Research Article

An Osgood Type Regularity Criterion for the 3D Boussinesq Equations

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We consider the three-dimensional Boussinesq equations, and obtain an Osgood type regularity criterion in terms of the velocity gradient.

1. Introduction

In this paper, we consider the following three-dimensional (3D) Boussinesq equations with the incompressibility condition:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \Delta \mathbf{u} + \nabla \pi &= \theta \mathbf{e}_3, \\ \theta_t + (\mathbf{u} \cdot \nabla) \theta - \Delta \theta &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0, \quad \theta(x, 0) &= \theta_0, \end{aligned} \quad (1)$$

where $\mathbf{u} = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the fluid velocity, $\pi = \pi(x, t)$ is a scalar pressure, and $\theta = \theta(x, t)$ is the scalar temperature, while $\mathbf{u}_0$ and θ_0 are the prescribed initial velocity and temperature, respectively, with $\nabla \cdot \mathbf{u}_0 = 0$.

In case $\theta = 0$, (1) reduces to the incompressible Navier-Stokes equations. The regularity of its weak solutions and the existence of global strong solutions are important open problems; see [1–3]. Starting with [4, 5], there have been a lot of literatures devoted to finding sufficient conditions (which now are called regularity criteria) to ensure the smoothness of the solutions; see [6–16] and so forth. Since the convective terms $(\mathbf{u} \cdot \nabla) \mathbf{u}$ are the same in the Navier-Stokes equations and Boussinesq equations, the authors also consider the regularity

conditions for (1). In particular, Qiu et al. [17] obtained Serrin type regularity condition:

$$\mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty. \quad (2)$$

The extension to the multiplier spaces was established by the same authors in [18]. For the Besov-type regularity criterion, Fan and Zhou [19] and Ishimura and Morimoto [20] showed the following regularity conditions:

$$\begin{aligned} \nabla \times \mathbf{u} &\in L^1(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^3)), \\ \nabla \mathbf{u} &\in L^1(0, T; L^\infty(\mathbb{R}^3)). \end{aligned} \quad (3)$$

Zhang [21, 22] then considers the regularity criterion in terms of the pressure or its gradient. The readers are also referred to [23] for generalized models.

Motivated by [24–26], we will improve (3) as in the following.

Theorem 1. Let $(\mathbf{u}_0, \theta_0) \in H^1(\mathbb{R}^3)$. Assume that $(\mathbf{u}, \theta)$ is the smooth solution to (1) with the initial data $(\mathbf{u}_0, \theta_0)$ for $0 \leq t < T$. If

$$\sup_{2 \leq q < \infty} \int_0^T \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} < \infty, \quad (4)$$

then the solution $(\mathbf{u}, \theta)$ can be extended after time $t = T$. Here, $\dot{\Delta}_k$ denotes the Fourier localization operator and $\Delta S_q = \sum_{l=-q}^q \dot{\Delta}_l$.

Remark 2. The Osgood type condition (4) is weaker than (3). Notice that, for $q \in [2, \infty)$, we have

$$\frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} \leq \frac{1}{q \ln q} \sum_{l=-q}^q \|\dot{\Delta}_l (\nabla \times \mathbf{u})\|_{L^\infty} \leq C \|\nabla \times \mathbf{u}\|_{\dot{B}_{\infty, \infty}^0}. \quad (5)$$

The rest of this paper is organized as follows. In Section 2, we recall the definition of Besov spaces and some interpolation inequalities. Section 3 is devoted to proving Theorem 1.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} dx. \quad (6)$$

Let us choose a nonnegative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \leq \hat{\varphi}(\xi) \leq 1, \quad \hat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases} \quad (7)$$

and let

$$\begin{aligned} \psi(x) &= \varphi(x) - 2^{-3} \varphi\left(\frac{x}{2}\right), \\ \varphi_j(x) &= 2^{3j} \varphi(2^j x), \quad \psi_j(x) = 2^{3j} \psi(2^j x), \quad j \in \mathbb{Z}. \end{aligned} \quad (8)$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and $\dot{\Delta}_j$ are, respectively, defined by

$$S_j f = \varphi_j * f, \quad \dot{\Delta}_j f = \psi_j * f. \quad (9)$$

Observe that $\dot{\Delta}_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \rightarrow 0, \quad \text{as } j \rightarrow -\infty; \quad S_j f \rightarrow f, \quad \text{as } j \rightarrow +\infty, \quad (10)$$

in the L^2 sense. By telescoping the series, we thus have the following Littlewood-Paley decomposition:

$$f = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j f, \quad (11)$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is the L^2 sense. Notice that

$$\dot{\Delta}_j f = \sum_{l=j-2}^{j+2} \dot{\Delta}_l \dot{\Delta}_j f = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f; \quad (12)$$

then from Young's inequality, it readily follows that

$$\|\dot{\Delta}_j f\|_{L^q} \leq C 2^{3j(1/p-1/q)} \|\dot{\Delta}_j f\|_{L^p}, \quad (13)$$

where $1 \leq p \leq q \leq \infty$ and C is an absolute constant independent of f and j .

Let $-\infty < s < \infty$, $1 \leq p, q \leq \infty$; the homogeneous Besov space $\dot{B}_{p,q}^s$ is defined by the full-dyadic decomposition such that

$$\dot{B}_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^3); \|f\|_{\dot{B}_{p,q}^s} < \infty \right\}, \quad (14)$$

where

$$\|f\|_{\dot{B}_{p,q}^s} = \left\| \left\{ 2^{js} \|\dot{\Delta}_j f\|_{L^p} \right\}_{j=-\infty}^{+\infty} \right\|_{\ell^q}, \quad (15)$$

and $\mathcal{S}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{S}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \hat{f}(0) = 0, \forall \alpha \in \mathbb{N}^3 \right\}. \quad (16)$$

Also, it is well known that

$$\dot{H}^s(\mathbb{R}^3) = \dot{B}_{2,2}^s(\mathbb{R}^3), \quad \forall s \in \mathbb{R}. \quad (17)$$

We refer to [27] for more detailed properties.

3. Proof of Theorem 1

This section is devoted to proving Theorem 1. From standard continuity arguments, we need to only provide the uniform H^1 bounds of the solution $(\mathbf{u}, \theta)$.

Taking the inner products of (1)₁ with $-\Delta \mathbf{u}$, (1)₂ with $-\Delta \theta$, we obtain by adding together that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 + \|\Delta(\mathbf{u}, \theta)\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \Delta \mathbf{u} \, dx - \int_{\mathbb{R}^3} \theta \Delta u_3 \, dx \\ &+ \int_{\mathbb{R}^3} [(\mathbf{u} \cdot \nabla) \theta] \cdot \Delta \theta \, dx \\ &= \int_{\mathbb{R}^3} \partial_k \theta \partial_k u_3 \, dx \\ &- \int_{\mathbb{R}^3} \partial_k u_j (\partial_j u_i \partial_k u_i + \partial_j \theta \partial_k \theta) \, dx \\ &\equiv I + J. \end{aligned} \quad (18)$$

For I , we use Hölder's inequality to get

$$I_1 \leq \frac{1}{2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2. \quad (19)$$

For J , applying the Littlewood-Paley decomposition as in (11), we get

$$\nabla \mathbf{u} = \sum_{l < -q} \dot{\Delta} \nabla \mathbf{u} + \sum_{l=-q}^q \dot{\Delta} \nabla \mathbf{u} + \sum_{l > q} \dot{\Delta} \nabla \mathbf{u}, \quad (20)$$

where q is positive integral to be determined later on. Plugging (20) into J , we see that

$$\begin{aligned} J &\leq \sum_{l < -q} \int_{\mathbb{R}^3} |\dot{\Delta}_l \nabla \mathbf{u}| \cdot |\nabla(\mathbf{u}, \theta)|^2 dx \\ &\quad + \int_{\mathbb{R}^3} \left| \sum_{l=-q}^q \dot{\Delta}_l \nabla \mathbf{u} \right| \cdot |\nabla(\mathbf{u}, \theta)|^2 dx \\ &\quad + \sum_{l > q} \int_{\mathbb{R}^3} |\dot{\Delta}_l \nabla \mathbf{u}| \cdot |\nabla(\mathbf{u}, \theta)|^2 dx \\ &\equiv J_1 + J_2 + J_3. \end{aligned} \quad (21)$$

For J_1 , we dominate as

$$\begin{aligned} J_1 &\leq \sum_{l < -q} \|\dot{\Delta}_l \nabla \mathbf{u}\|_{L^\infty} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\leq C \sum_{l < -q} 2^{3l/2} \|\dot{\Delta}_l \nabla \mathbf{u}\|_{L^2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \quad (\text{by (13)}) \\ &\leq C \left(\sum_{l < -q} 2^{(3l/2) \cdot 2} \right)^{1/2} \cdot \left(\sum_{l < -q} \|\dot{\Delta}_l \nabla \mathbf{u}\|_{L^2}^2 \right)^{1/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\leq C 2^{-3q/2} \|\nabla \mathbf{u}\|_{L^2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \quad (\text{by (17)}) \\ &= [C 2^{-q/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}]^3. \end{aligned} \quad (22)$$

For J_2 , we have

$$\begin{aligned} J_2 &= \int_{\mathbb{R}^3} |\bar{S}_q \nabla \mathbf{u}| \cdot |\nabla(\mathbf{u}, \theta)|^2 dx \\ &\leq \|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2. \end{aligned} \quad (23)$$

Finally, for J_3 , we estimate as

$$\begin{aligned} J_3 &\leq \sum_{l > q} \|\Delta_l \nabla \mathbf{u}\|_{L^3} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\leq C \sum_{l > q} 2^{l/2} \|\Delta_l \nabla \mathbf{u}\|_{L^2} \|\nabla(\mathbf{u}, \theta)\|_{L^2} \|\Delta(\mathbf{u}, \theta)\|_{L^2} \\ &\quad (\text{by (13) and Gagliardo-Nirenberg inequality}) \\ &\leq C \left(\sum_{l > q} 2^{-(l/2) \cdot 2} \right)^{1/2} \cdot \left(\sum_{l > q} 2^{l \cdot 2} \|\Delta_l \nabla \mathbf{u}\|_{L^2}^2 \right)^{1/2} \\ &\quad \times \|\nabla(\mathbf{u}, \theta)\|_{L^2} \|\Delta(\mathbf{u}, \theta)\|_{L^2} \\ &\leq [C 2^{-q/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}] \|\Delta(\mathbf{u}, \theta)\|_{L^2}^2 \quad (\text{by (17)}). \end{aligned} \quad (24)$$

Gathering (22), (23), and (24) together and plugging them into (21), we deduce

$$\begin{aligned} J &\leq [C 2^{-q/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}]^3 \\ &\quad + \|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\quad + [C 2^{-q/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}] \|\Delta(\mathbf{u}, \theta)\|_{L^2}^2. \end{aligned} \quad (25)$$

Substituting (19) and (25) into (18), we find

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 + \|\Delta(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\leq \frac{1}{2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 + [C 2^{-q/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}]^3 \\ &\quad + \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} \cdot q \ln q \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\quad + [C 2^{-q/2} \|\nabla(\mathbf{u}, \theta)\|_{L^2}] \|\Delta(\mathbf{u}, \theta)\|_{L^2}^2. \end{aligned} \quad (26)$$

Taking

$$q = \left\lfloor \frac{2}{\ln 2 \ln^+(C \|\nabla(\mathbf{u}, \theta)\|_{L^2})} \right\rfloor + 1, \quad (27)$$

where $[t]$ is the largest integer smaller than $t \in \mathbb{R}$ and $\ln^+ t = \ln(e + t)$, then (26) implies that

$$\begin{aligned} &\frac{d}{dt} \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 \\ &\leq \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 + C \\ &\quad + \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} \ln^+(\|\nabla(\mathbf{u}, \theta)\|_{L^2}) \ln^+(\|\nabla(\mathbf{u}, \theta)\|_{L^2}) \\ &\quad \times \|\nabla(\mathbf{u}, \theta)\|_{L^2}^2. \end{aligned} \quad (28)$$

Applying Gronwall inequality three times, we deduce

$$\begin{aligned} &\|\nabla(\mathbf{u}, \theta)\|_{L^2}^2 + \int_0^t \|\Delta(\mathbf{u}, \theta)\|_{L^2}^2 d\tau \\ &\leq C \exp \exp \exp \left(\int_0^t \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} d\tau \right). \end{aligned} \quad (29)$$

Recalling (4), we see the solution $(\mathbf{u}, \theta)$ is uniformly bounded in $L^\infty(0, T; H^1(\mathbb{R}^3))$. This completes the proof of Theorem 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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