that is,

$$S^l + A_{l-1}S^{l-1} + \cdots + A_0 = 0. \quad (2)$$

Let C be the companion matrix of $L(\lambda)$; that is,

$$C = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{l-1} \end{pmatrix}. \quad (3)$$

Then we have $\det(I\lambda - C) = \det(L(\lambda))$. If $S_1, S_2, \dots, S_l \in M_n(\mathbb{C})$ are the different l roots of $L(\lambda)$, it is clear that the Vandermonde matrix

$$W(S_1, S_2, \dots, S_l) = \begin{pmatrix} I & I & I & \cdots & I \\ S_1 & S_2 & S_3 & \cdots & S_l \\ S_1^2 & S_2^2 & S_3^2 & \cdots & S_l^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_1^{l-1} & S_2^{l-1} & S_3^{l-1} & \cdots & S_l^{l-1} \end{pmatrix} \quad (4)$$

plays a role in the investigation of the spectral analysis of $L(\lambda)$ [1].

Definition 1. We say that the l -tuple $(S_1, S_2, \dots, S_l)$ of complex square matrices of order n is complete if the Vandermonde matrix $W(S_1, S_2, \dots, S_l)$ is invertible. When $l = 2$ and $l = 3$, we call a complete l -tuple of matrices a complete pair and a complete triple, respectively.

Suppose that (S_1, S_2) is complete. Take $P = S_2 - S_1$. Then P is invertible. If we take

$$A_1 = -PS_2P^{-1} - S_1, \quad A_2 = PS_2P^{-1}S_1, \quad (5)$$

$$L(\lambda) = I\lambda^2 + A_1\lambda + A_2,$$

then $I\lambda - S_1, I\lambda - S_2$ form a complete pair of right divisors of $L(\lambda)$ (cf. Lemma 2.4 in [2]). The concept of complete pair was

introduced in [3, 4]. In [5], Lancaster gave some important applications of complete pairs to solutions of differential equations. Motivated by the results in [5], we are interested in the following question.

Question 1. Suppose that (S_1, S_2) is not a complete pair. Is it possible for us to find S_3 and a cubic matrix polynomial $L(\lambda)$ such that (S_1, S_2, S_3) is a complete triple and

$$I\lambda - S_1, \quad I\lambda - S_2, \quad I\lambda - S_3 \quad (6)$$

are right divisors of $L(\lambda)$?

If (S_1, S_2, S_3) is a complete triple, then

$$\begin{pmatrix} I & I \\ S_1 & S_2 \\ S_1^2 & S_2^2 \end{pmatrix} \quad (7)$$

is of full column rank. Note that if $S_1, S_2 \in M_n(\mathbb{C})$ and the rank of $S_1 - S_2$ is less than $n - 1$, we cannot always find S_3 such that (S_1, S_2, S_3) is a complete triple. For example, let $S_1, S_2 \in M_n(\mathbb{C})$ be defined as follows:

$$S_1 = \begin{pmatrix} A_1 & \alpha \\ \beta & A_2 \end{pmatrix}, \quad (8)$$

where A_1, A_2 are square matrices of orders $n - 2$ and 2 , respectively, and $S_2 - S_1$ is in the form

$$\begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

Then

$$S_2^2 - S_1^2 = \begin{pmatrix} 2A_1 + I_{n-2} & \alpha \\ \beta & 0 \end{pmatrix}. \quad (10)$$

If $\text{rank}(\alpha) = 1$, then the last two columns of $\begin{pmatrix} S_2 - S_1 \\ S_2^2 - S_1^2 \end{pmatrix}$ are linearly dependent. For any $S_3 \in M_n(\mathbb{C})$,

$$\begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (11)$$

cannot be invertible.

If $\text{rank}(\alpha) = 2$, then

$$\begin{pmatrix} I & I \\ S_1 & S_2 \\ S_1^2 & S_2^2 \end{pmatrix} \quad (12)$$

is of full column rank. From the condition that

$$\begin{pmatrix} I & I \\ S_1 & S_2 \\ S_1^2 & S_2^2 \end{pmatrix} \quad (13)$$

is of full column rank, we cannot conclude that $\text{rank}(S_2 - S_1) \geq n - 1$.

After the above discussion, we give the following definition.

Definition 2. For two matrices $S_1, S_2 \in M_n(\mathbb{C})$, one says they form an almost complete pair if the matrix

$$\begin{pmatrix} I & I \\ S_1 & S_2 \\ S_1^2 & S_2^2 \end{pmatrix} \quad (14)$$

is of full column rank and the rank of $S_1 - S_2$ is $n - 1$. For simplification, one says (S_1, S_2) is an almost complete pair of matrices of order n .

The following is the main result of this paper.

Theorem 3. Suppose that $S_1, S_2 \in M_n(\mathbb{C})$, $n \geq 2$, and (S_1, S_2) is an almost complete pair. Then one can find a matrix $S_3 \in M_n(\mathbb{C})$ such that (S_1, S_2, S_3) is a complete triple; that is,

$$\begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (15)$$

is invertible.

We prove this theorem in Section 2. In Section 3, we partially answer the question mentioned before and give an application of the main result to differential equations.

2. Proof of the Main Theorem

In this section, we prove Theorem 3. Note that there is an invertible matrix P such that $P^{-1}(S_2 - S_1)P$ is the Jordan normal form of $S_2 - S_1$. Let $Q = P \oplus P \oplus P$. Then the condition that

$$X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (16)$$

is invertible is equivalent to that

$$Q^{-1}XQ \quad (17)$$

is invertible. So from the beginning, we can assume that $S_2 - S_1$ is in Jordan normal form.

First we prove a lemma.

Lemma 4. Assume $S_1, S_2 \in M_n(\mathbb{C})$ and the pair (S_1, S_2) is an almost complete pair with $S_2 - S_1 = I_{n-1} \oplus 0$. Write

$$S_1 = \begin{pmatrix} A & \alpha \\ \beta & s \end{pmatrix}, \quad (18)$$

where $A \in M_{n-1}(\mathbb{C})$. Then α is not a zero vector.

Proof. By some computations, we have

$$S_2^2 - S_1^2 = \begin{pmatrix} 2A + I_{n-1} & \alpha \\ \beta & 0 \end{pmatrix}. \quad (19)$$

The last column of

$$\begin{pmatrix} S_2 - S_1 \\ S_2^2 - S_1^2 \end{pmatrix} \quad (20)$$

is $(0, 0, \dots, 0, \alpha, 0)^T$, which is a $2n$ -dimensional column vector. By the definition of almost complete pair, α is not a zero vector. $\square$

Now we prove the main theorem.

Proof. From now on, we write $S_1 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$. We denote by $P(i, j) \in M_{n-1}(\mathbb{R})$ the matrix which is obtained by exchanging the columns of i and j of $(n-1) \times (n-1)$ identity matrix.

Without loss of generality, we can assume $S_2 - S_1$ is in the Jordan normal form.

Case 1. Firstly, we consider the case that

$$S_2 - S_1 = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (21)$$

If $a_{1n} \neq 0$, then we take

$$S_3 = S_1 + \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}. \quad (22)$$

By some computation, the last columns of $S_2^2 - S_1^2$ and $S_3^2 - S_1^2$ are $(a_{1n}, a_{2n}, \dots, a_{n-1, n}, 0)^T$ and $(0, ta_{1n}, ta_{2n}, \dots, ta_{n-1, n})^T$, respectively, and the i th $(1 \leq i \leq n-2)$ column of $S_3^2 - S_1^2$ is

$$\begin{pmatrix} f_{i,1}(t), f_{i,2}(t), \dots, f_{i,i+1}(t), \\ t^2 + f_{i,i+2}(t), f_{i,i+3}(t), \dots, f_{i,n}(t) \end{pmatrix}^T, \quad (23)$$

and the $(n-1)$ th column of $S_3^2 - S_1^2$ is $(f_{n-1,1}(t), f_{n-1,2}(t), \dots, f_{n-1,n}(t))^T$, where $f_{i,1} = ta_{1, i+1}$ and $f_{i,j} = t(a_{j, i+1} + a_{j-1, i})$ ($1 \leq i \leq n-1$, $2 \leq j \leq n$). It is clear that

$$\det \begin{pmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{pmatrix} \quad (24)$$

is a polynomial in t . If $a_{1n} \neq 0$, then the degree of this polynomial is $2n-2$ and the leading term of this polynomial is $a_{1n}^2 t^{2n-2}$. If $|t|$ is sufficiently large, we can get that

$$X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (25)$$

is invertible. If $a_{1n} = 0$, by Lemma 4, then there must be some

$$a_{in} \neq 0, \quad (26)$$

where $2 \leq i \leq n-1$. So

$$\begin{pmatrix} P(1, i) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} P(1, i) & 0 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} (P(1, i)A)P(1, i) & P(1, i)\alpha \\ \beta P & 0 \end{pmatrix}, \quad (27)$$

where the first entry of the vector $P(1, i)\alpha$ is a_{in} . Take S_3 such that

$$(P(1, i) \oplus 1)(S_3 - S_1)(P(1, i) \oplus 1) = \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}. \quad (28)$$

Then

$$X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (29)$$

is invertible.

Case 2. Now we consider the case that $S_2 - S_1$ has the Jordan normal form

$$J(\lambda_1) \oplus J(\lambda_2) \oplus \dots \oplus J(\lambda_k) \oplus 0, \quad (30)$$

where $\lambda_i \neq 0$, $1 \leq i \leq k$. If $a_{1n} \neq 0$, then we take

$$S_3 = S_1 + \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}. \quad (31)$$

If $a_{1n} = 0$ and $a_{in} \neq 0$ ($1 < i < n$), then we take S_3 such that

$$(P(1, i) \oplus 1)(S_3 - S_1)(P(1, i) \oplus 1) = \begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}. \quad (32)$$

In any case when $|t|$ is sufficiently large, we can get that

$$X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (33)$$

is invertible.

Case 3. Now we consider the case that 0 is not a single root of characteristic polynomial of $S_2 - S_1$; that is, $S_2 - S_1$ has the Jordan normal form:

$$J(\lambda_1) \oplus J(\lambda_2) \oplus \dots \oplus J(\lambda_s) \oplus J(0). \quad (34)$$

If $n = 2$, then

$$S_2 - S_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (35)$$

By Lemma 4, $a_{12} \neq 0$. Take

$$S_3 = S_1 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (36)$$

Then

$$\det \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} = \det \begin{pmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{pmatrix} = -t^2 a_{12}^2 \neq 0. \quad (37)$$

We can conclude that

$$X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (38)$$

is invertible.

Firstly, we prove the case that the normal form itself is

$$\begin{pmatrix} 0 & 0 \\ tI_{n-1} & 0 \end{pmatrix}. \quad (39)$$

If $a_{1n} \neq 0$, then we take

$$S_3 = S_1 + \begin{pmatrix} t & 0 & 0 & 0 & \cdots & 0 & t \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & 0 \end{pmatrix}. \quad (40)$$

So

$$(S_3 - S_1)^2 = \begin{pmatrix} t^2 & 0 & 0 & \cdots & 0 & t^2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t^2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t^2 & 0 & 0 \end{pmatrix}. \quad (41)$$

By some computation,

$$\det X = \det \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (42)$$

is a polynomial in t , where the leading term of it is $a_{1n} t^{2n-1}$. So if $a_{1n} \neq 0$, then X is invertible when $|t|$ is sufficiently large. If $a_{1n} = 0$, then there must be an $i \in \{2, 3, \dots, n-1\}$ such that $a_{in} \neq 0$. If $i \in \{2, 3, \dots, n-2\}$, take S_3 such that

$$(P(1, i-1) \oplus 1)(S_3 - S_1)(P(1, i-1) \oplus 1) = \begin{pmatrix} t & 0 & 0 & 0 & \cdots & 0 & t \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & 0 \end{pmatrix}. \quad (43)$$

When $|t|$ is sufficiently large, we have X being invertible. If $i = n-1$, that is to say, the last column of $S_2^2 - S_1^2$ is $(0, 0, \dots, 0, a_{n-1n})^T$, then we take

$$S_3 = S_1 + \begin{pmatrix} t & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (44)$$

We have

$$(S_3 - S_1)^2 = \begin{pmatrix} t^2 & 0 & 0 & 0 & \cdots & 2t & 0 \\ 0 & t^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (45)$$

where the first entry of the last column of $S_3^2 - S_1^2$ is a_{n-1n} . So the leading term of the polynomial

$$\det X = \det \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (46)$$

is $a_{n-1n}^2 t^{2n-3}$. Thus

$$X = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (47)$$

is invertible when $|t|$ is sufficiently large.

Now we prove the general case that $S_2 - S_1$ has the Jordan normal form:

$$J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_s) \oplus J(0), \quad (48)$$

where $\lambda_i \neq 0$, $1 \leq i \leq s$. If there is some $a_{ni} \neq 0$, $m+1 \leq i \leq n$, we assume that $J(0)$ is a matrix of order $n-m$. For any square matrix B of order $n-m$, we can construct a square matrix A of order $n-m$ such that

$$\begin{pmatrix} J(0) & A \\ (J(0) + B)^2 - J^2(0) & (A+B)^2 - A^2 \end{pmatrix} \quad (49)$$

is invertible. Take

$$S_3 = S_1 + (tI_{n-m} \oplus A). \quad (50)$$

If $a_{in} = 0$, for all $m+1 \leq i \leq n$, there must be some j such that $1 \leq j \leq m$, $a_{jn} \neq 0$. Let $Y = XP$, where P is the matrix which is obtained by exchanging the columns $m+1$ and n of $n \times n$ identity matrix. Then $\det X = -\det Y$ and

$$Y = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_i) \oplus 0 \oplus C, \quad (51)$$

where the square matrix C of order $n - m + 1$ has the form

$$\begin{pmatrix} 0 & 1 \\ I_{n-m} & 0 \end{pmatrix}. \quad (52)$$

Let T be a matrix such that $T^{-1}CT$ has the Jordan normal form J . Then we can consider the matrix $Z = (I \oplus T^{-1})Y(I \oplus T)$. Note that Z has the form

$$Z = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_i) \oplus 0 \oplus J. \quad (53)$$

Denote $S = J(\lambda_1) \oplus J(\lambda_2) \oplus \cdots \oplus J(\lambda_i) \oplus 0$. For any square matrix B , we can construct a square matrix A of order $m + 1$ such that

$$\begin{pmatrix} S & A \\ (S+B)^2 - S^2 & (A+B)^2 - A^2 \end{pmatrix} \quad (54)$$

is invertible. Take $S_3 = S_1 + A \oplus tI_{m-1}$. Then we can conclude that

$$\begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (55)$$

is invertible. The proof is complete. $\square$

3. Applications

For given monic matrix polynomials $L_1(\lambda), L_2(\lambda), \dots, L_n(\lambda)$, the construction of a common multiple of them was presented in [2].

Suppose that (S_1, S_2) is an almost complete pair. By Theorem 3 in Section 1, we can find $S_3 \in M_n(\mathbb{C})$ such that (S_1, S_2, S_3) is a complete triple; that is,

$$\begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix} \quad (56)$$

is invertible. Let

$$W_m(S_1, S_2, S_3) = \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ \vdots & \vdots & \vdots \\ S_1^{m-1} & S_2^{m-1} & S_3^{m-1} \end{pmatrix}. \quad (57)$$

Then we can conclude that

$$\min \{m \geq 1 \mid \text{Ker } W_m = \text{Ker } W_{m+1}\} = 3. \quad (58)$$

By Theorem 9.11 of [2], there exists a monic matrix polynomial $L(\lambda)$ with degree 3 which is a common multiple of $I\lambda - S_1, I\lambda - S_2, I\lambda - S_3$.

As is well known, spectral method for the analysis of monic matrix polynomials was important (cf. [6–13]). The spectral method basically uses the construction of standard pairs and standard triples (e.g., [14, 15]) to solve differential equations.

Now we recall definitions of standard pair and standard triple for monic polynomials.

Definition 5 (see [2]). A pair of matrices (X, T) , where X is $n \times nl$ and T is $nl \times nl$, is called a standard pair for the monic matrix polynomial $L(\lambda)$ if the following conditions are satisfied:

(i) $\text{col}(XT^i)_{i=0}^{l-1}$ is nonsingular, where

$$\text{col}(XT^i)_{i=0}^{l-1} = \begin{pmatrix} X \\ XT \\ \vdots \\ XT^{l-1} \end{pmatrix}; \quad (59)$$

(ii) $\sum_{i=0}^{l-1} A_i XT^i + XT^l = 0$.

If T is in Jordan normal form, we call (X, T) a Jordan pair for the monic matrix polynomial $L(\lambda)$.

Definition 6 (see [2]). A triple of matrices (X, T, Y) , where X is $n \times nl$, T is $nl \times nl$, and Y is $nl \times n$, is called a standard triple for the monic matrix polynomial $L(\lambda)$ if (X, T) is a standard

pair for $L(\lambda)$, and $Y = \begin{pmatrix} X \\ XT \\ \vdots \\ XT^{l-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I \end{pmatrix}$.

Now we recall the explicit formulas for solutions of the following differential equation:

$$L\left(\frac{d}{dt}\right)x(t) = \frac{d^l x(t)}{dt^l} + \sum_{j=0}^{l-1} A_j \frac{d^j x(t)}{dt^j} = f(t). \quad (60)$$

Lemma 7 (Theorem 2.9 in [2]). The general solution of (60) is given by the formula

$$x(t) = Xe^{tT}c + X \int_a^t e^{(t-s)T} Y f(s) ds, \quad t \in [a, b], \quad (61)$$

where (X, T, Y) is a standard triple of $L(\lambda)$ and $c \in \mathbb{C}^{nl}$ is arbitrary. In particular, the general solution of the homogeneous equation

$$L\left(\frac{d}{dt}\right)x(t) = 0 \quad (62)$$

is given by the formula

$$x(t) = Xe^{tT}c, \quad c \in \mathbb{C}^{nl}. \quad (63)$$

Before we give the main result of this section, we introduce some notations. Suppose

$$J_1 = X_1^{-1}S_1X_1, \quad J_2 = X_2^{-1}S_2X_2, \quad J_3 = X_3^{-1}S_3X_3 \quad (64)$$

are the Jordan forms of S_1, S_2, S_3 , respectively. Write

$$X = (X_1 \ X_2 \ X_3), \quad J = J_1 \oplus J_2 \oplus J_3. \quad (65)$$

Theorem 8. Let (S_1, S_2, S_3) be a complete triple of $L(\lambda) = 0$, where $L(\lambda) = I\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$. Then every solution of

$$L\left(\frac{d}{dt}\right)u = f \quad (66)$$

is given by

$$u(t) = e^{S_1 t} c_1 + e^{S_2 t} c_2 + e^{S_3 t} c_3 + X \int_a^t e^{(t-s)J} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} f(s) ds, \quad t \in [a, b], \quad (67)$$

for some $c_1, c_2, c_3 \in \mathbb{C}^n$, where

$$\begin{pmatrix} Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ X_3^{-1} \end{pmatrix}, \quad (68)$$

$$Y_1 = -X_1^{-1} X_2 Y_2 - X_1^{-1} X_3 Y_3.$$

In particular, every solution of

$$L\left(\frac{d}{dt}\right)u = 0 \quad (69)$$

has the form

$$u(t) = e^{S_1 t} c_1 + e^{S_2 t} c_2 + e^{S_3 t} c_3, \quad (70)$$

where $c_1, c_2, c_3 \in \mathbb{C}^n$.

Proof. Using Lemma 7, we take

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = X^{-1} \begin{pmatrix} I & I & I \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \quad (71)$$

Then (X, J, Y) is a standard triple; it is easy to get the conclusion. $\square$

Remark 9. The above theorem and its proof are motivated by those in [5] (cf. also Sections 2.4 and 2.5 in [2]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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