Research Article

On Certain Subclass of Meromorphic Spirallike Functions Involving the Hypergeometric Function

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We introduce and investigate a new subclass $\mathcal{M}_m^1(\theta, \lambda, \eta)$ of meromorphic spirallike functions. Such results as integral representations, convolution properties, and coefficient estimates are proved. The results presented here would provide extensions of those given in earlier works. Several other results are also obtained.

1. Introduction

Let $\mathcal{M}$ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the punctured open unit disk:

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}. \quad (2)$$

Let $f, g \in \mathcal{M}$, where f is given by (1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n. \quad (3)$$

Then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n := (g * f)(z). \quad (4)$$

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathbb{U}), \quad (5)$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}). \quad (6)$$

For θ which is real with $|\theta| < \pi/2$, $0 \leq \gamma < 1$, we denote by $\mathcal{MS}^*(\theta, \gamma)$ and $\mathcal{MK}^*(\theta, \gamma)$ the subclasses of $f \in \mathcal{M}$ which are defined, respectively, by

$$\begin{aligned} \Re \left(e^{i\theta} \frac{zf'(z)}{f(z)} \right) &< -\gamma \cos \theta \quad (z \in \mathbb{U}^*), \\ \Re \left(e^{i\theta} \frac{(zf'(z))'}{f'(z)} \right) &< -\gamma \cos \theta \quad (z \in \mathbb{U}^*). \end{aligned} \quad (7)$$

By setting $\theta = 0$ in (7), we get the well-known subclasses of $f \in \mathcal{M}$ consisting of meromorphic functions which are starlike and convex of order γ ($0 \leq \gamma < 1$), respectively. For some recent investigations on meromorphic spirallike functions and related topics, see, for example, the earlier works [1–4] and the references cited therein.

For $\eta > 1$, Wang et al. [5] and Nehari and Netanyahu [6] introduced and studied the subclass $\mathcal{M}(\eta)$ of $\mathcal{M}$ consisting of functions f satisfying

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > -\eta \quad (z \in \mathbb{U}^*). \quad (8)$$

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, \quad (9)$$

which are analytic in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\theta, \delta, \gamma)$ if it satisfies the condition

$$\Re \left(\frac{e^{i\theta} z f'(z)}{(1-\delta) f(z) + \delta z f'(z)} \right) > \gamma \cos \theta \quad (10)$$

$$\left(z \in \mathbb{U}; |\theta| < \frac{\pi}{2}; 0 \leq \delta < 1; 0 \leq \gamma < 1 \right).$$

The function class $\mathcal{S}^*(\theta, \delta, \gamma)$ is introduced and studied recently by Orhan et al. [7]. An analogous of the class $\mathcal{S}^*(\theta, \delta, \gamma)$ has been studied by Murugusundaramoorthy [8].

For complex parameters $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ ($\beta_j \neq 0, -1, -2, \dots; j = 1, 2, \dots, m$) the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$$

$$:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \quad (11)$$

$$(l \leq m+1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $\mathbb{N}$ denotes the set of all positive integers and $(a)_k$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)(a+2) \cdots (a+n-1), & n \in \mathbb{N}; a \in \mathbb{C}. \end{cases} \quad (12)$$

For a function $f \in \mathcal{M}$, we consider a linear operator (which is a meromorphically modified version of the familiar Dziok-Srivastava linear operator [9, 10]:

$$\mathcal{H}_m^l[\alpha, \beta] f(z) = \mathcal{H}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z)$$

$$= [z^{-1} {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)] * f(z), \quad (13)$$

where $\alpha_s > 0, \beta_t > 0$ ($s = 1, \dots, l; t = 1, \dots, m; l \leq m+1; l, m \in \mathbb{N}_0$).

From the definition of the operator $\mathcal{H}_m^l[\alpha, \beta]f$, it is easy to observe that

$$\mathcal{H}_m^l[\alpha, \beta] f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \phi_n(\alpha, \beta; l; m) a_n z^n, \quad (14)$$

where

$$\phi_n(\alpha, \beta; l; m) = \frac{(\alpha_1)_{n+1} \cdots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \cdots (\beta_m)_{n+1}} \frac{1}{(n+1)!} \quad (15)$$

is a positive number for all $n \in \mathbb{N}$.

Recently, Aouf [11], Liu and Srivastava [12], and Raina and Srivastava [13] obtained many interesting results involving the linear operator $\mathcal{H}_m^l[\alpha, \beta]$. In particular, for

$$l = 2, \quad m = 1, \quad \alpha_1 = a, \quad \beta_1 = c, \quad \alpha_2 = 1, \quad (16)$$

we obtain the following linear operator:

$$\mathcal{L}(a, c) f(z) := \mathcal{H}[\alpha_1, 1; \beta_1] f(z) \quad (z \in \mathbb{U}^*), \quad (17)$$

which was introduced and investigated earlier by Liu and Srivastava [14] and was further studied in a subsequent investigation by Srivastava et al. [15]. It should also be remarked that the linear operator $\mathcal{H}_m^l[\alpha, \beta]$ is a generalization of other linear operators considered in many earlier investigations (see, e.g., [16–18]).

Using the operator $\mathcal{H}_m^l[\alpha, \beta]f$, we introduce the following class of meromorphic functions.

Definition 1. For $|\theta| < \pi/2$, $0 \leq \lambda < 1/2$, and $\eta > 1$, let $\mathcal{M}_m^l(\theta, \lambda, \eta)$ denote a subclass of $\mathcal{M}$ consisting of functions satisfying the condition that

$$\Re \left(\frac{e^{i\theta} (1-2\lambda) z (\mathcal{H}_m^l[\alpha, \beta] f(z))'}{(1-\lambda) \mathcal{H}_m^l[\alpha, \beta] f(z) + \lambda z (\mathcal{H}_m^l[\alpha, \beta] f(z))'} \right) > -\eta \cos \theta, \quad (z \in \mathbb{U}^*), \quad (18)$$

where $\mathcal{H}_m^l[\alpha, \beta]f$ is given by (13).

We note that, for $l = 2, m = 1, \alpha_1 = \alpha_2 = 1, \beta_1 = 1$, and $\theta = \lambda = 0$, the class $\mathcal{M}_1^2(0, 0, \eta)$ becomes the class $\mathcal{M}(\eta)$.

In the present paper, we aim at proving some interesting properties such as integral representations, convolution properties, and coefficient estimates for the class $\mathcal{M}_m^l(\theta, \lambda, \eta)$.

The following lemma will be required in our investigation.

Lemma 2. Suppose that the sequence $\{A_n\}_{n=1}^{\infty}$ is defined by

$$A_1 = \frac{(1-2\lambda)(\eta-1)\cos\theta}{(1-\lambda)\phi_1(\alpha, \beta; l; m)},$$

$$A_{n+1} = \frac{2(\eta-1)\cos\theta}{(n+2)(1-\lambda)\phi_{n+1}(\alpha, \beta; l; m)}$$

$$\times \left[1 - 2\lambda + \sum_{k=1}^n \phi_k(\alpha, \beta; l; m)(1-\lambda+k\lambda)A_k \right]. \quad (19)$$

Then

$$A_n = \frac{(1-2\lambda)(\eta-1)\cos\theta}{(1-\lambda)^n \phi_n(\alpha, \beta; l; m)}$$

$$\times \prod_{k=1}^{n-1} \frac{(k+1)(1-\lambda) + 2(1-\lambda+k\lambda)(\eta-1)\cos\theta}{k+2}, \quad (n \geq 2). \quad (20)$$

Proof. From (19), we have

$$\begin{aligned}
 & (n+2)(1-\lambda)\phi_{n+1}(\alpha, \beta; l; m)A_{n+1} \\
 &= 2(\eta-1)\cos\theta \left[1-2\lambda \right. \\
 & \quad \left. + \sum_{k=1}^n \phi_k(\alpha, \beta; l; m)(1-\lambda+k\lambda)A_k \right], \\
 & (n+1)(1-\lambda)\phi_n(\alpha, \beta; l; m)A_n \\
 &= 2(\eta-1)\cos\theta \left[1-2\lambda \right. \\
 & \quad \left. + \sum_{k=1}^{n-1} \phi_k(\alpha, \beta; l; m)(1-\lambda+k\lambda)A_k \right]. \quad (21)
 \end{aligned}$$

Combining (21), we find that

$$\frac{A_{n+1}}{A_n} = \frac{(n+1)(1-\lambda) + 2(1-\lambda+n\lambda)(\eta-1)\cos\theta}{(n+2)(1-\lambda)} \cdot \frac{\phi_n(\alpha, \beta; l; m)}{\phi_{n+1}(\alpha, \beta; l; m)}. \quad (22)$$

Thus, for $n \geq 2$, we deduce from (22) that

$$\begin{aligned}
 A_n &= \frac{A_n}{A_{n-1}} \cdots \frac{A_3}{A_2} \cdot \frac{A_2}{A_1} \cdot A_1 \\
 &= \frac{n(1-\lambda) + 2(1-n\lambda)(\eta-1)\cos\theta}{(n+1)(1-\lambda)} \cdots \\
 & \quad \times \frac{3(1-\lambda) + 2(1+\lambda)(\eta-1)\cos\theta}{4(1-\lambda)} \\
 & \quad \cdot \frac{2(1-\lambda) + 2(\eta-1)\cos\theta}{3(1-\lambda)} \\
 & \quad \cdot \frac{\phi_{n-1}(\alpha, \beta; l; m)}{\phi_n(\alpha, \beta; l; m)} \cdots \frac{\phi_2(\alpha, \beta; l; m)}{\phi_3(\alpha, \beta; l; m)} \\
 & \quad \cdot \frac{\phi_1(\alpha, \beta; l; m)}{\phi_2(\alpha, \beta; l; m)} \cdot \frac{(1-2\lambda)(\eta-1)\cos\theta}{(1-\lambda)\phi_1(\alpha, \beta; l; m)} \\
 &= \frac{(1-2\lambda)(\eta-1)\cos\theta}{(1-\lambda)^n \phi_n(\alpha, \beta; l; m)} \\
 & \quad \times \prod_{k=1}^{n-1} \frac{(k+1)(1-\lambda) + 2(1-\lambda+k\lambda)(\eta-1)\cos\theta}{k+2}. \quad (23)
 \end{aligned}$$

This completes the proof of Lemma 2. $\square$

2. Main Results

We begin by proving the following integral representation for the class $\mathcal{M}_m^l(\theta, \lambda, \eta)$.

Theorem 3. Let $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$. Then

$$\begin{aligned}
 f(z) &= \left(z^{-1} + \sum_{n=1}^{\infty} \phi_n^{-1}(\alpha, \beta; l; m) z^n \right) \\
 & \quad * \left[z^{-1} \cdot \exp \left(\int_0^z \left((1-2\lambda)(\eta-1) \right. \right. \right. \\
 & \quad \times (1+e^{-2i\theta})\omega(t)) \\
 & \quad \times \left([(1-\lambda)(1-\omega(t)) \right. \\
 & \quad \left. \left. \left. - \lambda(\eta-1)(1+e^{-2i\theta}) \right. \right. \right. \\
 & \quad \left. \left. \left. \times \omega(t) \right] t^{-1} \right) dt \right), \quad (24) \\
 & \quad (z \in \mathbb{U}^*),
 \end{aligned}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof. Suppose that $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$ and

$$\begin{aligned}
 \tau(z) &:= \left(\left(e^{i\theta}(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))' \right) \right. \\
 & \quad \times \left((1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) \right. \\
 & \quad \left. \left. + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))' \right)^{-1} \right) \\
 & \quad + \eta \cos\theta + i \sin\theta \times ((\eta-1)\cos\theta)^{-1}, \quad (25) \\
 & \quad (z \in \mathbb{U}).
 \end{aligned}$$

We know that $\tau \in \mathcal{P}$, which implies

$$\begin{aligned}
 & \left(\left(e^{i\theta}(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))' \right) \right. \\
 & \quad \times \left((1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))' \right)^{-1} \\
 & \quad \left. + \eta \cos\theta + i \sin\theta \right) \times ((\eta-1)\cos\theta)^{-1} \\
 &= \frac{1+\omega(z)}{1-\omega(z)}, \quad (26)
 \end{aligned}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. We find from (26) that

$$\begin{aligned}
 & \frac{(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))'}{(1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))'} \\
 &= \frac{-1 + [1 + (\eta-1)(1+e^{-2i\theta})]\omega(z)}{1-\omega(z)}, \quad (27)
 \end{aligned}$$

which follows

$$\begin{aligned} & \frac{(\mathcal{H}_m^l[\alpha, \beta] f(z))'}{\mathcal{H}_m^l[\alpha, \beta] f(z)} + \frac{1}{z} \\ &= \frac{(1-2\lambda)(\eta-1)(1+e^{-2i\theta})\omega(z)}{z[(1-\lambda)(1-\omega(z))-\lambda(\eta-1)(1+e^{-2i\theta})\omega(z)]}. \end{aligned} \quad (28)$$

Integrating both sides of (28) yields

$$\begin{aligned} & \log(z\mathcal{H}_m^l[\alpha, \beta] f(z)) \\ &= \int_0^z \frac{(1-2\lambda)(\eta-1)(1+e^{-2i\theta})\omega(t)}{[(1-\lambda)(1-\omega(t))-\lambda(\eta-1)(1+e^{-2i\theta})\omega(t)]t} dt. \end{aligned} \quad (29)$$

From (29), we obtain

$$\begin{aligned} & \mathcal{H}_m^l[\alpha, \beta] f(z) \\ &= z^{-1} \cdot \exp\left(\int_0^z \left((1-2\lambda)(1+e^{-2i\theta})(\eta-1)\omega(t)\right.\right. \\ & \quad \times \left.\left. \left[(1-\lambda)(1-\omega(t))-\lambda(\eta-1)(1+e^{-2i\theta})\omega(t)\right]^{-1}\right) dt\right). \end{aligned} \quad (30)$$

Thus, the assertion (24) of Theorem 3 follows directly from (30). $\square$

Next, we derive a convolution property for the class $\mathcal{M}_m^l(\theta, \lambda, \eta)$.

Theorem 4. Let $\xi \in \mathbb{C}$ and $|\xi| = 1$. Then $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$ if and only if

$$\begin{aligned} & f * \left\{ (1-\xi) \cdot \left[\frac{1}{z} - \sum_{n=1}^{\infty} n\phi_n(\alpha, \beta; l; m) z^n \right] \right. \\ & \quad + \left[\xi\eta + \xi(\eta-1)e^{-2i\theta} - 1 \right] \\ & \quad \cdot \left. \left[\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1-\lambda+n\lambda}{1-2\lambda} \phi_n(\alpha, \beta; l; m) z^n \right] \right\} \neq 0, \end{aligned} \quad (31)$$

$(z \in \mathbb{U}^*).$

Proof. From the definition (18), we know that $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$ if and only if

$$\begin{aligned} & \left(\left(e^{i\theta} (1-2\lambda) z (\mathcal{H}_m^l[\alpha, \beta] f(z))' \right) \right. \\ & \quad \times \left((1-\lambda) \mathcal{H}_m^l[\alpha, \beta] f(z) + \lambda z (\mathcal{H}_m^l[\alpha, \beta] f(z))' \right)^{-1} \\ & \quad \left. + \eta \cos \theta + i \sin \theta \right) \times ((\eta-1) \cos \theta)^{-1} \neq \frac{1+\xi}{1-\xi}, \end{aligned}$$

$(z \in \mathbb{U}^*; |\xi| = 1),$ (32)

which is equivalent to

$$\begin{aligned} & - (1-\xi) z (\mathcal{H}_m^l[\alpha, \beta] f(z))' + [\xi\eta + \xi(\eta-1)e^{-2i\theta} - 1] \\ & \cdot \frac{(1-\lambda) \mathcal{H}_m^l[\alpha, \beta] f(z) + \lambda z (\mathcal{H}_m^l[\alpha, \beta] f(z))'}{1-2\lambda} \neq 0. \end{aligned} \quad (33)$$

On the other hand, we find from (14) that

$$\begin{aligned} & - z (\mathcal{H}_m^l[\alpha, \beta] f(z))' \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} n\phi_n(\alpha, \beta; l; m) a_n z^n \\ &= f(z) * \left[\frac{1}{z} - \sum_{n=1}^{\infty} n\phi_n(\alpha, \beta; l; m) z^n \right], \end{aligned} \quad (34)$$

$$\begin{aligned} & \frac{(1-\lambda) \mathcal{H}_m^l[\alpha, \beta] f(z) + \lambda z (\mathcal{H}_m^l[\alpha, \beta] f(z))'}{1-2\lambda} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1-\lambda+n\lambda}{1-2\lambda} \phi_n(\alpha, \beta; l; m) a_n z^n \\ &= f(z) * \left[\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1-\lambda+n\lambda}{1-2\lambda} \phi_n(\alpha, \beta; l; m) z^n \right]. \end{aligned}$$

Combining (33) and (34), we get assertion (31) of Theorem 4. $\square$

Now, we discuss the coefficient estimates for functions in the class $\mathcal{M}_m^l(\theta, \lambda, \eta)$.

Theorem 5. Suppose that $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$. Then

$$\begin{aligned} & |a_1| \leq \frac{(1-2\lambda)(\eta-1)\cos\theta}{(1-\lambda)\phi_1(\alpha, \beta; l; m)}, \\ & |a_n| \leq \frac{(1-2\lambda)(\eta-1)\cos\theta}{(1-\lambda)^n \phi_n(\alpha, \beta; l; m)} \\ & \quad \times \prod_{k=1}^{n-1} \frac{(k+1)(1-\lambda) + 2(1-\lambda+k\lambda)(\eta-1)\cos\theta}{k+2}, \end{aligned}$$

$(n \in \mathbb{N} \setminus \{1\}).$ (35)

Proof. Let $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$. Then there exists $\tau \in \mathcal{P}$ such that

$$\frac{e^{i\theta} (1-2\lambda) z (\mathcal{H}_m^l[\alpha, \beta] f(z))'}{(1-\lambda) \mathcal{H}_m^l[\alpha, \beta] f(z) + \lambda z (\mathcal{H}_m^l[\alpha, \beta] f(z))'} \quad (36)$$

$$= (\eta-1) \cos \theta \tau(z) - \eta \cos \theta - i \sin \theta, \quad (z \in \mathbb{U}^*).$$

It follows from (36) that

$$e^{i\theta} (1-2\lambda) z (\mathcal{H}_m^l[\alpha, \beta] f(z))' = \left[(1-\lambda) \mathcal{H}_m^l[\alpha, \beta] f(z) + \lambda z (\mathcal{H}_m^l[\alpha, \beta] f(z))' \right] \times [(\eta-1) \cos \theta \tau(z) - \eta \cos \theta - i \sin \theta]. \quad (37)$$

Combining (1) and (37), we have

$$e^{i\theta} (1-2\lambda) \left[-\frac{1}{z} + \sum_{n=1}^{\infty} n \phi_n(\alpha, \beta; l; m) a_n z^n \right] = \left[(1-2\lambda) \frac{1}{z} + \sum_{n=1}^{\infty} (1-\lambda+n\lambda) \phi_n(\alpha, \beta; l; m) a_n z^n \right] \cdot [-e^{i\theta} + (\eta-1) \cos \theta \tau_1 z + (\eta-1) \cos \theta \tau_2 z^2 + \dots]. \quad (38)$$

Evaluating the coefficient of z^n in both sides of (38) yields

$$2e^{i\theta} (1-\lambda) \phi_1(\alpha, \beta; l; m) a_1 = (1-2\lambda) (\eta-1) \cos \theta \tau_2, \\ e^{i\theta} (n+1) (1-\lambda) \phi_n(\alpha, \beta; l; m) a_n = (\eta-1) \cos \theta \left[(1-2\lambda) \tau_{n+1} + \sum_{k=1}^{n-1} \phi_k(\alpha, \beta; l; m) (1-\lambda+k\lambda) a_k \tau_{n-k} \right], \quad (n \geq 2). \quad (39)$$

By observing the fact that $|\tau_n| \leq 2$ for $n \in \mathbb{N}$, we find from (39) that

$$|a_1| \leq \frac{(1-2\lambda) (\eta-1) \cos \theta}{(1-\lambda) \phi_1(\alpha, \beta; l; m)}, \quad (40)$$

$$|a_n| \leq \frac{2(\eta-1) \cos \theta}{(n+1) (1-\lambda) \phi_n(\alpha, \beta; l; m)} \times \left[1-2\lambda + \sum_{k=1}^{n-1} \phi_k(\alpha, \beta; l; m) (1-\lambda+k\lambda) |a_k| \right], \quad (n \geq 2). \quad (41)$$

Now we define the sequence $\{A_n\}_{n=1}^{\infty}$ as follows:

$$A_1 = \frac{(1-2\lambda) (\eta-1) \cos \theta}{(1-\lambda) \phi_1(\alpha, \beta; l; m)}, \\ A_{n+1} = \frac{2(\eta-1) \cos \theta}{(n+2) (1-\lambda) \phi_{n+1}(\alpha, \beta; l; m)} \times \left[1-2\lambda + \sum_{k=1}^n \phi_k(\alpha, \beta; l; m) (1-\lambda+k\lambda) A_k \right]. \quad (42)$$

In order to prove that

$$|a_n| \leq A_n \quad (n \in \mathbb{N}), \quad (43)$$

we use the principle of mathematical induction. Note that

$$|a_1| \leq A_1 = \frac{(1-2\lambda) (\eta-1) \cos \theta}{(1-\lambda) \phi_1(\alpha, \beta; l; m)}. \quad (44)$$

Therefore, assume that

$$|a_k| \leq A_k \quad (k = 1, 2, \dots, n; n \in \mathbb{N}). \quad (45)$$

Combining (41) and (42), we get

$$|a_{n+1}| \leq \frac{2(\eta-1) \cos \theta}{(n+2) (1-\lambda) \phi_{n+1}(\alpha, \beta; l; m)} \times \left[1-2\lambda + \sum_{k=1}^n \phi_k(\alpha, \beta; l; m) (1-\lambda+k\lambda) |a_k| \right] \leq \frac{2(\eta-1) \cos \theta}{(n+2) (1-\lambda) \phi_{n+1}(\alpha, \beta; l; m)} \times \left[1-2\lambda + \sum_{k=1}^n \phi_k(\alpha, \beta; l; m) (1-\lambda+k\lambda) A_k \right] = A_{n+1}. \quad (46)$$

Hence, by the principle of mathematical induction, we have

$$|a_n| \leq A_n \quad (n \in \mathbb{N}), \quad (47)$$

as desired. By means of Lemma 2 and (42), we know that (20) holds. Combining (47) and (20), we readily get the coefficient estimates asserted by Theorem 5. $\square$

Remark 6. By setting $\theta = 0, l = 2, m = 1, \alpha_1 = \alpha_2 = \beta_1 = 1$, and $\lambda = 0$ in Theorem 5, we get the corresponding result due to Wang et al. [5].

In what follows, we present some sufficient conditions for functions belonging to the class $\mathcal{M}_m^l(\theta, \lambda, \eta)$.

Theorem 7. Let ζ be a real number with $0 \leq \zeta < 1$. If $f \in \mathcal{M}$ satisfies the condition

$$\left| \frac{(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))'}{(1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))'} + 1 \right| \leq 1 - \zeta, \quad (z \in \mathbb{U}^*), \quad (48)$$

then $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$ provided that

$$\cos \theta \geq \frac{1-\zeta}{\eta-1}. \quad (49)$$

Proof. From (48), it follows that

$$\frac{(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))'}{(1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))'} = -1 + (1-\zeta)\omega(z), \quad (50)$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. Thus, we have

$$\begin{aligned} \Re \left(e^{i\theta} \frac{(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))'}{(1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))'} \right) \\ = \Re \left(e^{i\theta} (-1 + (1-\zeta)\omega(z)) \right) \\ = -\cos \theta + (1-\zeta) \Re \left(e^{i\theta} \omega(z) \right) \\ \geq -\cos \theta - (1-\zeta)|\omega(z)| \\ > -\cos \theta - (1-\zeta) \\ \geq -\eta \cos \theta, \end{aligned} \quad (51)$$

provided that $\cos \theta \geq (1-\zeta)/(\eta-1)$. This completes the proof of Theorem 7. $\square$

If we take $\zeta = 1 - (\eta-1)\cos \theta$ in Theorem 7, we obtain the following result.

Corollary 8. If $f \in \mathcal{M}$ satisfies the inequality

$$\left| \frac{(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))'}{(1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))'} + 1 \right| \leq (\eta-1)\cos \theta, \quad (52)$$

then $f \in \mathcal{M}_m^l(\theta, \lambda, \eta)$.

Theorem 9. If a function $f \in \mathcal{M}$ given by (1) satisfies the inequality

$$\sum_{n=1}^{\infty} [(1-\lambda)(1+n) \sec \theta + (\eta-1)(1-\lambda+n\lambda)] \times \phi_n(\alpha, \beta; l, m) |a_n| \leq (1-2\lambda)(\eta-1), \quad (53)$$

then it belongs to the class $\mathcal{M}_m^l(\theta, \lambda, \eta)$.

Proof. In virtue of Corollary 8, it suffices to show that condition (52) holds. We observe that

$$\begin{aligned} \left| \frac{(1-2\lambda)z(\mathcal{H}_m^l[\alpha, \beta]f(z))'}{(1-\lambda)\mathcal{H}_m^l[\alpha, \beta]f(z) + \lambda z(\mathcal{H}_m^l[\alpha, \beta]f(z))'} + 1 \right| \\ = (1-\lambda) \\ \times \left| \frac{\sum_{n=1}^{\infty} (1+n) \phi_n(\alpha, \beta; l, m) a_n z^{n+1}}{1-2\lambda + \sum_{n=1}^{\infty} (1-\lambda+n\lambda) \phi_n(\alpha, \beta; l, m) a_n z^{n+1}} \right| \\ < \frac{(1-\lambda) \sum_{n=1}^{\infty} (1+n) \phi_n(\alpha, \beta; l, m) |a_n|}{1-2\lambda - \sum_{n=1}^{\infty} (1-\lambda+n\lambda) \phi_n(\alpha, \beta; l, m) |a_n|}. \end{aligned} \quad (54)$$

The last expression is bounded by $(\eta-1)\cos \theta$, if

$$\begin{aligned} (1-\lambda) \sum_{n=1}^{\infty} (1+n) \phi_n(\alpha, \beta; l, m) |a_n| \\ \leq (\eta-1)\cos \theta \left[1-2\lambda - \sum_{n=1}^{\infty} (1-\lambda+n\lambda) \phi_n(\alpha, \beta; l, m) |a_n| \right], \end{aligned} \quad (55)$$

which is equivalent to

$$\begin{aligned} \sum_{n=1}^{\infty} [(1-\lambda)(1+n) \sec \theta + (\eta-1)(1-\lambda+n\lambda)] \\ \times \phi_n(\alpha, \beta; l, m) |a_n| \leq (1-2\lambda)(\eta-1). \end{aligned} \quad (56)$$

This completes the proof of Theorem 9. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] M. K. Aouf and B. A. Frasin, "Properties of some families of meromorphic multivalent functions involving certain linear operator," *Filomat*, vol. 24, no. 3, pp. 35–54, 2010.
- [2] M. Arif, J. Sokół, and M. Ayza, "Sufficient condition for functions to be in a class of meromorphic multivalent sakaguchi type spirallike functions," *Acta Mathematica Scientia*, vol. 34, no. 2, pp. 575–578, 2014.

- [3] L. Shi, Z.-G. Wang, and J.-P. Yi, "A new class of meromorphic functions associated with spirallike functions," *Journal of Applied Mathematics*, vol. 2012, Article ID 494917, 12 pages, 2012.
- [4] L. Shi, Z.-G. Wang, and M.-H. Zeng, "Some subclasses of multivalent spirallike meromorphic functions," *Journal of Inequalities and Applications*, vol. 336, 12 pages, 2013.
- [5] Z.-G. Wang, Y. Sun, and Z.-H. Zhang, "Certain classes of meromorphic multivalent functions," *Computers and Mathematics with Applications*, vol. 58, no. 7, pp. 1408–1417, 2009.
- [6] Z. Nehari and E. Netanyahu, "On the coefficients of meromorphic schlicht functions," *Proceedings of the American Mathematical Society*, vol. 8, no. 1, pp. 15–23, 1957.
- [7] H. Orhan, D. Răducanu, M. C. Çağlar, and M. Bayram, "Coefficient estimates and other properties for a class of spirallike functions associated with a differential operator," *Abstract and Applied Analysis*, vol. 2013, Article ID 415319, 7 pages, 2013.
- [8] G. Murugusundaramoorthy, "Subordination results for spirallike functions associated with the Srivastava-Attiya operator," *Integral Transforms and Special Functions*, vol. 23, no. 2, pp. 97–103, 2012.
- [9] J. Dziok and H. M. Srivastava, "Classes of analytic functions associated with the generalized hypergeometric function," *Applied Mathematics and Computation*, vol. 103, no. 1, pp. 1–13, 1999.
- [10] J. Dziok and H. M. Srivastava, "Certain subclasses of analytic functions associated with the generalized hypergeometric function," *Integral Transforms and Special Functions*, vol. 14, no. 1, pp. 7–18, 2003.
- [11] M. K. Aouf, "Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function," *Computers and Mathematics with Applications*, vol. 55, no. 3, pp. 494–509, 2008.
- [12] J.-L. Liu and H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hypergeometric function," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.
- [13] R. K. Raina and H. M. Srivastava, "A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions," *Mathematical and Computer Modelling*, vol. 43, no. 3-4, pp. 350–356, 2006.
- [14] J.-L. Liu and H. M. Srivastava, "A linear operator and associated families of meromorphically multivalent functions," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 2, pp. 566–581, 2001.
- [15] H. M. Srivastava, D.-G. Yang, and N.-E. Xu, "Some subclasses of meromorphically multivalent functions associated with a linear operator," *Applied Mathematics and Computation*, vol. 195, no. 1, pp. 11–23, 2008.
- [16] N. E. Cho and K. I. Noor, "Inclusion properties for certain classes of meromorphic functions associated with the Choi-Saigo-Srivastava operator," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 779–786, 2006.
- [17] J.-L. Liu and S. Owa, "Some families of meromorphic multivalent functions involving certain linear operator," *Indian Journal of Mathematics*, vol. 46, pp. 47–62, 2004.
- [18] K. Piejko and J. Sokół, "Subclasses of meromorphic functions associated with the Cho-Kwon-Srivastava operator," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1261–1266, 2008.