Research Article

Fixed Point Results for G - α -Contractive Maps with Application to Boundary Value Problems

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We unify the concepts of G -metric, metric-like, and b -metric to define new notion of generalized b -metric-like space and discuss its topological and structural properties. In addition, certain fixed point theorems for two classes of G - α -admissible contractive mappings in such spaces are obtained and some new fixed point results are derived in corresponding partially ordered space. Moreover, some examples and an application to the existence of a solution for the first-order periodic boundary value problem are provided here to illustrate the usability of the obtained results.

1. Introduction and Mathematical Preliminaries

The concept of a b -metric space was introduced by Czerwik [1]. After that, several interesting results about the existence of fixed point for single-valued and multivalued operators in (ordered) b -metric spaces have been obtained (see, e.g., [2–11]).

Definition 1 (see [1]). Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is a b -metric on X if, for all $x, y, z \in X$, the following conditions hold:

- (b₁) $d(x, y) = 0$ if and only if $x = y$,
- (b₂) $d(x, y) = d(y, x)$,
- (b₃) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b -metric space.

The concept of a generalized metric space, or a G -metric space, was introduced by Mustafa and Sims [12].

Definition 2 (see [12]). Let X be a nonempty set and $G : X \times X \times X \rightarrow \mathbb{R}^+$ a function satisfying the following properties:

- (G₁) $G(x, y, z) = 0$ if and only if $x = y = z$;

(G₂) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G₃) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;

(G₄) $G(x, y, z) = G(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all three variables);

(G₅) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G -metric on X and the pair (X, G) is called a G -metric space.

Definition 3 (see [13]). A metric-like on a nonempty set X is a mapping $\sigma : X \times X \rightarrow \mathbb{R}^+$ such that, for all $x, y, z \in X$, the following hold:

- (σ_1) $\sigma(x, y) = 0$ implies $x = y$;
- (σ_2) $\sigma(x, y) = \sigma(y, x)$;
- (σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

Below, we give some examples of metric-like spaces.

Example 4 (see [14]). Let $X = [0, 1]$. Then, the mapping $\sigma_1 : X \times X \rightarrow \mathbb{R}^+$ defined by $\sigma_1(x, y) = x + y - xy$ is a metric-like on X .

Example 5 (see [14]). Let $X = \mathbb{R}$; then the mappings $\sigma_i : X \times X \rightarrow \mathbb{R}^+ (i \in \{2, 3, 4\})$ defined by

$$\begin{aligned}\sigma_2(x, y) &= |x| + |y| + a, \\ \sigma_3(x, y) &= |x - b| + |y - b|, \\ \sigma_4(x, y) &= x^2 + y^2\end{aligned}\quad (1)$$

are metric-likes on X , where $a \geq 0$ and $b \in \mathbb{R}$.

Definition 6 (see [15]). Let X be a nonempty set and $s \geq 1$ a given real number. A function $\sigma_b : X \times X \rightarrow \mathbb{R}^+$ is a b -metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

$$\begin{aligned}(\sigma_b 1) \quad & \sigma_b(x, y) = 0 \text{ implies } x = y; \\ (\sigma_b 2) \quad & \sigma_b(x, y) = \sigma_b(y, x); \\ (\sigma_b 3) \quad & \sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)].\end{aligned}$$

A b -metric-like space is a pair (X, σ_b) such that X is a nonempty set and σ_b is a b -metric-like on X . The number s is called the coefficient of (X, σ_b) .

In a b -metric-like space (X, σ_b) if $x, y \in X$ and $\sigma_b(x, y) = 0$, then $x = y$, but the converse may not be true and $\sigma_b(x, x)$ may be positive for all $x \in X$. It is clear that every b -metric space is a b -metric-like space with the same coefficient s but not conversely in general.

Example 7 (see [8]). Let $X = \mathbb{R}^+$, let $p > 1$ be a constant, and let $\sigma_b : X \times X \rightarrow \mathbb{R}^+$ be defined by

$$\sigma_b(x, y) = (x + y)^p \quad \forall x, y \in X. \quad (2)$$

Then, (X, σ_b) is a b -metric-like space with coefficient $s = 2^{p-1}$.

The following propositions help us to construct some more examples of b -metric-like spaces.

Proposition 8 (see [8]). Let (X, σ) be a metric-like space and $\sigma_b(x, y) = [\sigma(x, y)]^p$, where $p > 1$ is a real number. Then, σ_b is a b -metric-like with coefficient $s = 2^{p-1}$.

From the above proposition and Examples 4 and 5, we have the following examples of b -metric-like spaces.

Example 9 (see [8]). Let $X = [0, 1]$. Then, the mapping $\sigma_{b1} : X \times X \rightarrow \mathbb{R}^+$ defined by $\sigma_{b1}(x, y) = (x + y - xy)^p$, where $p > 1$ is a real number, is a b -metric-like on X with coefficient $s = 2^{p-1}$.

Example 10 (see [8]). Let $X = \mathbb{R}$. Then, the mappings $\sigma_{bi} : X \times X \rightarrow \mathbb{R}^+ (i \in \{2, 3, 4\})$ defined by

$$\begin{aligned}\sigma_{b2}(x, y) &= (|x| + |y| + a)^p, \\ \sigma_{b3}(x, y) &= (|x - b| + |y - b|)^p, \\ \sigma_{b4}(x, y) &= (x^2 + y^2)^p\end{aligned}\quad (3)$$

are b -metric-like on X , where $p > 1$, $a \geq 0$, and $b \in \mathbb{R}$.

Each b -metric-like σ_b on X generates a topology τ_{σ_b} on X whose base is the family of all open σ_b -balls $\{B_{\sigma_b}(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_{\sigma_b}(x, \varepsilon) = \{y \in X : |\sigma_b(x, y) - \sigma_b(x, x)| < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Now, we introduce the concept of generalized b -metric-like space, or G_{σ_b} -metric space, as a proper generalization of both of the concepts of b -metric-like spaces and G -metric spaces.

Definition 11. Let X be a nonempty set. Suppose that a mapping $G_{\sigma_b} : X \times X \times X \rightarrow \mathbb{R}^+$ satisfies the following:

$$\begin{aligned}(G_{\sigma_b} 1) \quad & G_{\sigma_b}(x, y, z) = 0 \text{ implies } x = y = z; \\ (G_{\sigma_b} 2) \quad & G_{\sigma_b}(x, y, z) = G_{\sigma_b}(p\{x, y, z\}), \text{ where } p \text{ is any permutation of } x, y, z \text{ (symmetry in all three variables);} \\ (G_{\sigma_b} 3) \quad & G_{\sigma_b}(x, y, z) \leq s[G_{\sigma_b}(x, a, a) + G_{\sigma_b}(a, y, z)] \text{ for all } x, y, z, a \in X \text{ (rectangle inequality).}\end{aligned}$$

Then, G_{σ_b} is called a G_{σ_b} -metric and (X, G_{σ_b}) is called a generalized b -metric-like space.

The following proposition will be useful in constructing examples of a generalized b -metric-like space.

Proposition 12. Let (X, σ_b) be a b -metric-like space with coefficient s . Then,

$$\begin{aligned}G_{\sigma_b}^m(x, y, z) &= \max\{\sigma_b(x, y), \sigma_b(y, z), \sigma_b(z, x)\}, \\ G_{\sigma_b}^s(x, y, z) &= \sigma_b(x, y) + \sigma_b(y, z) + \sigma_b(z, x)\end{aligned}\quad (4)$$

are two generalized b -metric-like functions on X .

Proof. It is clear that $G_{\sigma_b}^m$ and $G_{\sigma_b}^s$ satisfy conditions $G_{\sigma_b} 1$ and $G_{\sigma_b} 2$ of Definition 11. So, we only show that $G_{\sigma_b} 3$ is satisfied by $G_{\sigma_b}^m$ and $G_{\sigma_b}^s$. Let $x, y, z, a \in X$. Then, using the triangular inequality in b -metric-like spaces, we have

$$\begin{aligned}G_{\sigma_b}^m(x, y, z) &= \max\{\sigma_b(x, y), \sigma_b(y, z), \sigma_b(z, x)\} \\ &\leq \max\{s(\sigma_b(x, a) + \sigma_b(a, y)), \\ &\quad \sigma_b(y, z), s(\sigma_b(z, a) + \sigma_b(a, x))\} \\ &\leq \max\{s(\sigma_b(x, a) + \sigma_b(a, y)), \\ &\quad s(\sigma_b(y, z) + \sigma_b(a, a)), \\ &\quad s(\sigma_b(z, a) + \sigma_b(a, x))\} \\ &= s \max\{\sigma_b(x, a), \sigma_b(a, a), \sigma_b(a, x)\} \\ &\quad + s \max\{\sigma_b(a, y), \sigma_b(y, z), \sigma_b(z, a)\} \\ &= s(G_{\sigma_b}^m(x, a, a) + G_{\sigma_b}^m(a, y, z)).\end{aligned}\quad (5)$$

Also,

$$\begin{aligned}
 G_{\sigma_b}^s(x, y, z) &= \sigma_b(x, y) + \sigma_b(y, z) + \sigma_b(z, x) \\
 &\leq s(\sigma_b(x, a) + \sigma_b(a, y)) \\
 &\quad + \sigma_b(y, z) + s(\sigma_b(z, a) + \sigma_b(a, x)) \\
 &\leq s(\sigma_b(x, a) + \sigma_b(a, y)) \\
 &\quad + s(\sigma_b(y, z) + \sigma_b(a, a)) \\
 &\quad + s(\sigma_b(z, a) + \sigma_b(a, x)) \\
 &= s(\sigma_b(x, a) + \sigma_b(a, a) + \sigma_b(a, x)) \\
 &\quad + s(\sigma_b(a, y), \sigma_b(y, z), \sigma_b(z, a)) \\
 &= s(G_{\sigma_b}^s(x, a, a) + G_{\sigma_b}^s(a, y, z)).
 \end{aligned} \tag{6}$$

□

According to the above proposition, we provide some examples of generalized b -metric-like spaces.

Example 13. Let $X = \mathbb{R}^+$, let $p > 1$ be a constant, and let $G_{\sigma_b}^m, G_{\sigma_b}^s : X \times X \times X \rightarrow \mathbb{R}^+$ be defined by

$$\begin{aligned}
 G_{\sigma_b}^m(x, y, z) &= \max \{(x + y)^p, (y + z)^p, (z + x)^p\}, \\
 G_{\sigma_b}^s(x, y, z) &= (x + y)^p + (y + z)^p + (z + x)^p,
 \end{aligned} \tag{7}$$

for all $x, y, z \in X$. Then, $(X, G_{\sigma_b}^m)$ and $(X, G_{\sigma_b}^s)$ are generalized b -metric-like spaces with coefficient $s = 2^{p-1}$. Note that, for $x = y = z > 0$, $G_{\sigma_b}^m(x, x, x) = (2x)^p > 0$ and $G_{\sigma_b}^s(x, x, x) = 3(2x)^p > 0$.

Example 14. Let $X = [0, 1]$. Then, the mappings $G_{\sigma_{b1}}^m, G_{\sigma_{b1}}^s : X \times X \times X \rightarrow \mathbb{R}^+$ defined by

$$\begin{aligned}
 G_{\sigma_{b1}}^m(x, y, z) &= \max \{(x + y - xy)^p, (y + z - yz)^p, (z + x - zx)^p\}, \\
 G_{\sigma_{b1}}^s(x, y, z) &= (x + y - xy)^p + (y + z - yz)^p + (z + x - zx)^p,
 \end{aligned} \tag{8}$$

where $p > 1$ is a real number, are generalized b -metric-like spaces with coefficient $s = 2^{p-1}$.

By some straight forward calculations, we can establish the following.

Proposition 15. Let X be a G_{σ_b} -metric space. Then, for each $x, y, z, a \in X$, it follows that:

- (1) $G_{\sigma_b}(x, y, y) > 0$ for $x \neq y$;
- (2) $G_{\sigma_b}(x, y, z) \leq s(G_{\sigma_b}(x, x, y) + G_{\sigma_b}(x, x, z))$;
- (3) $G_{\sigma_b}(x, y, y) \leq 2sG_{\sigma_b}(y, x, x)$;

$$(4) \quad G_{\sigma_b}(x, y, z) \leq sG_{\sigma_b}(x, a, a) + s^2G_{\sigma_b}(y, a, a) + s^2G_{\sigma_b}(z, a, a).$$

Definition 16. Let (X, G_{σ_b}) be a G_{σ_b} -metric space. Then, for any $x \in X$ and $r > 0$, the G_{σ_b} -ball with center x and radius r is

$$B_{G_{\sigma_b}}(x, r) = \{y \in X \mid |G_{\sigma_b}(x, x, y) - G_{\sigma_b}(x, x, x)| < +r\}. \tag{9}$$

The family of all G_{σ_b} -balls

$$\mathcal{F} = \{B_{G_{\sigma_b}}(x, r) \mid x \in X, r > 0\} \tag{10}$$

is a base of a topology $\tau(G_{\sigma_b})$ on X , which we call it G_{σ_b} -metric topology.

Definition 17. Let (X, G_{σ_b}) be a G_{σ_b} -metric space. Let $\{x_n\}$ be a sequence in X . Consider the following.

- (1) A point $x \in X$ is said to be a limit of the sequence $\{x_n\}$, denoted by $x_n \rightarrow x$, if $\lim_{n, m \rightarrow \infty} G_{\sigma_b}(x, x_n, x_m) = G_{\sigma_b}(x, x, x)$.
- (2) $\{x_n\}$ is said to be a G_{σ_b} -Cauchy sequence, if $\lim_{n, m \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m)$ exists (and is finite).
- (3) (X, G_{σ_b}) is said to be G_{σ_b} -complete if every G_{σ_b} -Cauchy sequence in X is G_{σ_b} -convergent to an $x \in X$ such that

$$\lim_{n, m \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) = \lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x, x) = G_{\sigma_b}(x, x, x). \tag{11}$$

Using the above definitions, one can easily prove the following proposition.

Proposition 18. Let (X, G_{σ_b}) be a G_{σ_b} -metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent:

- (1) $\{x_n\}$ is G_{σ_b} -convergent to x ;
- (2) $G_{\sigma_b}(x_n, x_n, x) \rightarrow G_{\sigma_b}(x, x, x)$, as $n \rightarrow \infty$;
- (3) $G_{\sigma_b}(x_n, x, x) \rightarrow G_{\sigma_b}(x, x, x)$, as $n \rightarrow \infty$;

Definition 19. Let (X, G_{σ_b}) and (X', G'_{σ_b}) be two generalized b -metric like spaces and let $f : (X, G_{\sigma_b}) \rightarrow (X', G'_{\sigma_b})$ be a mapping. Then, f is said to be G_{σ_b} -continuous at a point $a \in X$ if, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $|G_{\sigma_b}(a, a, x) - G_{\sigma_b}(a, a, a)| < \delta$ imply that $|G'_{\sigma_b}(f(a), f(a), f(x)) - G'_{\sigma_b}(f(a), f(a), f(a))| < \varepsilon$. The mapping f is G_{σ_b} -continuous on X if it is G_{σ_b} -continuous at all $a \in X$. For simplicity, we say that f is continuous.

Proposition 20. Let (X, G_{σ_b}) and (X', G'_{σ_b}) be two generalized b -metric like spaces. Then, a mapping $f : X \rightarrow X'$ is G_{σ_b} -continuous at a point $x \in X$ if and only if it is G_{σ_b} -sequentially continuous at x ; that is, whenever $\{x_n\}$ is G_{σ_b} -convergent to x , $\{f(x_n)\}$ is G_{σ_b} -convergent to $f(x)$.

We need the following simple lemma about the G_{σ_b} -convergent sequences in the proof of our main results.

Lemma 21. Let (X, G_{σ_b}) be a G_{σ_b} -metric space and suppose that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are G_{σ_b} -convergent to x , y , and z , respectively. Then, we have

$$\begin{aligned} & \frac{1}{s^3} G_{\sigma_b}(x, y, z) - \frac{1}{s^2} G_{\sigma_b}(x, x, x) \\ & - \frac{1}{s} G_{\sigma_b}(y, y, y) - G_{\sigma_b}(z, z, z) \\ & \leq \liminf_{n \rightarrow \infty} G_{\sigma_b}(x_n, y_n, z_n) \\ & \leq \limsup_{n \rightarrow \infty} G_{\sigma_b}(x_n, y_n, z_n) \\ & \leq s^3 G_{\sigma_b}(x, y, z) + s G_{\sigma_b}(x, x, x) \\ & + s^2 G_{\sigma_b}(y, y, y) + s^3 G_{\sigma_b}(z, z, z). \end{aligned} \quad (12)$$

In particular, if $\{y_n\} = \{z_n\} = a$ are constant, then

$$\begin{aligned} & \frac{1}{s} G_{\sigma_b}(x, a, a) - G_{\sigma_b}(x, x, x) \\ & \leq \liminf_{n \rightarrow \infty} G_{\sigma_b}(x_n, a, a) \leq \limsup_{n \rightarrow \infty} G_{\sigma_b}(x_n, a, a) \\ & \leq s G_{\sigma_b}(x, a, a) + s G_{\sigma_b}(x, x, x). \end{aligned} \quad (13)$$

Proof. Using the rectangle inequality, we obtain

$$\begin{aligned} G_{\sigma_b}(x, y, z) & \leq s G_{\sigma_b}(x, x_n, x_n) + s^2 G_{\sigma_b}(y, y_n, y_n) \\ & + s^3 G_{\sigma_b}(z, z_n, z_n) + s^3 G_{\sigma_b}(x_n, y_n, z_n), \\ G_{\sigma_b}(x_n, y_n, z_n) & \leq s G_{\sigma_b}(x_n, x, x) + s^2 G_{\sigma_b}(y_n, y, y) \\ & + s^3 G_{\sigma_b}(z_n, z, z) + s^3 G_{\sigma_b}(x, y, z). \end{aligned} \quad (14)$$

Taking the lower limit as $n \rightarrow \infty$ in the first inequality and the upper limit as $n \rightarrow \infty$ in the second inequality, we obtain the desired result.

If $\{y_n\} = \{z_n\} = a$, then

$$\begin{aligned} G_{\sigma_b}(x, a, a) & \leq s G_{\sigma_b}(x, x_n, x_n) + s G_{\sigma_b}(x_n, a, a), \\ G_{\sigma_b}(x_n, a, a) & \leq s G_{\sigma_b}(x_n, x, x) + s G_{\sigma_b}(x, a, a). \end{aligned} \quad (15)$$

Again taking the lower limit as $n \rightarrow \infty$ in the first inequality and the upper limit as $n \rightarrow \infty$ in the second inequality, we obtain the desired result. $\square$

2. Main Results

Samet et al. [16] defined the notion of α -admissible mappings and proved the following result.

Definition 22. Let T be a self-mapping on X and $\alpha : X \times X \rightarrow [0, \infty)$ a function. We say that T is an α -admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1. \quad (16)$$

Denote with Ψ' the family of all nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n th iterate of ψ .

Theorem 23. Let (X, d) be a complete metric space and T an α -admissible mapping. Assume that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad (17)$$

where $\psi \in \Psi'$. Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (ii) either T is continuous or, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, T has a fixed point.

For more details on α -admissible mappings, we refer the reader to [17–20].

Definition 24 (see [21]). Let (X, G) be a G -metric space, let f be a self-mapping on X , and let $\alpha : X^3 \rightarrow [0, \infty)$ be a function. We say that f is a G - α -admissible mapping if

$$x, y, z \in X, \quad \alpha(x, y, z) \geq 1 \implies \alpha(fx, fy, fz) \geq 1. \quad (18)$$

Motivated by [22], let $\mathcal{F}$ denote the class of all functions $\beta : [0, \infty) \rightarrow [0, 1/s)$ satisfying the following condition:

$$\limsup_{n \rightarrow \infty} \beta(t_n) = \frac{1}{s} \implies \limsup_{n \rightarrow \infty} t_n = 0. \quad (19)$$

Definition 25. Let $f : X \rightarrow X$ and $\alpha : X \times X \times X \rightarrow \mathbb{R}$. We say that f is a rectangular G - α -admissible mapping if

- (T1) $\alpha(x, y, z) \geq 1$ implies $\alpha(fx, fy, fz) \geq 1, x, y, z \in X$;
- (T2) $\left\{ \begin{array}{l} \alpha(x, y, y) \geq 1 \\ \alpha(y, z, z) \geq 1 \end{array} \right\}$ implies $\alpha(x, z, z) \geq 1, x, y, z \in X$.

From now on, let $\alpha : X^3 \rightarrow [0, \infty)$ be a function and

$$\begin{aligned} M_s(x, y, z) & = \max \left\{ G_{\sigma_b}(x, y, z), \right. \\ & \left. \frac{G_{\sigma_b}(x, fx, fx) G_{\sigma_b}(y, fy, fy) G_{\sigma_b}(z, fz, fz)}{1 + s^2 G_{\sigma_b}^2(fx, fy, fz)} \right\}. \end{aligned} \quad (20)$$

Theorem 26. Let (X, G_{σ_b}) be a G_{σ_b} -complete generalized b -metric-like space and let $f : X \rightarrow X$ be a rectangular G - α -admissible mapping. Suppose that

$$\begin{aligned} & s\alpha(x, y, z) G_{\sigma_b}(fx, fy, fz) \\ & \leq \beta(G_{\sigma_b}(x, y, z)) M_s(x, y, z), \end{aligned} \quad (21)$$

for all $x, y, z \in X$.

Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, fx_0) \geq 1$;
- (ii) f is continuous and, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha(x, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is a G - α -admissible mapping and $\alpha(x_0, x_1, x_1) = \alpha(x_0, fx_0, fx_0) \geq 1$, we deduce that $\alpha(x_1, x_2, x_2) = \alpha(fx_0, fx_1, fx_1) \geq 1$. Continuing this process, we get $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Step I. We will show that $\lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) = 0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = fx_n$. Thus, x_n is a fixed point of f . Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, then we can apply (21) which yields

$$\begin{aligned} sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) &= sG_{\sigma_b}(fx_{n-1}, fx_n, fx_n) \\ &\leq \beta(G_{\sigma_b}(x_{n-1}, x_n, x_n)) M_s(x_{n-1}, x_n, x_n) \\ &< \frac{1}{s} G_{\sigma_b}(x_{n-1}, x_n, x_n) \end{aligned} \quad (22)$$

$$\leq G_{\sigma_b}(x_{n-1}, x_n, x_n),$$

$$M_s(x_{n-1}, x_n, x_n)$$

$$\begin{aligned} &= \max \left\{ G_{\sigma_b}(x_{n-1}, x_n, x_n), \right. \\ &\quad \left(G_{\sigma_b}(x_{n-1}, fx_{n-1}, fx_{n-1}) \right. \\ &\quad \times G_{\sigma_b}(x_n, fx_n, fx_n) G_{\sigma_b}(x_n, fx_n, fx_n) \\ &\quad \times (1 + s^2 G_{\sigma_b}^2(fx_{n-1}, fx_n, fx_n))^{-1} \Big\} \\ &= \max \left\{ G_{\sigma_b}(x_{n-1}, x_n, x_n), \right. \\ &\quad \left(G_{\sigma_b}(x_{n-1}, x_n, x_n) \right. \\ &\quad \times G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) \\ &\quad \times (1 + s^2 G_{\sigma_b}^2(x_n, x_{n+1}, x_{n+1}))^{-1} \Big\} \\ &= G_{\sigma_b}(x_{n-1}, x_n, x_n). \end{aligned} \quad (23)$$

Therefore, $\{G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})\}$ is a decreasing and bounded sequence of nonnegative real numbers. Then, there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) = r$. Letting $n \rightarrow \infty$ in (22), we have

$$sr \leq r. \quad (24)$$

Since $s > 1$, we deduce that $r = 0$, that is

$$\lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) = 0. \quad (25)$$

By Proposition 15(2), we conclude that

$$\lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x_n, x_{n+1}) = 0. \quad (26)$$

Step II. Now, we prove that the sequence $\{x_n\}$ is a G_{σ_b} -Cauchy sequence. For this purpose, we will show that

$$\limsup_{m, n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) = 0. \quad (27)$$

Using the rectangular inequality with (21) (as $\alpha(x_n, x_m, x_m) \geq 1$, since f is a rectangular G - α -admissible mapping), we have

$$\begin{aligned} G_{\sigma_b}(x_n, x_m, x_m) &\leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) \\ &\quad + s^2 G_{\sigma_b}(x_{n+1}, x_{m+1}, x_{m+1}) + s^2 G_{\sigma_b}(x_{m+1}, x_m, x_m) \\ &\leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) \\ &\quad + s\beta(G_{\sigma_b}(x_n, x_m, x_m)) M_s(x_n, x_m, x_m) \\ &\quad + s^2 G_{\sigma_b}(x_{m+1}, x_m, x_m). \end{aligned} \quad (28)$$

Taking limit as $m, n \rightarrow \infty$ in the above inequality and applying (25) and (26), we have

$$\begin{aligned} \limsup_{m, n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) &\leq s \limsup_{n, m \rightarrow \infty} \beta(G_{\sigma_b}(x_n, x_m, x_m)) \limsup_{n, m \rightarrow \infty} M_s(x_n, x_m, x_m). \end{aligned} \quad (29)$$

Here,

$$\begin{aligned} G_{\sigma_b}(x_n, x_m, x_m) &\leq M_s(x_n, x_m, x_m) \\ &= \max \left\{ G_{\sigma_b}(x_n, x_m, x_m), \right. \\ &\quad \left. \frac{G_{\sigma_b}(x_n, fx_n, fx_n) [G_{\sigma_b}(x_m, fx_m, fx_m)]^2}{1 + s^2 G_{\sigma_b}^2(fx_n, fx_m, fx_m)} \right\} \\ &= \max \left\{ G_{\sigma_b}(x_n, x_m, x_m), \right. \\ &\quad \left. \frac{G_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) [G_{\sigma_b}(x_m, x_{m+1}, x_{m+1})]^2}{1 + s^2 G_{\sigma_b}^2(x_{n+1}, x_{m+1}, x_{m+1})} \right\}. \end{aligned} \quad (30)$$

Letting $m, n \rightarrow \infty$ in the above inequality, we get

$$\limsup_{m,n \rightarrow \infty} M_s(x_n, x_m, x_m) = \limsup_{m,n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m). \quad (31)$$

Hence, from (29) and (31), we obtain

$$\begin{aligned} \limsup_{m,n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) \\ \leq \limsup_{m,n \rightarrow \infty} \beta(G_{\sigma_b}(x_n, x_m, x_m)) \limsup_{m,n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m). \end{aligned} \quad (32)$$

If $\limsup_{m,n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) \neq 0$, then we get

$$\frac{1}{s} \leq \limsup_{m,n \rightarrow \infty} \beta(G_{\sigma_b}(x_n, x_m, x_m)). \quad (33)$$

Since $\beta \in \mathcal{F}$, we deduce that

$$\limsup_{m,n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) = 0, \quad (34)$$

which is a contradiction. Consequently, $\{x_n\}$ is a G_{σ_b} -Cauchy sequence in X . Since (X, G_{σ_b}) is G_{σ_b} -complete, there exists $u \in X$ such that $x_n \rightarrow u$, as $n \rightarrow \infty$. Now, from (34) and G_{σ_b} -completeness of X ,

$$\begin{aligned} \lim_{n \rightarrow \infty} G_{\sigma_b}(u, x_n, x_n) &= \lim_{m,n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) \\ &= G_{\sigma_b}(u, u, u) = 0. \end{aligned} \quad (35)$$

Step III. Now, we show that u is a fixed point of f .

Using the rectangle inequality, we get

$$G_{\sigma_b}(u, fu, fu) \leq sG_{\sigma_b}(u, fx_n, fx_n) + sG_{\sigma_b}(fx_n, fu, fu). \quad (36)$$

Letting $n \rightarrow \infty$ and using the continuity of f and (35), we obtain

$$\begin{aligned} G_{\sigma_b}(u, fu, fu) &\leq s \lim_{n \rightarrow \infty} G_{\sigma_b}(u, fx_n, fx_n) \\ &\quad + s \lim_{n \rightarrow \infty} G_{\sigma_b}(fx_n, fu, fu) \\ &= sG_{\sigma_b}(fu, fu, fu). \end{aligned} \quad (37)$$

Note that, from (21), as $\alpha(u, u, u) \geq 1$, we have

$$sG_{\sigma_b}(fu, fu, fu) \leq \beta(G_{\sigma_b}(u, u, u)) M_s(u, u, u), \quad (38)$$

where, by (37),

$$\begin{aligned} M_s(u, u, u) &= \max \left\{ G_{\sigma_b}(u, u, u), \right. \\ &\quad \left. \frac{G_{\sigma_b}(u, fu, fu) G_{\sigma_b}(u, fu, fu) G_{\sigma_b}(u, fu, fu)}{1 + s^2 G_{\sigma_b}^2(fu, fu, fu)} \right\} \\ &< G_{\sigma_b}(u, fu, fu). \end{aligned} \quad (39)$$

Hence, as $\beta(t) \leq 1$ for all $t \in [0, \infty)$, we have $sG_{\sigma_b}(fu, fu, fu) \leq G_{\sigma_b}(u, fu, fu)$. Thus, by (37), we obtain that $sG_{\sigma_b}(fu, fu, fu) = G_{\sigma_b}(u, fu, fu)$. But then, using (38), we get that

$$\begin{aligned} G_{\sigma_b}(u, fu, fu) &= sG_{\sigma_b}(fu, fu, fu) \\ &\leq \beta(G_{\sigma_b}(u, u, u)) M_s(u, u, u) \\ &< \frac{1}{s} G_{\sigma_b}(u, fu, fu), \end{aligned} \quad (40)$$

which is a contradiction. Hence, we have $fu = u$. Thus, u is a fixed point of f . $\square$

We replace condition (ii) in Theorem 26 by regularity of the space X .

Theorem 27. Under the same hypotheses of Theorem 26, instead of condition (ii), assume that whenever $\{x_n\}$ in X is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, one has $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then, f has a fixed point.

Proof. Repeating the proof of Theorem 26, we can construct a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow u \in X$ for some $u \in X$. Using the assumption on X , we have $\alpha(x_n, u, u) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we show that $u = fu$. By Lemma 21 and (35),

$$\begin{aligned} &s \left[\frac{1}{s} G_{\sigma_b}(u, fu, fu) - G_{\sigma_b}(u, u, u) \right] \\ &\leq \limsup_{n \rightarrow \infty} G_{\sigma_b}(x_{n+1}, fu, fu) \\ &\leq \limsup_{n \rightarrow \infty} \left(\beta(G_{\sigma_b}(x_n, u, u)) M_s(x_n, u, u) \right) \\ &\leq \frac{1}{s} \limsup_{n \rightarrow \infty} M_s(x_n, u, u), \end{aligned} \quad (41)$$

where

$$\begin{aligned} &\lim_{n \rightarrow \infty} M_s(x_n, u, u) \\ &= \lim_{n \rightarrow \infty} \max \left\{ G_{\sigma_b}(x_n, u, u), \right. \\ &\quad \left. \frac{[G_{\sigma_b}(x_n, fx_n, fx_n)] [G_{\sigma_b}(u, fu, fu)]^2}{1 + s^2 G_{\sigma_b}^2(fx_n, fu, fu)} \right\} \\ &= \lim_{n \rightarrow \infty} \max \left\{ G_{\sigma_b}(x_n, u, u), \right. \\ &\quad \left. \frac{[G_{\sigma_b}(x_n, x_{n+1}, x_{n+1})] [G_{\sigma_b}(u, fu, fu)]^2}{1 + s^2 G_{\sigma_b}^2(x_{n+1}, fu, fu)} \right\} \\ &= \max \{ G_{\sigma_b}(u, u, u), 0 \} = 0 \text{ (see (25) and (35))}. \end{aligned} \quad (42)$$

Therefore, we deduce that $G_{\sigma_b}(u, fu, fu) \leq 0$. Hence, we have $u = fu$. $\square$

A mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \rightarrow 0$, as $n \rightarrow \infty$ for any $t \in [0, \infty)$ (see, e.g., [23, 24] for more details and examples).

Definition 28. A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be a (c)-comparison function if

- (c₁) φ is increasing,
- (c₂) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$ for $k \geq k_0$ and any $t \in [0, \infty)$.

Later, Berinde [5] introduced the notion of (b)-comparison function as a generalization of (c)-comparison function.

Definition 29 (see [5]). Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled:

- (1) φ is increasing;
- (2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$ for $k \geq k_0$ and any $t \in [0, \infty)$.

Let Ψ_b be the class of all (b)-comparison functions $\varphi : [0, \infty) \rightarrow [0, \infty)$. It is clear that the notion of (b)-comparison function coincides with (c)-comparison function for $s = 1$.

Lemma 30 (see [25]). If $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a (b)-comparison function, then we have the following:

- (1) the series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;
- (2) the function $b_s : [0, \infty) \rightarrow [0, \infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t)$, $t \in [0, \infty)$, is increasing and continuous at 0.

Remark 31. It is easy to see that if $\varphi \in \Psi_b$, then we have $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$ and φ is continuous at 0.

In the next example, we present a class of (b)-comparison functions.

Example 32. Any function of the form $\psi(t) = \ln((a/s)t + 1)$ for all $t \in [0, \infty)$ where $0 < a < 1$ is a (b)-comparison function.

Proof. From the part (1) of Lemma 30, the necessary condition is that the series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$. But, for each $t > 0$ and $k \geq 1$, we have

$$\begin{aligned} s^k \varphi^k(t) &= s^k \varphi(\varphi^{k-1}(t)) \\ &= s^k \ln\left(\frac{a}{s} \varphi^{k-1}(t) + 1\right) \leq s^k \frac{a}{s} \varphi^{k-1}(t) \\ &\leq \dots \leq s^k \left(\frac{a}{s}\right)^k t = a^k t. \end{aligned} \quad (43)$$

So, according to the comparison test of the series, we should have $a < 1$. On the other hand, we have

$$\begin{aligned} s^{k+1} \varphi^{k+1}(t) &= s^{k+1} \varphi(\varphi^k(t)) \\ &= s^{k+1} \ln\left(\frac{a}{s} \varphi^k(t) + 1\right) \\ &\leq s^{k+1} \frac{a}{s} \varphi^k(t) = s^k a \varphi^k(t). \end{aligned} \quad (44)$$

Therefore, for any convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ and each $k \geq k_0 = 1$, we have

$$s^{k+1} \varphi^{k+1}(t) \leq as^k \varphi^k(t) \leq as^k \varphi^k(t) + v_k. \quad (45)$$

□

For example, for $s = 2$ and $a = 1/2$, the function $\psi(t) = \ln(t/4 + 1)$ is a (b)-comparison function.

Theorem 33. Let (X, G_{σ_b}) be a G_{σ_b} -complete generalized b-metric-like space and let $f : X \rightarrow X$ be a G - α -admissible mapping. Suppose that

$$s\alpha(x, y, z) G_{\sigma_b}(fx, fy, fz) \leq \psi(M_s(x, y, z)), \quad (46)$$

for all $x, y, z \in X$ where $\psi \in \Psi_b$ and

$$\begin{aligned} M_s(x, y, z) &= \max \left\{ G_{\sigma_b}(x, y, z), \right. \\ &\quad \left. \frac{G_{\sigma_b}(x, x, fx) G_{\sigma_b}(y, y, fy) G_{\sigma_b}(z, z, fz)}{1 + 2s^2 G_{\sigma_b}^2(fx, fy, fz)} \right\}. \end{aligned} \quad (47)$$

Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, f^2x_0) \geq 1$;
- (ii)
 - (a) f is continuous and, for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$, one has $\alpha(x, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;
 - (b) assume that whenever $\{x_n\}$ in X is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, one has $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, f^2x_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is a G - α -admissible mapping and $\alpha(x_0, x_1, x_2) = \alpha(x_0, fx_0, f^2x_0) \geq 1$, we deduce that $\alpha(x_1, x_2, x_3) = \alpha(fx_0, fx_1, fx_2) \geq 1$. Continuing this process, we get $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, for all $n \in \mathbb{N}$, we assume that $x_n \neq x_{n+1}$.

Step I (Cauchyess of $\{x_n\}$). As $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \geq 0$, using condition (46), we obtain

$$\begin{aligned} sG_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) &\leq s\alpha(x_{n-1}, x_n, x_{n+1})G_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) \\ &= s\alpha(x_{n-1}, x_n, x_{n+1})G_{\sigma_b}(fx_{n-1}, fx_n, fx_{n+1}) \\ &\leq \psi(M_s(x_{n-1}, x_n, x_{n+1})). \end{aligned} \quad (48)$$

Using Proposition 15(2) as $x_n \neq x_{n+1}$, we get

$$\begin{aligned} M_s(x_{n-1}, x_n, x_{n+1}) &= \max \left\{ G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\quad (G_{\sigma_b}(x_{n-1}, x_{n-1}, fx_{n-1}) \\ &\quad \times G_{\sigma_b}(x_n, x_n, fx_n) \\ &\quad \times G_{\sigma_b}(x_{n+1}, x_{n+1}, fx_{n+1})) \\ &\quad \times (1 + 2s^2 G_{\sigma_b}^2(fx_{n-1}, fx_n, fx_{n+1}))^{-1} \Big\} \\ &= \max \left\{ G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\quad (G_{\sigma_b}(x_{n-1}, x_{n-1}, x_n) \\ &\quad \times G_{\sigma_b}(x_n, x_n, x_{n+1}) \\ &\quad \times G_{\sigma_b}(x_{n+1}, x_{n+1}, x_{n+2})) \\ &\quad \times (1 + 2s^2 G_{\sigma_b}^2(x_n, x_{n+1}, x_{n+2}))^{-1} \Big\} \\ &\leq \max \left\{ G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\quad \left. \frac{2sG_{\sigma_b}(x_{n-1}, x_n, x_{n+1})G_{\sigma_b}^2(x_n, x_{n+1}, x_{n+2})}{1 + 2s^2 G_{\sigma_b}^2(x_n, x_{n+1}, x_{n+2})} \right\} \\ &= G_{\sigma_b}(x_{n-1}, x_n, x_{n+1}). \end{aligned} \quad (49)$$

Hence,

$$\begin{aligned} G_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) &\leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) \leq \psi(G_{\sigma_b}(x_{n-1}, x_n, x_{n+1})). \end{aligned} \quad (50)$$

By induction, since $x_n \neq x_{n+1}$, we get that

$$\begin{aligned} G_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) &\leq \psi(G_{\sigma_b}(x_{n-1}, x_n, x_{n+1})) \\ &\leq \psi^2(G_{\sigma_b}(x_{n-2}, x_{n-1}, x_n)) \\ &\leq \dots \leq \psi^n(G_{\sigma_b}(x_0, x_1, x_2)). \end{aligned} \quad (51)$$

Let $\varepsilon > 0$ be arbitrary. Then, there exists a natural number N such that

$$\sum_{n=N}^{\infty} s^n \psi^n(G_{\sigma_b}(x_0, x_1, x_2)) < \frac{\varepsilon}{2s}. \quad (52)$$

Let $m > n \geq N$. Then, by the rectangular inequality and Proposition 15(2) as $x_n \neq x_{n+1}$, we get

$$\begin{aligned} G_{\sigma_b}(x_n, x_m, x_m) &\leq sG_{\sigma_b}(x_n, x_{n+1}, x_{n+1}) + s^2 G_{\sigma_b}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \dots + s^{m-n-1} G_{\sigma_b}(x_{m-1}, x_m, x_m) \\ &\leq 2s(sG_{\sigma_b}(x_n, x_n, x_{n+1}) + s^2 G_{\sigma_b}(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad + \dots + s^{m-n-1} G_{\sigma_b}(x_{m-1}, x_{m-1}, x_m)) \\ &\leq 2s(sG_{\sigma_b}(x_n, x_{n+1}, x_{n+2}) + s^2 G_{\sigma_b}(x_{n+1}, x_{n+2}, x_{n+3}) \\ &\quad + \dots + s^{m-n-1} G_{\sigma_b}(x_{m-1}, x_m, x_{m+1})) \\ &\leq 2s \sum_{k=n}^{m-2} s^{k-n+1} \psi^k(G_{\sigma_b}(x_0, x_1, x_2)) \\ &\leq 2s \sum_{k=n}^{\infty} s^k \psi^k(G_{\sigma_b}(x_0, x_1, x_2)) < \varepsilon. \end{aligned} \quad (53)$$

Consequently, $\{x_n\}$ is a G_{σ_b} -Cauchy sequence in X . Since (X, G_{σ_b}) is G_{σ_b} -complete, so there exists $u \in X$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} G_{\sigma_b}(u, x_n, x_n) &= \lim_{m, n \rightarrow \infty} G_{\sigma_b}(x_n, x_m, x_m) \\ &= G_{\sigma_b}(u, u, u) = 0. \end{aligned} \quad (54)$$

Step II. Now, we show that u is a fixed point of f . Suppose to the contrary, that is, $fu \neq u$, then, we have $G_{\sigma_b}(u, u, fu) > 0$.

Let the part (a) of (ii) holds.

Using the rectangle inequality, we get

$$G_{\sigma_b}(u, u, fu) \leq sG_{\sigma_b}(fu, fx_n, fx_n) + sG_{\sigma_b}(fx_n, u, u). \quad (55)$$

Letting $n \rightarrow \infty$ and using the continuity of f , we get

$$G_{\sigma_b}(u, u, fu) \leq sG_{\sigma_b}(fu, fu, fu). \quad (56)$$

From (46) and part (a) of condition (ii), we have

$$sG_{\sigma_b}(fu, fu, fu) \leq \psi(M_s(u, u, u)), \quad (57)$$

where, by using (56), we have

$$\begin{aligned} M_s(u, u, u) &= \max \left\{ G_{\sigma_b}(u, u, u), \right. \\ &\quad \left. \frac{G_{\sigma_b}(u, u, fu)G_{\sigma_b}(u, u, fu)G_{\sigma_b}(u, u, fu)}{1 + 2s^2 G_{\sigma_b}^2(fu, fu, fu)} \right\} \\ &\leq G_{\sigma_b}(u, u, fu). \end{aligned} \quad (58)$$

Hence, from properties of ψ , $sG_{\sigma_b}(fu, fu, fu) \leq G_{\sigma_b}(u, u, fu)$. Thus, by (56), we obtain that

$$G_{\sigma_b}(u, u, fu) = sG_{\sigma_b}(fu, fu, fu). \quad (59)$$

Moreover, (57) yields that $G_{\sigma_b}(u, u, fu) \leq \psi(M_s(u, u, u)) \leq \psi(G_{\sigma_b}(u, u, fu))$. This is impossible, according to our assumptions on ψ . Hence, we have $fu = u$. Thus, u is a fixed point of f .

Now, let part (b) of (ii) holds.

As $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $\alpha(x_n, x, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we show that $u = fu$. By (46), we have

$$sG_{\sigma_b}(fu, fu, x_n) \leq s\alpha(u, u, x_{n-1})G_{\sigma_b}(fu, fu, fx_{n-1}) \leq \psi(M_s(u, u, x_{n-1})), \quad (60)$$

where

$$M_s(u, u, x_{n-1}) = \max \left\{ G_{\sigma_b}(u, u, x_{n-1}), \frac{[G_{\sigma_b}(u, u, fu)]^2 G_{\sigma_b}(x_{n-1}, x_{n-1}, fx_{n-1})}{1 + [sG_{\sigma_b}(fu, fu, fx_{n-1})]^2} \right\}. \quad (61)$$

Letting $n \rightarrow \infty$ in the above inequality and using (54) and Lemma 21, we get

$$\lim_{n \rightarrow \infty} M_s(u, u, x_{n-1}) = 0. \quad (62)$$

Again, taking the upper limit as $n \rightarrow \infty$ in (60) and using (62) and Lemma 21, we obtain

$$\begin{aligned} s \left[\frac{1}{s} G_{\sigma_b}(u, fu, fu) - G_{\sigma_b}(u, u, u) \right] &\leq s \limsup_{n \rightarrow \infty} G_{\sigma_b}(x_n, fu, fu) \\ &\leq \limsup_{n \rightarrow \infty} \psi(M_s(u, u, x_{n-1})) \\ &= \psi \left(\limsup_{n \rightarrow \infty} M_s(u, u, x_{n-1}) \right) \\ &= \psi(0) = 0. \end{aligned} \quad (63)$$

So, we get $G_{\sigma_b}(u, fu, fu) = 0$. That is, $fu = u$. $\square$

Let $(X, G_{\sigma_b}, \preceq)$ be a partially ordered G_{σ_b} -metric-like space. We say that T is an increasing mapping on X if $\forall x, y \in X, x \preceq y \Rightarrow T(x) \preceq T(y)$ [26]. Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [27–30] and references therein). From the results proved above, we derive the following new results in partially ordered G_{σ_b} -metric-like space.

Theorem 34. Let $(X, G_{\sigma_b}, \preceq)$ be a partially ordered G_{σ_b} -complete generalized b -metric-like space and let $f : X \rightarrow X$ be an increasing mapping. Suppose that

$$sG_{\sigma_b}(fx, fy, fz) \leq \beta(G_{\sigma_b}(x, y, z))M_s(x, y, z), \quad (64)$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$, where

$$M_s(x, y, z) = \max \left\{ G_{\sigma_b}(x, y, z), \frac{G_{\sigma_b}(x, fx, fx)G_{\sigma_b}(y, fy, fy)G_{\sigma_b}(z, fz, fz)}{1 + s^2 G_{\sigma_b}^2(fx, fy, fz)} \right\}. \quad (65)$$

Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;
- (ii) f is continuous or assume that whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$, one has $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

Proof. Define $\alpha : X \times X \times X \rightarrow [0, +\infty)$ by

$$\alpha(x, y, z) = \begin{cases} 1, & \text{if } x \preceq y \preceq z \\ 0, & \text{otherwise.} \end{cases} \quad (66)$$

First, we prove that f is a triangular α -admissible mapping. Hence, we assume that $\alpha(x, y, z) \geq 1$. Therefore, we have $x \preceq y \preceq z$. Since f is increasing, we get $fx \preceq fy \preceq fz$; that is, $\alpha(fx, fy, fz) \geq 1$. Also, let $\alpha(x, y, y) \geq 1$ and $\alpha(y, z, z) \geq 1$; then $x \preceq y$ and $y \preceq z$. Consequently, we deduce that $x \preceq z$; that is, $\alpha(x, z, z) \geq 1$. Thus, f is a triangular α -admissible mapping. Since f satisfies (64) so, by the definition of α , we have

$$s\alpha(x, y, z)G_{\sigma_b}(fx, fy, fz) \leq \beta(G_{\sigma_b}(x, y, z))M_s(x, y, z) \quad (67)$$

for all $x, y, z \in X$. Therefore, f satisfies the contractive condition (21). From (i), there exists $x_0 \in X$ such that $x_0 \preceq fx_0$; that is, $\alpha(x_0, fx_0, fx_0) \geq 1$. According to (ii), we conclude that all the conditions of Theorems 26 and 27 are satisfied and so f has a fixed point. $\square$

Similarly, using Theorem 33, we can prove following result.

Theorem 35. Let $(X, G_{\sigma_b}, \preceq)$ be an ordered G_{σ_b} -complete generalized b -metric-like space and let $f : X \rightarrow X$ be an increasing mapping. Suppose that

$$sG_{\sigma_b}(fx, fy, fz) \leq \psi(M_s(x, y, z)), \quad (68)$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ where $\psi \in \Psi_b$ and

$$M_s(x, y, z) = \max \left\{ G_{\sigma_b}(x, y, z), \frac{G_{\sigma_b}(x, x, fx)G_{\sigma_b}(y, y, fy)G_{\sigma_b}(z, z, fz)}{1 + 2s^2 G_{\sigma_b}^2(fx, fy, fz)} \right\}. \quad (69)$$

Also, suppose that the following assertions hold:

- (i) there exists $x_0 \in X$ such that $x_0 \leq fx_0$;
- (ii)

- (a) f is continuous;
- (b) assume that whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$, one has $x_n \leq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then, f has a fixed point.

We conclude this section by presenting some examples that illustrate our results.

Example 36. Let $X = [0, (-1 + \sqrt{5})/2]$ be endowed with the usual ordering on $\mathbb{R}$. Define the generalized b -metric-like function G_{σ_b} given by

$$G_{\sigma_b}(x, y, z) = \max\{(x+y)^2, (y+z)^2, (z+x)^2\} \quad (70)$$

with $s = 2$. Consider the mapping $f : X \rightarrow X$ defined by $f(x) = (1/4)x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}$ and the function $\beta \in \mathcal{F}$ given by $\beta(t) = (1/2)e^{-t}, t > 0$, and $\beta(0) \in [0, 1/2]$. It is easy to see that f is an increasing function on X . We show that f is G_{σ_b} -continuous on X . By Proposition 20, it is sufficient to show that f is G_{σ_b} -sequentially continuous on X . Let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x, x) = G_{\sigma_b}(x, x, x)$, so we have $\max\{(\lim_{n \rightarrow \infty} x_n + x)^2, 4x^2\} = 4x^2$ and, equivalently, $\lim_{n \rightarrow \infty} x_n = \alpha \leq x$. On the other hand, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} G_{\sigma_b}(fx_n, fx, fx) &= \lim_{n \rightarrow \infty} \max\{(fx_n + fx)^2, 4f^2x^2\} \\ &= \lim_{n \rightarrow \infty} \max\left\{\frac{1}{16}\left(x_n\sqrt{e^{-(x_n+(-1+\sqrt{5})/2)^2}} + x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2, \right. \\ &\quad \left.\frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\right\} \\ &= \max\left\{\frac{1}{16}\left(\lim_{n \rightarrow \infty} x_n\sqrt{e^{-(\lim_{n \rightarrow \infty} x_n+(-1+\sqrt{5})/2)^2}} + x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2, \right. \\ &\quad \left.\frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\right\} \\ &= \max\left\{\frac{1}{16}\left(\alpha\sqrt{e^{-(\alpha+(-1+\sqrt{5})/2)^2}} + x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2, \right. \\ &\quad \left.\frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2}\right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4}x^2e^{-(x+(-1+\sqrt{5})/2)^2} \\ &= G_{\sigma_b}(fx, fx, fx). \end{aligned}$$

(71)

So, f is G_{σ_b} -sequentially continuous on X .

For all elements $x, y, z \in X$, and the fact that $g(x) = x^2e^{-x^2}$ is an increasing function on X , we have

$$\begin{aligned} sG_{\sigma_b}(fx, fy, fz) &= 2 \max\{(fx + fy)^2, (fy + fz)^2, (fz + fx)^2\} \\ &= 2 \max\left\{\left(\frac{1}{4}x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}} + \frac{1}{4}y\sqrt{e^{-(y+(-1+\sqrt{5})/2)^2}}\right)^2, \right. \\ &\quad \left(\frac{1}{4}y\sqrt{e^{-(y+(-1+\sqrt{5})/2)^2}} + \frac{1}{4}z\sqrt{e^{-(z+(-1+\sqrt{5})/2)^2}}\right)^2, \\ &\quad \left(\frac{1}{4}z\sqrt{e^{-(z+(-1+\sqrt{5})/2)^2}} + \frac{1}{4}x\sqrt{e^{-(x+(-1+\sqrt{5})/2)^2}}\right)^2\right\} \\ &\leq 2 \max\left\{\left(\frac{1}{4}x\sqrt{e^{-(x+y)^2}} + \frac{1}{4}y\sqrt{e^{-(y+x)^2}}\right)^2, \right. \\ &\quad \left(\frac{1}{4}y\sqrt{e^{-(y+z)^2}} + \frac{1}{4}z\sqrt{e^{-(z+y)^2}}\right)^2, \\ &\quad \left(\frac{1}{4}z\sqrt{e^{-(z+x)^2}} + \frac{1}{4}x\sqrt{e^{-(x+z)^2}}\right)^2\right\} \\ &= \frac{1}{8} \max\{(x+y)^2e^{-(x+y)^2}, (y+z)^2e^{-(y+z)^2}, \\ &\quad (z+x)^2e^{-(z+x)^2}\} \\ &= \frac{1}{8}e^{-\max\{(x+y)^2, (y+z)^2, (z+x)^2\}} \\ &\quad \times \max\{(x+y)^2, (y+z)^2, (z+x)^2\} \\ &\leq \frac{1}{2}e^{-\max\{(x+y)^2, (y+z)^2, (z+x)^2\}} \\ &\quad \times \max\{(x+y)^2, (y+z)^2, (z+x)^2\} \\ &= \beta(G_{\sigma_b}(x, y, z))G_{\sigma_b}(x, y, z) \\ &\leq \beta(G_{\sigma_b}(x, y, z))M_s(x, y, z). \end{aligned} \quad (72)$$

Hence, f satisfies all the assumptions of Theorem 34 and thus it has a fixed point (which is $u = 0$).

Example 37. Let $X = [0, 1]$ with the usual ordering on $\mathbb{R}$. Define the generalized b -metric-like function G_{σ_b} given by $G_{\sigma_b}(x, y, z) = \max\{(x+y)^2, (y+z)^2, (z+x)^2\}$ with $s = 2$. Consider the mapping $f : X \rightarrow X$ defined by $f(x) =$

$\ln(1 + xe^{-x}/4)$ and the function $\psi \in \Psi_b$ given by $\psi(t) = (1/8)t$, $t \geq 0$. It is easy to see that f is increasing function. Now, we show that f is a G_{σ_b} -continuous function on X .

Let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} G_{\sigma_b}(x_n, x, x) = G_{\sigma_b}(x, x, x)$, so we have $\max\{(\lim_{n \rightarrow \infty} x_n + x)^2, 4x^2\} = 4x^2$ and, equally, $\lim_{n \rightarrow \infty} x_n = \alpha \leq x$. On the other hand, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} G_{\sigma_b}(fx_n, fx, fx) \\ &= \lim_{n \rightarrow \infty} \max\left\{(fx_n + fx)^2, 4f^2x\right\} \\ &= \max\left\{\left(\ln\left(1 + \frac{\lim_{n \rightarrow \infty} x_n e^{-\lim_{n \rightarrow \infty} x_n}}{4}\right) + \ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2, \right. \\ & \quad \left. 4\left(\ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\right\} \\ &= \max\left\{\left(\ln\left(1 + \frac{\alpha e^{-\alpha}}{4}\right) + \ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2, \right. \\ & \quad \left. 4\left(\ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\right\} \\ &= 4\left(\ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2 \\ &= G_{\sigma_b}(fx, fx, fx). \end{aligned} \quad (73)$$

So, f is G_{σ_b} -sequentially continuous on X .

For all elements $x, y, z \in X$, we have

$$\begin{aligned} & sG_{\sigma_b}(fx, fy, fz) \\ &= 2 \max\left\{\left(\ln\left(1 + \frac{xe^{-x}}{4}\right) + \ln\left(1 + \frac{ye^{-y}}{4}\right)\right)^2, \right. \\ & \quad \left(\ln\left(1 + \frac{ye^{-y}}{4}\right) + \ln\left(1 + \frac{ze^{-z}}{4}\right)\right)^2, \\ & \quad \left.\left(\ln\left(1 + \frac{ze^{-z}}{4}\right) + \ln\left(1 + \frac{xe^{-x}}{4}\right)\right)^2\right\} \\ &\leq 2 \max\left\{\left(\frac{xe^{-x}}{4} + \frac{ye^{-y}}{4}\right)^2, \left(\frac{ye^{-y}}{4} + \frac{ze^{-z}}{4}\right)^2, \right. \\ & \quad \left.\left(\frac{ze^{-z}}{4} + \frac{xe^{-x}}{4}\right)^2\right\} \\ &\leq \frac{1}{8} \max\{(x+y)^2, (y+z)^2, (z+x)^2\} \\ &= \psi(G_{\sigma_b}(x, y, z)) \\ &\leq \psi(M_s(x, y, z)). \end{aligned} \quad (74)$$

Hence, f satisfies all the assumptions of Theorem 35 and thus it has a fixed point (which is $u = 0$).

3. Application

In this section, we present an application of our results to establish the existence of a solution to a periodic boundary value problem (see [30, 31]).

Let $X = C([0, T])$ be the set of all real continuous functions on $[0, T]$. We first endow X with the b -metric-like

$$\sigma_b(u, v) = \sup_{t \in [0, T]} (|u(t)| + |v(t)|)^p \quad (75)$$

for all $u, v \in X$ where $p > 1$ and then we endow it with the generalized b -metric-like G_{σ_b} defined by

$$\begin{aligned} G_{\sigma_b}(u, v, w) &= \max\left\{\sup_{t \in [0, T]} (|u(t)| + |v(t)|)^p, \right. \\ & \quad \sup_{t \in [0, T]} (|v(t)| + |w(t)|)^p, \\ & \quad \left. \sup_{t \in [0, T]} (|w(t)| + |u(t)|)^p\right\}. \end{aligned} \quad (76)$$

Clearly, (X, G_{σ_b}) is a complete generalized b -metric-like space with parameter $K = 2^{p-1}$. We equip $C([0, T])$ with a partial order given by

$$x \leq y \quad \text{iff } x(t) \leq y(t) \quad \forall t \in [0, T]. \quad (77)$$

Moreover, as in [30], it is proved that $(C([0, T]), \leq)$ is regular; that is, whenever $\{x_n\}$ in X is an increasing sequence such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $x_n \leq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Consider the first-order periodic boundary value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x(T), \quad (78)$$

where $t \in I = [0, T]$, with $T > 0$, and $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

A lower solution for (78) is a function $\alpha \in C^1[0, T]$ such that

$$\begin{aligned} \alpha'(t) &\leq f(t, \alpha(t)), \\ \alpha(0) &\leq \alpha(T), \end{aligned} \quad (79)$$

where $t \in I = [0, T]$.

Assume that there exists $\lambda > 0$ such that, for all $x, y \in C[0, T]$, we have

$$\begin{aligned} & |f(t, x(t)) + \lambda x(t)| + |f(t, y(t)) + \lambda y(t)| \\ & \leq \frac{\lambda}{2^{p-1}} \sqrt[p]{\ln\left(\frac{a}{2^{p-1}}(|x(t)| + |y(t)|)^p + 1\right)}, \end{aligned} \quad (80)$$

where $0 \leq a < 1$. Then, the existence of a lower solution for (78) provides the existence of a solution of (78).

Problem (78) can be rewritten as

$$\begin{aligned} x'(t) + \lambda x(t) &= f(t, x(t)) + \lambda x(t), \\ x(0) &= x(T). \end{aligned} \quad (81)$$

Consider

$$\begin{aligned} x'(t) + \lambda x(t) &= \delta_x(t) = F(t, x(t)), \\ x(0) &= x(T), \end{aligned} \quad (82)$$

where $t \in I$.

Using variation of parameters formula, we get

$$x(t) = x(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \delta_x(s) ds \quad (83)$$

which yields

$$x(T) = x(0)e^{-\lambda T} + \int_0^T e^{-\lambda(T-s)} \delta_x(s) ds. \quad (84)$$

Since $x(0) = x(T)$, we get

$$x(0) [1 - e^{-\lambda T}] = e^{-\lambda T} \int_0^T e^{\lambda s} \delta_x(s) ds \quad (85)$$

or

$$x(0) = \frac{1}{e^{\lambda T} - 1} \int_0^T e^{\lambda s} \delta_x(s) ds. \quad (86)$$

Substituting the value of $x(0)$ in (83), we arrive at

$$x(t) = \int_0^T G(t, s) \delta_x(s) ds \quad (87)$$

where

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s \leq t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t \leq s \leq T. \end{cases} \quad (88)$$

Now, define the operator $S : C[0, T] \rightarrow C[0, T]$ as

$$Sx(t) = \int_0^T G(t, s) F(s, x(s)) ds. \quad (89)$$

The mapping S is increasing [31]. Note that if $u \in C[0, T]$ is a fixed point of S , then $u \in C^1[0, T]$ is a solution of (78).

Let $x, y, z \in C[0, T]$. Then, we have

$$\begin{aligned} & 2^{p-1} [|Sx(t)| + |Sy(t)|] \\ &= 2^{p-1} \left[\left| \int_0^T G(t, s) F(s, x(s)) ds \right| \right. \\ & \quad \left. + \left| \int_0^T G(t, s) F(s, y(s)) ds \right| \right] \\ &\leq 2^{p-1} \int_0^T |G(t, s)| \\ & \quad \times [|F(s, x(s))| + |F(s, y(s))|] ds \\ &\leq 2^{p-1} \int_0^T |G(t, s)| \frac{\lambda}{2^{p-1}} \\ & \quad \times \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} (|x(t)| + |y(t)|)^p + 1 \right)} ds \end{aligned}$$

$$\begin{aligned} &\leq \lambda \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} \sigma_b(x, y) + 1 \right)} \\ & \quad \times \left[\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right] \\ &= \lambda \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} \sigma_b(x, y) + 1 \right)} \\ & \quad \times \left[\frac{1}{\lambda (e^{\lambda T} - 1)} \left(e^{\lambda(T+s-t)} \Big|_0^t + e^{\lambda(s-t)} \Big|_t^T \right) \right] \\ &= \lambda \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} \sigma_b(x, y) + 1 \right)} \\ & \quad \times \left[\frac{1}{\lambda (e^{\lambda T} - 1)} \left(e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1 \right) \right] \\ &= \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} \sigma_b(x, y) + 1 \right)} \\ &\leq \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} G_{\sigma_b}(x, y, z) + 1 \right)} \\ &\leq \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right)}. \end{aligned} \quad (90)$$

Similarly,

$$\begin{aligned} 2^{p-1} |Sy(t)| + |Sz(t)| &\leq \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right)}, \\ 2^{p-1} |Sx(t)| + |Sz(t)| &\leq \sqrt[p]{\ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right)}, \end{aligned} \quad (91)$$

where

$$\begin{aligned} &M(x, y, z) \\ &= \max \left\{ G_{\sigma_b}(x, y, z), \right. \\ & \quad \left. \frac{G_{\sigma_b}(x, Sx, Sx) G_{\sigma_b}(y, Sy, Sy) G_{\sigma_b}(z, Sz, Sz)}{1 + 2^{2p-1} G_{\sigma_b}^2(Sx, Sy, Sz)} \right\}. \end{aligned} \quad (92)$$

Equivalently,

$$\begin{aligned} (2^{p-1} |Sx(t)| + |Sy(t)|)^p &\leq \ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right), \\ (2^{p-1} |Sy(t)| + |Sz(t)|)^p &\leq \ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right), \\ (2^{p-1} |Sx(t)| + |Sz(t)|)^p &\leq \ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right), \end{aligned} \quad (93)$$

which yields that

$$\begin{aligned} 2^{p-1} \sigma_b(Sx, Sy) &\leq \ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right), \\ 2^{p-1} \sigma_b(Sy, Sz) &\leq \ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right), \\ 2^{p-1} \sigma_b(Sx, Sz) &\leq \ln \left(\frac{a}{2^{p-1}} M(x, y, z) + 1 \right). \end{aligned} \quad (94)$$

Finally, it is easy to obtain that

$$2^{p-1}G_{\sigma_b}(Sx, Sy, Sz) \leq \ln\left(\frac{a}{2^{p-1}}M(x, y, z) + 1\right). \quad (95)$$

Finally, since α is a lower solution for (78), so it is easy to show that $\alpha \leq S(\alpha)$ [31].

Hence, the hypotheses of Theorem 35 are satisfied, with $\psi(t) = \ln((a/2^{p-1})t + 1)$ where $0 \leq a < 1$. Hence, there exists a fixed point $\hat{x} \in C[0, T]$ such that $S\hat{x} = \hat{x}$ which is a solution to periodic boundary value problem (78).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] S. Czerwik, "Contraction mappings in b -metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.
- [2] V. Parvaneh, J. R. Roshan, and S. Radenovic, "Existence of tripled coincidence points in ordered b -metric spaces and an application to a system of integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 130, 2013.
- [3] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, and W. Shatanawi, "Common fixed points of almost generalized contractive mappings in ordered b -metric spaces," *Fixed Point Theory and Applications*, vol. 2013, article 159, 2013.
- [4] H. Aydi, M. F. Bota, E. Karapinar, and S. Moradi, "A common fixed point for weak ϕ -contractions on b -metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.
- [5] V. Berinde, "Sequences of operators and fixed points in quasi-metric spaces," *Studia Universitatis "Babes-Bolyai", Mathematica*, vol. 16, no. 4, pp. 23–27, 1996.
- [6] N. Hussain, D. Đorić, Z. Kadelburg, and S. Radenović, "Suzuki-type fixed point results in metric type spaces," *Fixed Point Theory and Applications*, vol. 2012, article 126, 2012.
- [7] N. Hussain and M. H. Shah, "KKM mappings in cone b -metric spaces," *Computers and Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.
- [8] N. Hussain, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Fixed points of contractive mappings in b -metric-like spaces," *The Scientific World Journal*, vol. 2014, Article ID 471827, 15 pages, 2014.
- [9] M. A. Khamsi and N. Hussain, "KKM mappings in metric type spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 73, no. 9, pp. 3123–3129, 2010.
- [10] M. A. Khamsi, "Remarks on cone metric spaces and fixed point theorems of contractive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 315398, 7 pages, 2010.
- [11] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Some common fixed point results in ordered partial b -metric spaces," *Journal of Inequalities and Applications*, vol. 2013, article 562, 2013.
- [12] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.
- [13] A. Amini-Harandi, "Metric-like spaces, partial metric spaces and fixed points," *Fixed Point Theory and Applications*, vol. 2012, article 204, 2012.
- [14] N. Shobkolaei, S. Sedghi, J. R. Roshan, and N. Hussain, "Suzuki type fixed point results in metric-like spaces," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 143686, 9 pages, 2013.
- [15] M. A. Alghamdi, N. Hussain, and P. Salimi, "Fixed point and coupled fixed point theorems on b -metric-like spaces," *Journal of Inequalities and Applications*, vol. 2013, article 402, 2013.
- [16] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for $\alpha - \psi$ -contractive type mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.
- [17] P. Salimi and E. Karapinar, "Suzuki-Edelstein type contractions via auxiliary functions," *Mathematical Problems in Engineering*, vol. 2013, Article ID 648528, 8 pages, 2013.
- [18] P. Salimi, C. Vetro, and P. Vetro, "Fixed point theorems for twisted (α, β) - ψ -contractive type mappings and applications," *Filomat*, vol. 27, no. 4, pp. 605–615, 2013.
- [19] P. Salimi, C. Vetro, and P. Vetro, "Some new fixed point results in non-Archimedean fuzzy metric spaces," *Nonlinear Analysis: Modelling and Control*, vol. 18, no. 3, pp. 344–358, 2013.
- [20] N. Hussain, P. Salimi, and A. Latif, "Fixed point results for single and set-valued α - η - ψ -contractive mappings," *Fixed Point Theory and Applications*, vol. 2013, article 212, 2013.
- [21] M. A. Alghamdi and E. Karapinar, "G- β - ψ contractive-type mappings and related fixed point theorems," *Journal of Inequalities and Applications*, vol. 2013, article 70, 2013.
- [22] M. Geraghty, "On contractive mappings," *Proceedings of the American Mathematical Society*, vol. 40, pp. 604–608, 1973.
- [23] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.
- [24] N. Hussain, Z. Kadelburg, S. Radenovic, and F. R. Al-Solamy, "Comparison functions and fixed point results in partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 605781, 15 pages, 2012.
- [25] V. Berinde, "Generalized contractions in quasimetric spaces," in *Seminar on Fixed Point Theory*, Preprint no. 3, pp. 3–9, 1993.
- [26] L. Ćirić, M. Abbas, R. Saadati, and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 12, pp. 5784–5789, 2011.
- [27] R. P. Agarwal, N. Hussain, and M. A. Taoudi, "Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 15 pages, 2012.
- [28] N. Hussain, A. R. Khan, and R. P. Agarwal, "Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.
- [29] N. Hussain and M. A. Taoudi, "Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 196, 2013.

- [30] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.
- [31] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 7-8, pp. 3403–3410, 2009.