Research Article
On Rationality of Kneading Determinants

Sheng Chen and Chao Xia
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
Correspondence should be addressed to Sheng Chen; schenhit@gmail.com
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In this section, $R$ denotes a ring with identity $I$. We study conditions under which $I-\lambda A$ and $I-\lambda B$ are left coprime or right coprime, where $A, B \in R$. As applications, we get sufficient conditions under which the Kneading determinant of a finite rank pair of operators on an infinite dimensional space is rational.

1. Introduction

If $A$ is a $k \times k$ matrix with rational coefficients, one has the well-known identity between formal power series:

$$\exp \sum_{n \geq 1} -\frac{\text{tr}(A^n)}{n} z^n = \det(I-zA),$$

where $\text{tr}(A^n)$ denotes the trace of matrix $A^n$ (matrix $A$ raised to the $n$th power) and $I$ denotes the $k \times k$ identity matrix. This identity plays a significant role in the discussion of an important problem in dynamical systems theory. For more details see [1, 2].

We denote by $H$ the infinite dimension vector space over $Q$, the space of linear forms on $H$ will be denoted, as usual, by $H^*$, and the space of all linear endomorphisms on $H$ will be denoted by $L(H)$. If $\psi \in L(H)$ and $n$ is a nonnegative integer, the $n$th iterate $\psi^n$ is defined recursively by $\psi^0 = 1d_H \in L(H)$, $\psi^n = \psi \circ \psi^{n-1}$, for $n \geq 1$.

The subspace of $L(H)$ whose elements are the linear endomorphism on $H$ with finite rank will be denoted by $L_{FR}(H)$.

Let $q$ be a positive integer; we use the symbol $\bar{h}$ to denote an element of $H^q = H \times H \times \cdots \times H$ ($q$ times) and the symbol $\bar{\alpha}$ to denote an element of $H^q$; that is

$$\bar{h} = (h_1, \ldots, h_q), \quad \bar{\alpha} = (\alpha_1, \ldots, \alpha_q). \quad (2)$$

Given $\bar{h} \in H^q$ and $\bar{\alpha} \in H^{q}$, we define the finite rank endomorphism $\bar{\alpha} \circ \bar{h} \in L_{FR}(H)$ and the matrix $M(\bar{\alpha}, \bar{h}) \in Q^{q \times q}$ by setting

$$\bar{\alpha} \circ \bar{h} = \sum_{p=1}^{q} \alpha_p \circ h_p, \quad (3)$$

with the usual notation

$$\alpha \in H^*, h \in H : (\alpha \circ h)(x) = \alpha(x) \ u, \quad x \in H,$$

$$M(\bar{\alpha}; \bar{h}) = \begin{pmatrix} \alpha_1(h_1) & \cdots & \alpha_1(h_q) \\ \vdots & \ddots & \vdots \\ \alpha_q(h_1) & \cdots & \alpha_q(h_q) \end{pmatrix}. \quad (4)$$

Definition 1 (see [3]). A pair $(\varphi, \psi) \in L(H) \times L(H)$ is said to have finite rank if $\psi - \varphi \in L_{FR}(H)$.

Notice that if a pair $(\varphi, \psi)$ has finite rank, then the pair $(\varphi^n, \psi^n)$ also has finite rank for all $n \geq 1$, and therefore the trace of $\varphi^n - \psi^n$ is defined.

Definition 2 (see [3]). For any pair $(\varphi, \psi) \in L(H) \times L(H)$ with finite rank, we define the Kneading determinant of $(\varphi, \psi)$ as the following invertible element of $Q[[z]]$:

$$\Delta_{(\varphi, \psi)} = \exp \sum_{n \geq 1} -\frac{\text{tr}(\varphi^n - \psi^n)}{n} z^n. \quad (5)$$
Remark 3. Kneading determinant was first studied by Milnor and Thurston in [4].

Let $\mathbb{Q}[z]^{|q|}$ be the ring of the $q \times q$ matrices whose entries lie in $\mathbb{Q}[z]$. If $\varphi \in L(\mathcal{H})$, $\nu \in \mathcal{H}^q$, and $\alpha \in \mathcal{H}^q$, we define the matrix $M_\varphi(\alpha; \nu) \in \mathbb{Q}[z]^{|q|}$ by

$$
M_\varphi(\alpha; \nu) = \left( \begin{array}{ccccc}
\sum_{n=0}^d \alpha_1^n (u_1) z^n & \cdots & \sum_{n=0}^d \alpha_{q_1^n} (u_{q_1}) z^n \\
\vdots & \ddots & \vdots \\
\sum_{n=0}^d \alpha_2^n (u_2) z^n & \cdots & \sum_{n=0}^d \alpha_{q_2^n} (u_{q_2}) z^n
\end{array} \right).
$$

Lemma 4 (see [3]). Let $(\varphi, \psi) \in L(\mathcal{H}) \times L(\mathcal{H})$, $\nu \in \mathcal{H}^q$, and $\alpha \in \mathcal{H}^q$ such that $\psi - \varphi$ has finite rank. Denote by $I$ the $q \times q$ identity matrix. Then,

$$
\Delta_{(\varphi, \psi)} = \det \left( I - z M_\varphi(\alpha; \nu) \right)
$$

holds in $\mathbb{Q}[z]$. In general, any power series can be the Kneading determinant of some pair $(\varphi, \psi)$ with finite rank (see [3]). So it is interesting to study conditions under which the Kneading determinant is a rational power series.

Remark 3. In this section, $R$ denotes a ring with identity $I$. We discuss conditions under which $I - A\lambda$ and $I - B\lambda$ are left coprime or right coprime, where $A, B \in R$.

We say $I - A\lambda$ and $I - B\lambda$ are left coprime if there exist polynomials $X(\lambda), Y(\lambda) \in R[\lambda]$ such that

$$
X(\lambda)(I - A\lambda) + Y(\lambda)(I - B\lambda) = I. \tag{8}
$$

Proposition 6. $I - A\lambda$ and $I - B\lambda$ are left coprime if and only if there exist $X_0, X_1, \ldots, X_m \in R$ such that

$$
X_m H + X_{m-1} H B + X_{m-2} H B^2 + \cdots + X_2 H B^m + B^{m+1} = 0, \tag{9}
$$

for $H = A - B$.

Proof. If $I - A\lambda$ and $I - B\lambda$ are left coprime, there exist $X(\lambda) = X_0 + X_1 \lambda + \cdots + X_m \lambda^m$ and $Y(\lambda) = Y_0 + Y_1 \lambda + \cdots + Y_m \lambda^m$ such that

$$
X(\lambda)(I - A\lambda) + Y(\lambda)(I - B\lambda) = I. \tag{10}
$$

We can assume that $s$ equals $m$. And we have

$$
X_0 + Y_0 = I,
$$

$$
X_i A + Y_i B - X_{i+1} - Y_{i+1} = 0, \quad i = 1, 2, \ldots, m - 1, \tag{11}
$$

$$
X_m A + Y_m B = 0.
$$

So,

$$
Y_0 = I - X_0. \tag{12}
$$

If we write $H = A - B$, then,

$$
Y_1 = -X_1 + X_0 H + B,
$$

$$
Y_2 = -X_2 + X_1 H + X_0 H B + B^2, \tag{13}
$$

and let

$$
Y_m = -X_m + X_{m-1} H + \cdots + X_0 H B^{m-1} + B^m.
$$

So,

$$
X_m H + X_{m-1} H B + X_{m-2} H B^2 + \cdots + X_0 H B^m + B^{m+1} = 0. \tag{14}
$$

Conversely, if there exist $X_0, X_1, \ldots, X_m \in R$ such that

$$
X_m H + X_{m-1} H B + X_{m-2} H B^2 + \cdots + X_0 H B^m + B^{m+1} = 0, \tag{15}
$$

let

$$
X(\lambda) = X_0 + X_1 \lambda + \cdots + X_m \lambda^m,
$$

$$
Y(\lambda) = Y_0 + Y_1 \lambda + \cdots + Y_m \lambda^m. \tag{17}
$$

Then,

$$
X(\lambda)(I - A\lambda) + Y(\lambda)(I - B\lambda) = I. \tag{18}
$$

Now we give the definition of right coprime. We say $I - A\lambda$ and $I - B\lambda$ are right coprime if there exist polynomials $X(\lambda), Y(\lambda) \in R[\lambda]$ such that

$$
(I - A\lambda) X(\lambda) + (I - B\lambda) Y(\lambda) = I. \tag{19}
$$

Proposition 7. $I - A\lambda$ and $I - B\lambda$ are right coprime if and only if there exist $X_0, X_1, \ldots, X_s \in R$ such that

$$
H X_s + B H X_{s-1} + B^2 H X_{s-2} + \cdots + B^s H X_0 + B^{s+1} = 0, \tag{20}
$$

for $H = A - B$.

Proof. It is similar to the proof of Proposition 6.  \qed
3. Proof of the Main Theorem

In this section, $\varphi, \psi \in L(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional vector space over $\mathbb{Q}$, and we denote by $L(\mathcal{H})$ the ring of linear transforms on $\mathcal{H}$.

Lemma 8. Suppose that $I - \varphi \lambda$ and $I - \psi \lambda$ are left coprime or right coprime, $h = \psi - \varphi$ is of finite rank, and $W = \text{Im}(h)$. Then there exists a finite dimensional space $\bar{W}$ containing $W$ such that $(\psi^k - \varphi^k)(\mathcal{H}) \subset \bar{W}$ with $k = 1, 2, \ldots$

Proof. First we assume that if $I - \varphi \lambda$ and $I - \psi \lambda$ are right coprime, then by Proposition 7 we have $X_0, X_1, \ldots, X_s \in L(\mathcal{H})$ such that

$$
hX_m + \varphi hX_{m-1} + \varphi^2 hX_{m-2} + \cdots + \varphi^m hX_0 + \varphi^{m+1} = 0.
$$

(21)

Write $\bar{W} = W + \varphi W + \varphi^2 W + \cdots + \varphi^m W$; $\bar{W}$ is an invariant subspace of $\mathcal{H}$ under $h$. It is not very hard to check that $\psi(\bar{W}) = \varphi^{m+1} W + \varphi W + \varphi^2 W + \cdots + \varphi^m W \subset \bar{W}$. So $\bar{W}$ is invariant under the operator $\varphi$.

We use induction to prove the conclusion.

If $k = 1$, then $(\varphi^k - \psi^k)(\mathcal{H}) = W$.

Suppose that for $k \leq l$ we have $(\varphi^k - \psi^k)(\mathcal{H}) \subset W + \varphi W + \cdots + \varphi^m W$. Then for $k = l + 1$, we have $(\varphi^k - \psi^k)(\mathcal{H}) = \psi(\varphi^l - \psi^l)(\mathcal{H}) - h\psi^l(\mathcal{H}) \subset \phi(\bar{W}) + W \subset \bar{W}$. We get the conclusion in this case.

Now we assume that $I - \varphi \lambda$ and $I - \psi \lambda$ are left coprime; then there exist $s \in \mathbb{Z}$ and $x_0, x_1, \ldots, x_s \in L(\mathcal{H})$ such that

$$
x_s h + x_{s-1} h\varphi + x_{s-2} h\varphi^2 + \cdots + x_0 h\varphi^s + \varphi^{s+1} = 0.
$$

(22)

holds. Take

$$
\bar{W} = W + x_0 W + \cdots + x_s W,
$$

(23)

Notice when $l \geq s + 1$, $\varphi^l(\mathcal{H}) \subset x_0 W + x_1 W + \cdots + x_s W$.

We use induction to prove the conclusion.

If $k = 1$, then $(\varphi^k - \psi^k)(\mathcal{H}) = W$.

Suppose that for $k \leq l$ we have $(\varphi^k - \psi^k)(\mathcal{H}) \subset W + \varphi W + \cdots + \varphi^m W$. Then for $k = l + 1$, we have $(\varphi^k - \psi^k)(\mathcal{H}) = \psi(\varphi^l - \psi^l)(\mathcal{H}) - h\varphi^l(\mathcal{H}) \subset \psi(W) + W \subset \bar{W}$.

The proof is complete.

Now we will give the proof of Theorem 5.

Proof. If $I - \varphi \lambda$ and $I - \psi \lambda$ are coprime, by Lemma 8, there is a finite dimensional space $\bar{W}$ such that

$$
(\psi^k - \varphi^k)(\mathcal{H}) \subset \bar{W},
$$

(24)

and we have $\varphi(\bar{W}) \subset \bar{W}$. Define

$$
\bar{\varphi} = \text{Pr}(\varphi)|_{\bar{W}} : \bar{W} \rightarrow \bar{W}.
$$

(25)

Suppose $q = \text{rank}(\varphi - \psi)$; we denote by $I_q$ the identity operator. Then $p(z) = \det(I_q - z\bar{\varphi}) \in \mathbb{Q}[z]$. For any $i, j \in \{1, 2, \ldots, q\}$, we have $\sum_{n \geq 0} a_n z^n = f_{ij}(z)/p(z)$ for some $f_{ij}(z) \in \mathbb{Q}[z]$. For more details see [5]. So by Lemma 4, $\det(I - zM_q(\bar{\varphi}))$ is rational; we get the conclusion.

Example 9. Suppose $\mathcal{H}$ is the ring of countable infinite matrix with finite nonzero entries in each column.

Let

$$
\varphi = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots \\
A & 0 & 0 & \cdots \\
0 & A & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix},
$$

$$
h = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix},
$$

(26)

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\alpha_1, \alpha_2, \alpha_3, \ldots$ are three-dimensional row vectors. We see that $(\varphi - h)^3 = 0$. So $I - \psi \lambda$ and $I - \varphi \lambda$ are left coprime. It is easy to check that

$$
\Delta_{(\varphi, \psi)} = \frac{1 - 2z}{1 - z}.
$$

(27)

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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