Research Article

Equivalent Pairs of Words and Points of Connection

Qaiser Mushtaq and Abdul Razaq

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Correspondence should be addressed to Abdul Razaq; makenqau@gmail.com

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Higman has defined coset diagrams for $\text{PGL}(2, \mathbb{Z})$. The coset diagrams are composed of fragments, and the fragments are further composed of two or more circuits. A condition for the existence of a certain fragment γ in a coset diagram is a polynomial f in $\mathbb{Z}[z]$, obtained by choosing a pair of words $F[w_i, w_j]$ such that both w_i and w_j fix a vertex v in γ . Two pairs of words are equivalent if and only if they have the same polynomial. In this paper, we find distinct pairs of words that are equivalent. We also show there are certain fragments, which have the same orientations as those of their mirror images.

1. Introduction

The modular group $\text{PSL}(2, \mathbb{Z})$ [1–3] an interesting group of hyperbolic isometries and has a finite presentation

$$\langle x, y : x^2 = y^3 = 1 \rangle, \quad (1)$$

where x and y are the linear fractional transformations defined by $z \rightarrow -1/z$ and $z \rightarrow (z-1)/z$, respectively. The modular group $\text{PSL}(2, \mathbb{Z})$ is of great interest in many fields of mathematics, for example, group theory, graph theory, and number theory. By adjoining a new element $t : z \rightarrow 1/z$ with x and y , we obtain a presentation

$$\langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \rangle \quad (2)$$

of the extended modular group $\text{PGL}(2, \mathbb{Z})$.

The Hecke group $H(\lambda_q)$ is the discrete group generated by $z \rightarrow -1/z$, $z \rightarrow -1/(z + \lambda_q)$, where $\lambda_q = 2 \cos(\pi/q)$. It is well known that the modular group can be generalized to the Hecke groups. For $q = 3$, the resulting Hecke group $H(\lambda_3)$ is the modular group $\text{PSL}(2, \mathbb{Z})$.

If q is a power of a prime p , then the projective line over the finite field F_q , which is $F_q \cup \{\infty\}$, is denoted by $\text{PL}(F_q)$. The group $\text{PGL}(2, q)$ has its customary meaning, a group of all linear fractional transformations $z \rightarrow (az + b)/(cz + d)$ such that $a, b, c, d \in F_q$ and $ad - bc \neq 0$, while $\text{PSL}(2, q)$ is its subgroup consisting of all those, where $ad - bc$ is a non-zero square in F_q .

There is a well-known relationship between the action of $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{R}$ and continued fractions. There are many articles on the connection between the geodesics on the modular surface and continued fractions which are particularly important in the theory of approximation of real numbers by rational [4, 5]. Important results have been obtained using these ideas in a very well-written article [6]. A good account on relationship between continued fractions and indefinite binary quadratic forms is also given in [7]. In [8], Mushtaq has shown a relationship between reduced indefinite binary quadratic forms and orbits of the modular group. There is another interesting article [9] of Mushtaq and Hayat on the topic of Pell numbers, Pell-Lucas numbers, and modular group. In [10], Bong and Mushtaq determine the Fibonacci and Locus numbers through the action of modular group on real quadratic fields.

In 1978, G. Higman introduced a new type of graph called a coset diagram for $\text{PGL}(2, \mathbb{Z})$. The three cycles of y are denoted by small triangles whose vertices are permuted counterclockwise by y and any two vertices which are interchanged by x are joined by an edge. The fixed points of x and y are denoted by heavy dots. Notice $(yt)^2 = 1$ is equivalent to $t yt = y^{-1}$, which means that t reverses the orientation of the triangles representing the three cycles of y (as reflection does); because of this, there is no need to make the diagram more complicated by introducing t -edges. For details about coset diagrams, one can refer to [11–14].

Two homomorphisms α and β from $\text{PGL}(2, \mathbb{Z})$ to $\text{PGL}(2, q)$ are called conjugate if $\beta = \alpha\rho$ for some inner automorphism ρ on $\text{PGL}(2, q)$. We call α to be non-degenerate if neither of x, y lies in the kernel of α . In [15], it has been shown that the conjugacy classes of non-degenerate homomorphisms from $\text{PGL}(2, \mathbb{Z})$ to $\text{PGL}(2, q)$ are in one to one correspondence with the elements $\theta \neq 0, 3$ of F_q under the correspondence which maps each class to its parameter θ . As in [15], the coset diagram corresponding to the action of $\text{PGL}(2, \mathbb{Z})$ on $\text{PL}(F_q)$ via a homomorphism α with parameter θ is denoted by $D(\theta, q)$. Each conjugacy class is represented by a unique coset diagram, unique in the sense that the diagram remains invariant except that labels of vertices change from α to β .

By a circuit in a coset diagram for an action of $\text{PGL}(2, \mathbb{Z})$ on $\text{PL}(F_q)$, we will mean a closed path of triangles and edges. Let $n_1, n_2, \dots, n_{2k}$ be a sequence of positive integers, the circuit which contains a vertex, fixed by $w = (xy)^{n_1}(xy^{-1})^{n_2} \dots (xy^{-1})^{n_{2k}} \in \text{PSL}(2, \mathbb{Z})$ for some $k \geq 1$; we will mean the circuit in which n_1 triangles have one vertex inside the circuit and n_2 triangles have one vertex outside the circuit and so on. Since it is a cycle $(n_1, n_2, \dots, n_{2k})$, so it does not make any difference if n_1 triangles have one vertex outside the circuit and n_2 triangles have one vertex inside the circuit and so on. The circuit of the type $(n_1, n_2, \dots, n_{2k'}, n_1, n_2, \dots, n_{2k'}, \dots, n_1, n_2, \dots, n_{2k'})$ is called a periodic circuit and the length of its period is $2k'$. A circuit that is not of this type is non-periodic. A circuit is called simple, if each vertex of the circuit is fixed by a unique word w or its inverse w^{-1} .

2. Joining of Circuits

Consider two non-periodic and simple circuits $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$. Let v_i be any vertex in $(n_1, n_2, \dots, n_{2k})$ fixed by a word w_i and let v_j be any vertex in $(m_1, m_2, \dots, m_{2k})$ fixed by a word w_j . In order to connect these two circuits at v_i and v_j , we choose, without loss of generality $(n_1, n_2, \dots, n_{2k})$, and apply w_j on v_i in such a way that w_j ends at v_i . Consequently, we get a fragment denoted by γ . In other words, by γ , we will mean a nonsimple fragment whose one vertex $v = v_i = v_j$ is fixed by a pair of words w_i, w_j , denoted by $F[w_i, w_j]$.

Example 1. We connect the vertex v , fixed by $(xy)(xy^{-1})^3(xy)^3$, in $(4, 3)$ with the vertex u , fixed by $(xy)^3(xy^{-1})^2$ of $(3, 2)$, and compose a fragment γ as shown in Figures 1, 2, and 3.

One can see that the vertex $v = u$ in γ (Figure 3) is fixed by a pair of words

$$F[(xy)(xy^{-1})^3(xy)^3, (xy)^3(xy^{-1})^2]. \quad (3)$$

The action of $\text{PGL}(2, \mathbb{Z})$ on $\text{PL}(F_{q^2})$ yields two components, namely, $\text{PL}(F_q)$ and $\text{PL}(F_{q^2}) \setminus \text{PL}(F_q)$. For sake of simplicity, let $\overline{\text{PL}(F_q)}$ denote the complement $\text{PL}(F_{q^2}) \setminus \text{PL}(F_q)$.

The coset diagram $D(\theta, q)$ is made of fragments. It is therefore necessary to ask when a fragment exists in $D(\theta, q)$. In [16], this question is answered in the following way.

Theorem 2. Given a fragment, there is a polynomial f in $\mathbb{Z}[z]$ such that

- (i) if the fragment occurs in $D(\theta, q)$, then $f(\theta) = 0$;
- (ii) if $f(\theta) = 0$, then the fragment, or a homomorphic image of it occurs in $D(\theta, q)$ or in $\overline{\text{PL}(F_q)}$.

In [16], the method of calculating a polynomial from a fragment is given. Here, we describe this method briefly. Since a fragment is composed of two non-periodic and connected circuits $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$ with a common fixed vertex, say, v , then there is a pair of words, for instance,

$$F[w_i, w_j] = F[(xy)^{l_1}(xy^{-1})^{l_2} \dots (xy^{-1})^{l_{2k_1}}, (xy)^{m_1}(xy^{-1})^{m_2} \dots (xy^{-1})^{m_{2k_2}}] \quad (4)$$

such that $(v)w_i = v$ and $(v)w_j = v$. Let X and Y be the matrices corresponding to x and y of $\text{PGL}(2, q)$. Then, w_i and w_j can be expressed as

$$W_i = (XY)^{l_1}(XY^{-1})^{l_2} \dots (XY^{-1})^{l_{2k_1}}, \quad (5)$$

$$W_j = (XY)^{m_1}(XY^{-1})^{m_2} \dots (XY^{-1})^{m_{2k_2}},$$

where $k_1, k_2 > 0$. By using (3.1) to (3.7) of [16], the matrices W_i and W_j can be expressed linearly as

$$W_i = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY, \quad (6)$$

$$W_j = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

such that λ_i and μ_i , for $i = 0, 1, 2, 3$, are polynomials in r and Δ , where r is the trace of XY and Δ is its determinant. Since v is a common fixed vertex of w_i and w_j , therefore the 2×2 matrices W_i and W_j have a common eigenvector. Then, by Lemma 3.1 of [16], the algebra generated by W_i and W_j has dimension 3. The algebra contains I, W_i, W_j , and $W_i W_j$ and so these are linearly dependent. Using (3.1) to (3.7) of [16], the matrix $W_i W_j$ is expressed as

$$W_i W_j = \nu_0 I + \nu_1 X + \nu_2 Y + \nu_3 XY, \quad (7)$$

where ν_i , for $i = 0, 1, 2, 3$, can be calculated in terms of the λ_i and μ_i , using (3.1) to (3.7) of [16]. The condition that I, W_i, W_j , and $W_i W_j$ are linearly dependent is expressed as

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = 0. \quad (8)$$

FIGURE 1

FIGURE 2

FIGURE 3

If we calculate v_1, v_2, v_3 in terms of λ_i and μ_i and substitute in (8), we find that it is equivalent to

$$\begin{aligned} & (\lambda_2\mu_3 - \mu_2\lambda_3)^2 + \Delta(\lambda_3\mu_1 - \mu_3\lambda_1)^2 + (\lambda_1\mu_2 - \mu_1\lambda_2)^2 \\ & + r(\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_3\mu_1 - \mu_3\lambda_1) \\ & + (\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_1\mu_2 - \mu_1\lambda_2) = 0. \end{aligned} \quad (9)$$

This gives a homogeneous equation in Δ and r . By substituting $\Delta\theta$ for r^2 , we get an equation in θ .

Two pairs of words $F[w_i, w_j]$ and $F[w_k, w_l]$ are equivalent if and only if they have the same polynomial. If two pairs of words $F[w_i, w_j]$ and $F[w_k, w_l]$ are equivalent, then we write $F[w_i, w_j] \sim F[w_k, w_l]$.

In this paper, we find distinct pairs of words that are equivalent. We also show there are certain fragments, which have the same orientations as those of their mirror images.

3. Main Results

Theorem 3. In the above notation, the polynomial obtained from a fragment γ is unique.

Proof. Let v_i and v_k be any two vertices of a fragment γ , such that v_i is fixed by a pair of words $F[w_i, w_j]$ and v_k is fixed by a pair of words $F[w_k, w_l]$. Suppose $f(\theta)$ is a polynomial obtained by choosing $F[w_i, w_j]$ and $g(\theta)$ is a polynomial obtained by choosing $F[w_k, w_l]$. Now, if $f(\theta) = 0$, then, by Theorem 2, the fragment γ or its homomorphic image occurs in $D(\theta, q)$ or in $\overline{\text{PL}(F_q)}$. So, there exists a vertex in $D(\theta, q)$ or in $\overline{\text{PL}(F_q)}$ which is fixed by $F[w_k, w_l]$. Again, by Theorem 2, we have $g(\theta) = 0$. Similarly, it can be proved that if $g(\theta) = 0$, then $f(\theta) = 0$. Hence, $f(\theta) = g(\theta)$. $\square$

Remark 4. If a vertex v is fixed by a word $w_i \in \text{PSL}(2, \mathbb{Z})$, then the vertex $(v)w$ is fixed by the conjugate $w^{-1}w_iw$ of w_i .

We know that γ is a fragment created by joining a vertex v_i of $(n_1, n_2, \dots, n_{2k})$ with the vertex v_j of $(m_1, m_2, \dots, m_{2k})$. Since the same polynomial is evolved for all the points of connection for the fragment γ , therefore it is important to know all the points of connection for the fragment γ . Following theorem is useful in finding all points of connection of $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$ for the fragment γ .

Theorem 5. Let fragment γ be constructed by joining a vertex v_i of $(n_1, n_2, \dots, n_{2k})$ with the vertex v_j of $(m_1, m_2, \dots, m_{2k})$. Then, γ is obtainable also, if the vertex $(v_i)w$ of $(n_1, n_2, \dots, n_{2k})$ is joined with the vertex $(v_j)w$ of $(m_1, m_2, \dots, m_{2k})$.

Proof. Suppose vertex v_i is fixed by a word w_i and vertex v_j is fixed by a word w_j . Then, one vertex of the fragment γ is fixed by w_i and w_j . Now, we join the vertex $(v_i)w$ of $(n_1, n_2, \dots, n_{2k})$ with the vertex $(v_j)w$ of $(m_1, m_2, \dots, m_{2k})$ and compose a fragment δ . By Remark 4, the vertex $(v)w$ of fragment δ is fixed by $w^{-1}w_iw$ and $w^{-1}w_jw$, whereas the vertex $((v)w)w^{-1} = v$ of fragment δ is fixed by $w(w^{-1}w_iw)w^{-1} = w_i$ and $w(w^{-1}w_jw)w^{-1} = w_j$. This shows that γ and δ are the same fragments. $\square$

Since γ and δ are the same fragments, and a unique polynomial is obtained from a fragment, hence $F[w_i, w_j] \sim F[w^{-1}w_iw, w^{-1}w_jw]$.

Corollary 6. Let P be the set of words such that for any $w \in P$, both vertices $(v_i)w$ and $(v_j)w$ lie on the circuits $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$. Let s be the total number of points of connection of the circuits to compose γ . Then, $s = |P|$.

Example 7. As in Example 1, the vertex v in (4, 3) is connected with the vertex u in (3, 2), and a fragment γ is evolved. Then, one can see that

$$P = \{y, y^{-1}, e, x, xy, xy^{-1}, xyx, xyxy, xyxy^{-1}\} \quad (10)$$

is the set of words such that, for any $w \in P$, both vertices $(v)w$ and $(u)w$ lie on (4, 3) and (3, 2). Since $|P| = 9$, there are nine points of connection of (4, 3) and (3, 2) composing γ . Hence, 9 vertices of one circuit are merged in another circuit. In other words, 3 triangles of one circuit are merged in another circuit.

The number of points of connection of the circuits for a fragment is always multiple of three and plays a significant role because they are directly related to the structure of the fragment. The following theorem illustrates this relation.

Theorem 8. Let r be the total number of triangles in a fragment γ and s the total number of points of connection of the circuits $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$. Then, $r = \sum_{j=1}^{2k} n_j + \sum_{j=1}^{2k} m_j - (1/3)s$.

Proof. Since circuits $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$ have $\sum_{j=1}^{2k} n_j$ and $\sum_{j=1}^{2k} m_j$ number of triangles, respectively, and there are s number of points of connection of $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$ composing γ , therefore, by Corollary 6, there are s number of words w such that $(v_i)w$, $(v_j)w$ lie on $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$, respectively. So, s number of vertices of $(m_1, m_2, \dots, m_{2k})$ are merged in $(n_1, n_2, \dots, n_{2k})$, and remaining $3 \sum_{j=1}^{2k} m_j - s$ number of vertices are included in $(n_1, n_2, \dots, n_{2k})$ to compose γ . So, the total number of vertices in γ is $(3 \sum_{j=1}^{2k} m_j - s + 3 \sum_{j=1}^{2k} n_j)$. This implies that $r = \sum_{j=1}^{2k} n_j + \sum_{j=1}^{2k} m_j - (1/3)s$. $\square$

Remark 9. Consider a fragment γ such that one vertex v of γ is fixed by $F[w_i, w_j]$. Then, obviously, v is fixed by $F[w_i^{-1}, w_j^{-1}]$. So, $F[w_i, w_j] \sim F[w_i^{-1}, w_j^{-1}]$. Also, it is trivial that $F[w_i, w_j] \sim F[w_j, w_i]$.

The following Theorem shows that $(n_1, n_2, \dots, n_{2k})$ and $(m_1, m_2, \dots, m_{2k})$ are not a unique pair of circuits for the fragment γ .

Theorem 10. The following pairs of words are equivalent:

$$\begin{aligned} F[w_i, w_j], \quad F[w_i, w_i w_j], \quad F[w_i, w_i w_j^{-1}], \\ F[w_i, w_j w_i], \quad F[w_i, w_j w_i^{-1}]. \end{aligned} \quad (11)$$

Proof. Let v_i, v_j , and v_k be arbitrary vertices of the circuits $(n_1, n_2, \dots, n_{2k})$, $(m_1, m_2, \dots, m_{2k})$, and $(m'_1, m'_2, \dots, m'_{2k})$,

respectively. Let the words w_i, w_j , and $w_i w_j$ fix the vertices v_i, v_j , and v_k , respectively. We join a vertex v_i fixed by w_i of the circuit $(n_1, n_2, \dots, n_{2k})$ with the vertex v_j fixed by w_j of the circuit $(m_1, m_2, \dots, m_{2k})$ to compose γ . Also, we join a vertex v_i fixed by w_i of the circuit $(n_1, n_2, \dots, n_{2k})$ with the vertex fixed by $w_i w_j$ of the $(m'_1, m'_2, \dots, m'_{2k})$ to compose a fragment δ . So, in δ , v_i is a common fixed vertex of $F[w_i, w_i w_j]$. The fragment δ is different from the fragment γ if and only if, in fragment δ , the word $w_i w_j$ fixes the vertex v_i in the following way: w_i maps the vertex v_i to some vertex $a \neq v_i$, and then w_j maps the vertex a to the vertex v_i ; that is, $v_i \xrightarrow{w_i} a \xrightarrow{w_j} v_i$. Since w_i is a composition of linear fractional transformations x and y , therefore $a = v_i$. This implies that γ and δ are the same fragments, since a unique polynomial is obtained from a fragment. Hence, $F[w_i, w_j] \sim F[w_i, w_i w_j]$.

Similarly, it is easy to prove that

$$\begin{aligned} F[w_i, w_j] &\sim F[w_i, w_i w_j^{-1}], \quad F[w_i, w_j] \sim F[w_i, w_j w_i], \\ F[w_i, w_j] &\sim F[w_i, w_j w_i^{-1}]. \end{aligned} \quad (12)$$

Since $\sim$ is an equivalence relation, hence

$$\begin{aligned} F[w_i, w_j] &\sim F[w_i, w_i w_j] \sim F[w_i, w_i w_j^{-1}] \\ &\sim F[w_i, w_j w_i] \sim F[w_i, w_j w_i^{-1}]. \end{aligned} \quad (13)$$

$\square$

Example 11. Theorem 10 shows that the fragment γ , which has a vertex fixed by $F[w_i, w_j]$, and the fragment δ' , which has a vertex fixed by $F[w_i, w_j w_i]$, are the same. So, the fragment, which has a vertex fixed by $F[(xy)^2(xy^{-1}), (xy^{-1})(xy)^3]$, and the fragment, which has a vertex fixed by $F[(xy)^2(xy^{-1}), (xy^{-1})(xy)^5(xy^{-1})]$, are the same. Since the words $(xy)^2(xy^{-1})$, $(xy^{-1})(xy)^3$, and $(xy^{-1})(xy)^5(xy^{-1})$ correspond to the circuits (2, 1), (3, 1), and (5, 2), so the fragment (Figure 4) is obtained, not only by joining the vertex v_1 in (2, 1) (Figure 5) with the vertex v_2 in (3, 1) (Figure 6) but also by joining the vertex v_1 in (2, 1) (Figure 7) with the vertex u_2 in (5, 2) (Figure 8).

Theorem 12. In a coset diagram for the action of $PGL(2, \mathbb{Z})$ on $PL(F_q)$, if a vertex v is fixed by w , then the vertex $(v)t$ is fixed by w^* .

Proof. Let $w = xy^{\eta_1} xy^{\eta_2} \dots xy^{\eta_{n-1}} xy^{\eta_n}$ be a word such that

$$(v) xy^{\eta_1} xy^{\eta_2} \dots xy^{\eta_{n-1}} xy^{\eta_n} = v. \quad (14)$$

Since $(xt)^2 = (yt)^2 = 1$, this implies that

$$yt = ty^{-1}, \quad y^{-1}t = ty, \quad xt = tx. \quad (15)$$

Now, by using (14) and (15), we have $(v)t = ((v)xy^{\eta_1} xy^{\eta_2} \dots xy^{\eta_{n-1}} xy^{\eta_n})t = ((v)t)xy^{-\eta_1} xy^{-\eta_2} \dots xy^{-\eta_{n-1}} xy^{-\eta_n} = ((v)t)w^*$. This shows that $xy^{-\eta_1} xy^{-\eta_2} \dots xy^{-\eta_{n-1}} xy^{-\eta_n}$ also fixes a point $(v)t$. $\square$

FIGURE 4

FIGURE 5

FIGURE 6

FIGURE 7

FIGURE 8

FIGURE 9

Theorem 13. If a vertex v is fixed by a word w in a circuit (n_1, n_2) , then there exists a vertex v^* in (n_1, n_2) such that v^* is fixed by w^* .

Proof. Consider a simple circuit (n_1, n_2) .

In Figure 9, one can see that the vertex u_1 is fixed by $(xy)^{n_1}(xy^{-1})^{n_2}$ and u_{3n_2} is fixed by $(xy^{-1})^{n_1}(xy)^{n_2}$. So, $u_{3n_2} = u_1^*$. Also, the vertex u_2 is fixed by $y(xy)^{n_1}(xy^{-1})^{n_2-1}xy$ and u_{3n_2-1} is fixed by $y^{-1}(xy^{-1})^{n_1}(xy)^{n_2-1}xy^{-1}$. So, $u_{3n_2-1} = u_2^*$. Similarly, $u_{3n_2-2} = u_3^*$, $u_{3n_2-3} = u_4^*$, $\dots$.

The vertices v_1 and v_{3n_1} are fixed by $(xy)^{n_2}(xy^{-1})^{n_1}$ and $(xy^{-1})^{n_2}(xy)^{n_1}$, respectively, implying that $v_{3n_1} = v_1^*$. Also, the vertices v_2 and v_{3n_1-1} are fixed by $y(xy)^{n_2}(xy^{-1})^{n_1-1}xy$ and $y^{-1}(xy^{-1})^{n_2}(xy)^{n_1-1}xy^{-1}$, respectively, implying that $v_{3n_1-1} = v_2^*$. Similarly, $v_{3n_1-2} = v_3^*$, $v_{3n_1-3} = v_4^*$, $\dots$. $\square$

Let the homomorphic image of the fragment (Figure 10) occur in the coset diagram $D(\theta, q)$. Since $D(\theta, q)$ admits the axis of symmetry, the mirror image of the fragment under the permutation t will also occur as shown in Figure 11.

Proposition 14. If the fragment γ has one vertex v fixed by $F[w_i, w_j]$, then its mirror image γ^* has one vertex fixed by $F[w_i^*, w_j^*]$.

Proof. Let γ be the fragment with one vertex v fixed by $F[w_i, w_j]$. Since γ^* is the mirror image of γ , therefore γ^* has

FIGURE 10

one vertex $(v)t$. Then, by Theorem 12, the vertex $(v)t$ is fixed by $F[w_i^*, w_j^*]$. $\square$

Theorem 15. *The polynomials obtained from the fragment γ and its mirror image γ^* are the same.*

Proof. Let $f(\theta)$ be the polynomial obtained by choosing a pair of words $F[w_i, w_j]$ from γ and let $g(\theta)$ be the polynomial obtained by choosing a pair of words $F[w_i^*, w_j^*]$ from γ^* . Let $f(\theta) = 0$. Then, by Theorem 2, there is a vertex say v in $D(\theta, q)$ or in $\overline{\text{PL}(F_q)}$ which is fixed by $F[w_i, w_j]$. Then, by Theorem 12, the vertex $(v)t$ is fixed by $F[w_i^*, w_j^*]$. So, by Theorem 2, $g(\theta) = 0$. Similarly, it can be proved that if $g(\theta) = 0$, then $f(\theta) = 0$. Hence, $f(\theta) = g(\theta)$. $\square$

Since a unique polynomial is obtained from a fragment γ and its mirror image γ^* , hence $F[w_i, w_j] \sim F[w_i^*, w_j^*]$.

Consider two circuits (n_1, n_2) and (m_1, m_2) . In general, fragment γ and its mirror image γ^* do not have the same orientation. There are certain fragments which have the same orientations as those of their mirror images. These kinds of fragments have vertical symmetry and may have fixed points of t . The following Theorem shows how these fragments are composed.

Theorem 16. (i) *The fragment composed by joining a vertex u_1 , fixed by*

$$(xy^{-1})^{(n_1-1)/2}(xy)^{n_2}(xy^{-1})^{(n_1+1)/2} \quad (16)$$

in (n_1, n_2) , with the vertex u_2 , fixed by $(xy^{-1})^{(m_1-1)/2}(xy)^{m_2}(xy^{-1})^{(m_1+1)/2}$ in (m_1, m_2) , has the same orientation as that of its mirror image.

(ii) *The fragment composed by joining a vertex u_1 , fixed by*

$$(xy^{-1})^{n_1/2}(xy)^{n_2}(xy^{-1})^{n_1/2} \quad (17)$$

in (n_1, n_2) , with the vertex u_2 , fixed by $(xy^{-1})^{m_1/2}(xy)^{m_2}(xy^{-1})^{m_1/2}$ in (m_1, m_2) , has the same orientation as that of its mirror image.

(iii) *The fragment composed by joining a vertex v_1 , fixed by*

$$(xy^{-1})^{n_1/2}(xy)^{n_2}(xy^{-1})^{n_1/2} \quad (18)$$

in (n_1, n_2) , with the vertex v_2 , fixed by $(xy)^{m_1/2}(xy^{-1})^{m_2}(xy)^{m_1/2}$ in (m_1, m_2) , has the same orientation as that of its mirror image.

Proof. (i) Let γ_1 be the fragment which has one vertex say u , fixed by

$$(xy^{-1})^{(n_1-1)/2}(xy)^{n_2}(xy^{-1})^{(n_1+1)/2}, \quad (19)$$

$$(xy^{-1})^{(m_1-1)/2}(xy)^{m_2}(xy^{-1})^{(m_1+1)/2}.$$

Then, by Proposition 14, its mirror image fragment γ_1^* contains a vertex, say, u^* , fixed by $(xy)^{(n_1-1)/2}(xy^{-1})^{n_2}(xy)^{(n_1+1)/2}$ and $(xy)^{(m_1-1)/2}(xy^{-1})^{m_2}(xy)^{(m_1+1)/2}$. By Remark 4, the vertex $(u^*)y$ of fragment γ_1^* is fixed by

$$y(xy)^{(n_1-1)/2}(xy^{-1})^{n_2}(xy)^{(n_1+1)/2}y^{-1} = \left((xy^{-1})^{(n_1-1)/2}(xy)^{n_2}(xy^{-1})^{(n_1+1)/2} \right)^{-1} \quad (20)$$

and $y(xy)^{(m_1-1)/2}(xy^{-1})^{m_2}(xy)^{(m_1+1)/2}y^{-1} = ((xy^{-1})^{(m_1-1)/2}(xy)^{m_2}(xy^{-1})^{(m_1+1)/2})^{-1}$. This implies that $u = (u^*)y$. Hence fragment γ_1 has the same orientation as that of its mirror image γ_1^* .

(ii) Let γ_2 be the fragment which has one vertex, say, v , fixed by

$$(xy^{-1})^{n_1/2}(xy)^{n_2}(xy^{-1})^{n_1/2}, \quad (21)$$

$$(xy^{-1})^{m_1/2}(xy)^{m_2}(xy^{-1})^{m_1/2}.$$

Then, its mirror image fragment γ_2^* has one vertex, say, v^* , fixed by

$$(xy)^{n_1/2}(xy^{-1})^{n_2}(xy)^{n_1/2}, \quad (xy)^{m_1/2}(xy^{-1})^{m_2}(xy)^{m_1/2}. \quad (22)$$

The vertex $(v^*)x$ of fragment γ_2^* is fixed by

$$x(xy)^{n_1/2}(xy^{-1})^{n_2}(xy)^{n_1/2}x = \left((xy^{-1})^{n_1/2}(xy)^{n_2}(xy^{-1})^{n_1/2} \right)^{-1} \quad (23)$$

and $x(xy)^{m_1/2}(xy)^{m_2}(xy)^{m_1/2}x = ((xy^{-1})^{m_1/2}(xy)^{m_2}(xy^{-1})^{m_1/2})^{-1}$. This implies that $v = (v^*)x$. Hence, γ_2 has the same orientation as that of its mirror image γ_2^* .

(iii) Let γ_3 be the fragment which has one vertex, say, v , fixed by

$$(xy^{-1})^{n_1/2}(xy)^{n_2}(xy^{-1})^{n_1/2}, \quad (xy)^{m_1/2}(xy^{-1})^{m_2}(xy)^{m_1/2}. \quad (24)$$

Then, its mirror image fragment γ_3^* has one vertex, say, v^* , fixed by

$$(xy)^{n_1/2}(xy^{-1})^{n_2}(xy)^{n_1/2}, \quad (xy^{-1})^{m_1/2}(xy)^{m_2}(xy^{-1})^{m_1/2}. \quad (25)$$

FIGURE 11

The vertex $(v^*)x$ of fragment γ_3^* is fixed by

$$\begin{aligned} & x(xy)^{n_1/2}(xy^{-1})^{n_2}(xy)^{n_1/2}x \\ &= \left((xy^{-1})^{n_1/2}(xy)^{n_2}(xy^{-1})^{n_1/2} \right)^{-1} \end{aligned} \quad (26)$$

and $x(xy^{-1})^{m_1/2}(xy)^{m_2}(xy^{-1})^{m_1/2}x = ((xy)^{m_1/2}(xy^{-1})^{m_2}(xy)^{m_1/2})^{-1}$. This implies that $v = (v^*)x$. Hence, γ_3 has the same orientation as that of its mirror image γ_3^* . $\square$

4. Motivation and Open Problems

If we join a pair of circuits at a certain point, we get a fragment and hence a polynomial. Since each such polynomial splits linearly in a suitable Galois [16] and corresponding to each zero, we get a triplet $(\bar{x}, \bar{y}, \bar{t})$ [15] which is a group. This shows that each pair of circuits can be related to a group. A pair of circuits has finitely many points of connections. Two distinct points of connection may or may not give the same fragment. So, it is important to ask the following question.

Problem 17. How many fragments (polynomials) are obtained if we join a pair of circuits (n_1, n_2) and (m_1, m_2) at all points of connection?

In order to give the answer of the above question, we first have to find those pair of words which are equivalent. In other words, we have to find those points of connection, at which we get the same polynomial. This issue is resolved in this paper. We will give the answer of the above question in an other paper. After that, one can establish a connection between a class of groups and a pair of circuits (n_1, n_2) and (m_1, m_2) , which is indeed a great development.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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