Research Article

Some New Type Sigma Convergent Sequence Spaces and Some New Inequalities

Kuddusi Kayaduman¹ and Mehmet Şengönül²

¹ Department of Mathematics, Faculty of Arts and Sciences, G. Antep University, 27310 Şehitkamil-Gaziantep, Turkey

² Department of Mathematics, Faculty of Arts and Sciences, Nevşehir Hacı Bektaş Veli University, 50300 Nevşehir, Turkey

Correspondence should be addressed to Mehmet Şengönül; msengonul@yahoo.com

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We have discussed some important problems about the spaces $\widetilde{V}_\sigma$ and $\widetilde{V}_{0\sigma}$ of Cesàro sigma convergent and Cesàro null sequence.

1. Introduction and Preliminaries

In the theory of the sequence spaces, by using the matrix domain of a particular limitation method, so many sequence spaces have been built and published in famous maths journals. By reviewing the literature, one can reach them easily (for instance, see Başar et al. [1], Kirişçi and Başar [2], Şengönül and Başar [3], Altay [4], Mohiuddine and Alotaibi [5], and numerous). As known, the method to obtain a new sequence space by using convergence field of an infinite matrix is an old method in the theory of sequence spaces. But, the study of convergence field of an infinite matrix in the space of σ -convergent sequences is quite new. For example, quite recently, Kayaduman and Şengönül introduced the spaces $\widetilde{f}$ and $\widetilde{f}_0$ consisting of the sequences $x = (x_k)$ such that $(\sum_{j=0}^k (x_j/(j+1))) \in f$, f_0 and gave some important results on those spaces in [6]. Furthermore, in [7], Şengönül and Kayaduman have introduced the spaces $\widehat{f}$ and $\widehat{f}_0$ consisting of the sequences $x = (x_k)$ such that $(\sum_{j=0}^k (r_j x_j / R_k)) \in f$, f_0 .

After here, we will pass to the preliminaries for our study. We will denote the space of all real or complex valued sequences by w and we will write ℓ_∞ , c , c_0 , ℓ_1 , cs , and bs for the spaces of all bounded, convergent, null sequences, absolutely convergent series, convergent series, and bounded series,

respectively. Each linear subspace of w is called a sequence space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbb{N} = \{0, 1, 2, \dots\}$. Then, we can say that A defines a matrix mapping from λ to μ , and we denote it by writing $A : \lambda \rightarrow \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, that is A -transform of x , in μ where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \quad (1)$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ . By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \rightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}. \quad (2)$$

If we take $\lambda = c$, then c_A is called convergence domain of A . We write the limit of Ax as $\lim_A x = \lim_{n \rightarrow \infty} \sum_k a_{nk} x_k$, and A is called regular if $\lim_A x = \lim x$ for each convergent sequence x .

The sets ℓ_∞ and c are Banach spaces with the norm $\|x_n\| = \sup_n |x_n|$. Let σ be a one-to-one mapping from $\mathbb{N}$ into itself.

A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ -mean if and only if

- (i) $\phi(x) \geq 0$ for all sequence $x = (x_n)$ with nonnegative terms,
- (ii) $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$,
- (iii) $\phi(x_{\sigma(k)}) = \phi(x)$ for all $x \in \ell_\infty$. Throughout this paper, we consider the mapping σ such that $\sigma^p(k) \neq k$ for all positive integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ is the p th iterate of σ at k . Thus, a σ -mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ (see [8]). Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ -means are equal; that is,

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_{p \rightarrow \infty} t_{pn}(x) = s \text{ uniformly in } n \right\}, \quad (3)$$

where

$$t_{pn}(x) = \frac{(x_n + Tx_n + \dots + T^p x_n)}{(p+1)}, \quad t_{-1,n}(x) = 0, \quad (4)$$

[9]. We say that a bounded sequence $x = (x_k)$ is σ -convergent if $x \in V_\sigma$. In case $\sigma(k) = k+1$, a σ -mean is often called a Banach limit and V_σ is reduced to the set of almost sequences, introduced by Lorentz (see [10]). If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. By Z , we denote the set of σ -convergent sequences with σ -limit zero. It is well known [11] that $x \in \ell_\infty$ if and only if $Tx - x \in Z$.

Let K be a subset of $\mathbb{N}$. The natural density δ of K is defined by $\delta(K) = \lim_{n \rightarrow \infty} (1/n) |\{k \leq n : k \in K\}|$, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every ε , $\delta(\{k : |x_k - l| \geq \varepsilon\}) = 0$ (see [12]). In this case, we write $st\text{-}\lim x = l$. We will also write st and st_0 to denote the sets of all statistically convergent sequences and statistically null sequences. Let us consider the following functionals defined on ℓ_∞ :

$$\begin{aligned} l(x) &= \liminf_{k \rightarrow \infty} x_k, & L(x) &= \limsup_{k \rightarrow \infty} x_k, \\ q_\sigma(x) &= \limsup_{p \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{p+1} \sum_{i=0}^p x_{\sigma^i(n)}, \\ L^*(x) &= \limsup_{p \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{p+1} \sum_{i=0}^p x_{n+i}. \end{aligned} \quad (5)$$

In [13], the σ -core of a real bounded sequence x is defined as the closed interval $[-q_\sigma(-x), q_\sigma(x)]$, and also the inequalities $q_\sigma(Ax) \leq L(x)$ (σ -core of $Ax \subseteq K$ -core of x), $q_\sigma(Ax) \leq q_\sigma(x)$ (σ -core of $Ax \subseteq \sigma$ -core of x), for all $x \in \ell_\infty$, have been studied. Here, the Knopp core, in short K -core of x , is the interval $[l(x), L(x)]$ (see [14]). When $\sigma(n) = n+1$, since $q_\sigma(x) = L^*(x)$, σ -core of x is reduced to the Banach core, in short B -core of x defined by the interval $[-L^*(-x), L^*(x)]$, (see [15]). The concepts of B -core and σ -core have been

studied by many authors [8, 15–19] and Fridy and Orhan [12] have introduced the notions of statistical boundedness, statistical limit superior (or briefly $st\text{-}\lim \sup$), and statistical limit inferior (or briefly $st\text{-}\lim \inf$), defined that the statistical core (or briefly st -core) of a statistically bounded sequence is the closed interval $[st\text{-}\lim \inf x, st\text{-}\lim \sup x]$, and also determined necessary and sufficient conditions for a matrix A to yield $K\text{-core}(Ax) \subseteq st\text{-core}(x)$ for all $x \in \ell_\infty$.

In this paper, we define the spaces of Cesàro sigma convergent and Cesàro null sequences and give some interesting theorems.

2. Some New Type Sigma Convergent Sequence Spaces

Definition 1. A bounded sequence $x = (x_k)$ is said to be Cesàro sigma convergent to the number α if and only if $\lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=0}^k (x_{\sigma^j(p)}) / ((n+1)(k+1)) = \alpha$ uniformly in p , and the set of all such sequences is denoted with $\tilde{V}_\sigma$. If $\alpha = 0$, then we write $\tilde{V}_{0\sigma}$ instead of $\tilde{V}_\sigma$; that is,

$$\begin{aligned} \tilde{V}_\sigma &= \left\{ x \in \ell_\infty : \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=0}^k \frac{x_{\sigma^j(p)}}{(n+1)(k+1)} = \alpha \text{ uniformly in } p \right\}, \\ \tilde{V}_{0\sigma} &= \left\{ x \in \ell_\infty : \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{j=0}^k \frac{x_{\sigma^j(p)}}{(n+1)(k+1)} = 0 \text{ uniformly in } p \right\}, \end{aligned} \quad (6)$$

where $\alpha = \sigma_C\text{-}\lim x$, respectively.

Definition 2. A bounded sequence $x = (x_k)$ is said to be Cesàro sigma bounded if and only if $\sup_n \left| \sum_{k=0}^n \sum_{j=0}^k (x_{\sigma^j(p)}) / ((n+1)(k+1)) \right| < \infty$, and the set of all such sequences is denoted with $\tilde{V}_\sigma^\infty$; that is,

$$\tilde{V}_\sigma^\infty = \left\{ x \in \ell_\infty : \sup_n \left| \sum_{k=0}^n \sum_{j=0}^k \frac{x_{\sigma^j(p)}}{(n+1)(k+1)} \right| < \infty \right\}. \quad (7)$$

With the notation of (2), we can write $\tilde{V}_\sigma = (V_\sigma)_C$ and $\tilde{V}_{0\sigma} = (V_{0\sigma})_C$, where C denotes the Cesàro matrix of one order. Define the sequence $y = (y_n)$, which will be frequently used as the C -transform of a sequence $x = (x_k)$; that is,

$$y_k = \sum_{i=0}^k \frac{x_i}{k+1} \quad \forall k \in \mathbb{N}. \quad (8)$$

Clearly, if we take $\sigma(p) = p+1$, then the spaces $\tilde{V}_\sigma$ and $\tilde{V}_{0\sigma}$ are reduced to the spaces $\tilde{f}$ and $\tilde{f}_0$, respectively. Also, we note that the inclusions $\tilde{V}_{0\sigma} \subset \tilde{V}_\sigma \subset \tilde{V}_\sigma^\infty$ hold.

Now, we begin with the following theorem.

Theorem 3. *The sequence spaces $\widetilde{V}_\sigma$ and $\widetilde{V}_{0\sigma}$ are linearly isomorphic to the spaces V_σ and $V_{0\sigma}$, respectively; that is, $\widetilde{V}_\sigma \cong V_\sigma$ and $\widetilde{V}_{0\sigma} \cong V_{0\sigma}$.*

Proof. We consider only the spaces $\widetilde{V}_\sigma$ and V_σ . In order to prove the fact $\widetilde{V}_\sigma \cong V_\sigma$, we should show the existence of a linear bijection between the spaces $\widetilde{V}_\sigma$ and V_σ . Consider the transformation of C defined with the notation of (8) from $\widetilde{V}_\sigma$ to V_σ by $x \mapsto y = Cx$. The linearity of C is clear. Further, it is trivial that $x = \theta = (0, 0, \dots)$ whenever $Cx = \theta$ and hence C is injective. Let $y \in V_\sigma$ and define the sequence x by

$$x_{\sigma^i(p)} = (i+1)y_{\sigma^i(p)} - iy_{\sigma^{i-1}(p)}, \quad (i \in \mathbb{N}). \quad (9)$$

Then, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{j=0}^n \sum_{i=0}^j \frac{x_{\sigma^i(p)}}{(n+1)(j+1)} \\ = \lim_{n \rightarrow \infty} \sum_{j=0}^n \sum_{i=0}^j \frac{(i+1)y_{\sigma^i(p)} - iy_{\sigma^{i-1}(p)}}{(n+1)(j+1)} \\ = \lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{y_{\sigma^j(p)}}{n+1} \text{ uniformly in } p, \end{aligned} \quad (10)$$

which shows that $x \in \widetilde{V}_\sigma$. Consequently, we see that C is surjective. Hence, C is linear bijection which therefore shows that the spaces $\widetilde{V}_\sigma$ and V_σ are linearly isomorphic, as desired. This completes the proof. The fact that the spaces $\widetilde{V}_{0\sigma}$ and $V_{0\sigma}$ are linearly isomorphic can also be proved by the similar way, so we omit it. $\square$

Remark 4 (see [20]). The spaces V_σ and $V_{0\sigma}$ are BK-spaces with the norm $\|\cdot\|_{\ell_\infty}$.

Theorem 5. *The sets $\widetilde{V}_\sigma^\infty$, $\widetilde{V}_\sigma$, and $\widetilde{V}_{0\sigma}$ are linear spaces with the coordinatewise addition and scalar multiplication which are BK-spaces with the norm*

$$\|x\|_{\widetilde{V}_\sigma} = \sup_n \left| \sum_{j=0}^n \sum_{i=0}^j \frac{x_{\sigma^i(p)}}{(n+1)(j+1)} \right|. \quad (11)$$

Proof. The proof of first part of the theorem is easy. We will prove second part of the theorem. Since (8) holds, the spaces V_σ and $V_{0\sigma}$ are BK-spaces with the norm $\|\cdot\|_{\ell_\infty}$, (see Remark 4), the matrix C is normal and Theorem 4.3.2 of Wilansky [21] gives the fact that the spaces $\widetilde{V}_\sigma$ and $\widetilde{V}_{0\sigma}$ are BK-spaces. $\square$

It is known that, if λ and μ are normed spaces, then the set $G(T) = \{(x, T(x)) : x \in \lambda\}$ is subspace of $\lambda \times \mu$ and the set $G(T)$ is normed space with the norm

$$\|(x, T(x))\|_{G(T)} = \|x\|_\lambda + \|y\|_\mu, \quad (12)$$

where T is a linear transformation from λ to μ .

Theorem 6. *The graph $G(C)$ of the transformation C is closed subspace in $\widetilde{V}_\sigma \times V_\sigma$.*

Proof. We know that the spaces $\widetilde{V}_\sigma$ and V_σ are Banach spaces (see, Theorem 5 and Remark 4) and the transformation C is linear and continuous from $\widetilde{V}_\sigma$ to V_σ . Let us suppose that the sequence $(x_n, C(x_n))$ is convergent to (x, y) in $G(C)$ for $x \in \widetilde{V}_\sigma$ and $y \in V_\sigma$. With this supposition, and from the equality

$$\|(x_n, C(x_n)) - (x, y)\|_{G(C)} = \|x_n - x\|_{\widetilde{V}_\sigma} + \|C(x_n) - y\|_{V_\sigma}, \quad (13)$$

we see that $x_n \rightarrow x$ and $C(x_n) \rightarrow y$ as $n \rightarrow \infty$. Also, since C is continuous and from the definition of the sequential continuous, we obtain that $Cx = y$ and this completes the proof. $\square$

3. Some Matrix Mappings Related to the Space $\widetilde{V}_\sigma$

Following Başar [22], we start with giving short knowledge on the dual summability methods of the new type.

Let us suppose that the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$ which are connected by the relation (8) to the sequences $z = (z_n)$ and $t = (t_n)$, respectively; that is,

$$\begin{aligned} z_n &= (Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}), \\ t_n &= (By)_n = \sum_k b_{nk} y_k, \quad (n \in \mathbb{N}). \end{aligned} \quad (14)$$

It is clear here that the method B is applied to the C -transform of the sequence $x = (x_k)$ while the method A is directly applied to the entries of the sequence $x = (x_k)$. So, the methods A and B are essentially different. Let us assume that the matrix product BC exists which is a much weaker assumption than the conditions on the matrix B belonging to any matrix class, in general. The methods A and B in (14) are called *dual summability methods of the new type* if z_n reduces to t_n (or t_n reduces to z_n) under the application of formal summation by parts. This leads us to the fact that BC exists and is equal to A and $(BC)x = B(Cx)$ formally holds, if one side exists. This statement is equivalent to the following relation between the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$:

$$\begin{aligned} a_{nk} &:= \sum_{j=k}^{\infty} \frac{b_{nj}}{j+1} \quad \text{or} \\ b_{nk} &:= (k+1)(a_{nk} - a_{n,k+1}) = (k+1)\Delta a_{nk}, \end{aligned} \quad (15)$$

for all $n, k \in \mathbb{N}$.

Lemma 7 (see [9]). $A \in (\ell_\infty, V_\sigma)$, if and only if

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \quad (16)$$

$$\sigma\text{-}\lim a_{nk} = \alpha_k \quad \text{for each } k, \quad (17)$$

$$\lim_p \sum_k \frac{1}{p+1} \left| \sum_{i=0}^p (a_{\sigma^i(n),k} - \alpha_k) \right| \text{ uniformly in } n. \quad (18)$$

Lemma 8 (see [9]). $A \in (c, V_\sigma)$, if and only if (16) and (17) hold, and

$$\sigma\text{-}\lim_n \sum_k a_{nk} = \alpha. \quad (19)$$

Lemma 9. $A \in (V_\sigma, V_\sigma)$, if and only if (16) and (17) hold, and

$$A(T - I) \in (m, V_\sigma), \quad (20)$$

where I is identity matrix.

Now, we give the following theorem concerning to the dual matrices of the new type.

Theorem 10. Suppose that the entries of the infinite matrices $D = (d_{nk})$ and $E = (e_{nk})$ are connected with the relation

$$e_{nk} = \sum_{j=0}^n \frac{d_{jk}}{n+1}, \quad (n, k \in \mathbb{N}), \quad (21)$$

and let μ be any given sequence space. Then, $D \in (\mu : \widetilde{V}_\sigma)$, if and only if $E \in (\mu : V_\sigma)$.

Proof. Let $x = (x_k) \in \mu$ and consider the following equality with (21):

$$\sum_{j=0}^n \sum_{k=0}^m \frac{d_{jk} x_{\sigma^k(p)}}{n+1} = \sum_{k=0}^m e_{nk} x_{\sigma^k(p)}; \quad (m, n, k \in \mathbb{N}), \quad (22)$$

which yields as $m \rightarrow \infty$ that $Dx \in \widetilde{V}_\sigma$, whenever $x \in \mu$, if and if only $Ex \in V_\sigma$, whenever $x \in \mu$. This step completes the proof. $\square$

If we take $\sigma(p) = p + 1$, then Theorem 10 is reduced to Theorem 5.2 of Kayaduman and Şengönül, [6].

Now, right here, we have stated two theorem which are natural consequences of the Lemma 7, Lemma 8, and Theorem 10.

Theorem 11. Let $A = (a_{nk})$ be an infinite matrix real or complex numbers. Then, $A = (a_{nk}) \in (V_\sigma : \widetilde{V}_\sigma)$ if and only if

- (1) $\sup_{n \in \mathbb{N}} \sum_k |\sum_{j=k}^\infty (a_{nj}/(j+1))| < \infty$,
- (2) $\sigma\text{-}\lim_{n \rightarrow \infty} \sum_k \sum_{j=k}^\infty (a_{nj}/(j+1)) = \alpha$,
- (3) $A(T - I) \in (\ell_\infty : \widetilde{V}_\sigma)$,

where I is the identity matrix.

Theorem 12. Let $A = (a_{nk})$ be an infinite matrix real or complex numbers. Then, $A = (a_{nk}) \in (\ell_\infty : \widetilde{V}_\sigma)$ if and only if

- (4) $\sup_{n \in \mathbb{N}} \sum_k |\sum_{j=k}^\infty (a_{nj}/(j+1))| < \infty$,
- (5) $\sigma\text{-}\lim_{n \rightarrow \infty} \sum_{j=k}^\infty (a_{nj}/(j+1)) = \alpha_k$, exists for each fixed $k \in \mathbb{N}$,
- (6) $\lim_{m \rightarrow \infty} \sum_k |\sum_{i=0}^m \sum_{j=k}^\infty (a_{\sigma^i(n),j}/(m+1)(j+1)) - \alpha_k| = 0$, uniformly in n .

Proof. The proof is clear from Theorem 10 and Lemma 7. $\square$

Theorem 13. Let $A = (a_{nk})$ be an infinite matrix real or complex numbers. Then, $A = (a_{nk}) \in (c : \widetilde{V}_\sigma)$ if and only if

- (7) $\sup_{n \in \mathbb{N}} \sum_k |\sum_{j=k}^\infty (a_{nj}/(j+1))| < \infty$,
- (8) $\sigma\text{-}\lim_{n \rightarrow \infty} \sum_{j=k}^\infty (a_{nj}/(j+1)) = \alpha_k$ exists for each fixed $k \in \mathbb{N}$,
- (9) $\sigma\text{-}\lim_{n \rightarrow \infty} \sum_k \sum_{j=k}^\infty (a_{nj}/(j+1)) = \alpha$.

Proof. The proof is clear from Theorem 10 and Lemma 8. $\square$

Furthermore, from the Lemma 3.2 of [23], we have the following proposition.

Proposition 14. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation

$$a_{nk} := \sum_{j=k}^\infty \frac{b_{nj}}{j+1}, \quad (n, k \in \mathbb{N}), \quad (23)$$

and let μ be any given sequence space. Then, if $A \in (\widetilde{V}_\sigma : \mu)$, then

- (10) $\alpha_k = \lim_n \sum_{j=k}^\infty (b_{nj}/(j+1))$ exists for every $k \in \mathbb{N}$,
- (11) $\sup_n (\sum_k |\sum_{j=k}^\infty (b_{nj}/(j+1)) - \alpha_k|) < \infty$,
- (12) $\alpha = \sum_k |\alpha_k| < \infty$.

4. Some Inequalities

In this section, we use matrices classes $(\ell_\infty : \widetilde{V}_\sigma)$, $(c : \widetilde{V}_\sigma)$, and $(V_\sigma : \widetilde{V}_\sigma)$ to show the following inequalities: $\widetilde{q}_\sigma(Ax) \leq L(x)$, $\widetilde{q}_\sigma(Ax) \leq q_\sigma(x)$, and $\widetilde{q}_\sigma(Ax) \leq \beta(x)$, which are analogues of Knopp's core theorem.

Definition 15. Let $x \in \ell_\infty$. Then, σ_C -core of x is defined by the closed interval $[-\widetilde{q}_\sigma(-x), \widetilde{q}_\sigma(x)]$, where

$$\widetilde{q}_\sigma(x) = \limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_{k=0}^n \sum_{j=0}^k \frac{x_{\sigma^j(p)}}{(n+1)(k+1)}. \quad (24)$$

Therefore, it is easy to see that σ_C -core of x is α if and only if $\sigma_C\text{-}\lim x = \alpha$.

We need the following lemma due to Das [17] for the proof of next theorem.

Lemma 16. Let $\|C\| = \|c_{ni}(p)\| < \infty$ and let $\lim_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} |c_{ni}(p)| = 0$. Then, there is a $y = (y_i) \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$\limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i c_{ni}(p) y_i = \limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i |c_{ni}(p)|. \quad (25)$$

Lemma 17. Let P and Q be sublinear functionals on a linear space X . Then, $\{X, P\} \subset \{X, Q\}$ if and only if $P(x) \leq Q(x)$ for all $x \in X$.

Theorem 18. $\tilde{q}_\sigma(Ax) \leq L(x)$ for all $x \in \ell_\infty$ if and only if $A \in (c : \tilde{V}_\sigma)_{\text{reg}}$ and

$$\lim_{n \rightarrow \infty} \sum_i \bar{a}(n, i, p) = 1 \text{ uniformly in } p, \quad (26)$$

where $\bar{a}(n, i, p) = \sum_{k=0}^n \sum_{j=0}^k (a_{\sigma^j(p), i} / (n+1)(k+1))$ for all $n, i, p \in \mathbb{N}$.

Proof. Consider the following.

Sufficiency. Since $x \in \ell_\infty$, it is known that for any given $\epsilon > 0$, there exists a positive integer i_0 such that $x_i \leq L(x) + \epsilon$ whenever $i \geq i_0$. Now, let us write

$$\begin{aligned} \sum_i \bar{a}(n, i, p) x_i &= \sum_{i < i_0} \bar{a}(n, i, p) x_i + \sum_{i \geq i_0} \bar{a}(n, i, p)^+ x_i \\ &\quad - \sum_{i \geq i_0} \bar{a}(n, i, p)^- x_i. \end{aligned} \quad (27)$$

Then, by hypothesis, the first and the last sums on the right-hand side of (27) tend to zero, as $n \rightarrow \infty$, uniformly in p . Therefore, we obtain by applying the operator $\limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}}$ to the equality (27) that

$$\tilde{q}_\sigma(Ax) \leq [L(x) + \epsilon] \limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i \bar{a}(n, i, p). \quad (28)$$

Since (26) holds, we have $\tilde{q}_\sigma(Ax) \leq L(x)$.

Necessity. We observe by inserting $-x = (-x_i)$ in place of $x = (x_i)$ in the inequality $\tilde{q}_\sigma(Ax) \leq L(x)$ that

$$\ell(x) \leq -\tilde{q}_\sigma(-Ax) \leq \tilde{q}_\sigma(Ax) \leq L(x). \quad (29)$$

If $x \in c$, then $\ell(x) = L(x) = \lim x$ and so $-\tilde{q}_\sigma(-Ax) = \tilde{q}_\sigma(Ax)$ which means that $\sigma_C\text{-lim} Ax = \lim x$. Hence, $A \in (c : \tilde{V}_\sigma)_{\text{reg}}$. It is clear that

$$\liminf_{n \rightarrow \infty} \sum_i |\bar{a}(n, i, p)| \geq \liminf_{n \rightarrow \infty} \sum_i \bar{a}(n, i, p) = 1, \quad (30)$$

uniformly in p . On the other hand, if we choose the sequence of matrices $\mathcal{A} = \{\bar{a}_{in}(p)\}$ defined by $\bar{a}_{in}(p) = \bar{a}(n, i, p)$, for all $n, i, p \in \mathbb{N}$, then clearly $\mathcal{A}$ satisfies the conditions of Lemma 16. So, there exists a $y = (y_i) \in \ell_\infty$ with $\|y\| \leq 1$ and

$$\limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i |\bar{a}(n, i, p)| = \limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i \bar{a}(n, i, p) y_i. \quad (31)$$

Thus, we derive by inserting this y in the hypothesis that

$$\limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i |\bar{a}(n, i, p)| = \tilde{q}_\sigma(Ay) \leq L(y) \leq \|y\| \leq 1. \quad (32)$$

Combining this result by the fact in (30), we obtain the required condition. $\square$

In the special case $\sigma(p) = p + 1$, we also have the following.

Theorem 19 (see [6]). $B_{C\text{-core}}(Ax) \subseteq K\text{-core}(x)$ for all $x \in \ell_\infty$ if and only if $A \in (c : \tilde{f})_{\text{reg}}$ and

$$\lim_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i \frac{1}{n+1} \left| \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k a_{j+p, i} \right| = 1. \quad (33)$$

Corollary 20. $\tilde{q}_\sigma(Ax) \leq q_\sigma(x)$ for all $x \in \ell_\infty$ if and only if $A \in (V_\sigma : \tilde{V}_\sigma)_{\text{reg}}$ and (26) holds.

Proof. Consider the following.

Necessity. By the similar way used in the proof of necessity of Theorem 18, one can see that $A \in (V_\sigma : \tilde{V}_\sigma)_{\text{reg}}$. On the other hand, since $q_\sigma(x) \leq L(x)$ for any sequence x , condition (26) follows from Theorem 18.

Sufficiency. Suppose that $(V_\sigma : \tilde{V}_\sigma)_{\text{reg}}$ and (26) hold. Since $(V_\sigma : \tilde{V}_\sigma)_{\text{reg}}$ implies that $A \in (c : \tilde{V}_\sigma)_{\text{reg}}$, from Theorem 18,

$$u(x) = \inf_{z \in V_{0\sigma}} \tilde{q}_\sigma(Ax + z) \leq \inf_{z \in V_{0\sigma}} L(x + z) = v(x). \quad (34)$$

On the other hand, since $\tilde{q}_\sigma(Az) = -\tilde{q}_\sigma(-Az) = 0$ for $z \in V_{0\sigma}$, we have

$$u(x) \geq \inf_{z \in V_{0\sigma}} \{\tilde{q}_\sigma(Ax) + [-\tilde{q}_\sigma(-Az)]\} = \tilde{q}_\sigma(Ax). \quad (35)$$

Now, combining (34) and (35), we obtain $\tilde{q}_\sigma(Ax) \leq v(x)$ for all $x \in \ell_\infty$. Since $v(x) = q_\sigma(x)$ [13], the proof is completed. $\square$

In the special case $\sigma(p) = p + 1$, we also have the following theorem.

Theorem 21 (see [6]). One can see that $B_{C\text{-core}}(Ax) \subseteq B\text{-core}(x)$ for all $x \in \ell_\infty$ if and only if $A \in (f : \tilde{f})_{\text{reg}}$ and (33) holds.

Corollary 22. $A \in (st \cap \ell_\infty : \tilde{V}_\sigma)_{\text{reg}}$ if and only if $A \in (c : \tilde{V}_\sigma)_{\text{reg}}$ and

$$\lim_{n \rightarrow \infty} \sum_{i \in E} \frac{1}{n+1} \left| \sum_{k=0}^n \sum_{j=0}^k \frac{a_{\sigma^j(p), i}}{k+1} \right| = 0 \text{ uniformly in } p, \quad (36)$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

Proof. Let $A \in (st \cap \ell_\infty : \tilde{V}_\sigma)_{\text{reg}}$. Then, $A \in (c : \tilde{V}_\sigma)_{\text{reg}}$ immediately follows from the fact that $c \subset st \cap \ell_\infty$. Now, define a sequence $t = (t_i)$ via $x \in \ell_\infty$ as

$$t_i = \begin{cases} x_i, & i \in E, \\ 0, & i \notin E, \end{cases} \quad (37)$$

where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. Then, $st\text{-lim} t_i = 0$ and $t \in st_0$, so we have $At \in \tilde{V}_{0\sigma}$. On the other hand, since $(At)_n = \sum_{i \in E} a_{ni} t_i$, the matrix $B = (b_{ni})$, defined by

$$b_{ni} = \begin{cases} a_{ni}, & i \in E, \\ 0, & i \notin E, \end{cases} \quad (38)$$

for all n , must belong to the class $(\ell_\infty : \tilde{V}_{0\sigma})$. Hence, the necessity of (36) follows from (4) part of Theorem 12. Conversely, suppose that $A \in (c : \tilde{V}_\sigma)_{\text{reg}}$ and (36) holds. Let $x \in st \cap \ell_\infty$ and let $st\text{-}\lim x = \ell$. Write $E = \{i : |x_i - \ell| \geq \varepsilon\}$ for any given $\varepsilon > 0$, so that $\delta(E) = 0$. Since $A \in (c, \tilde{V}_\sigma)_{\text{reg}}$ and $\sigma_C\text{-}\lim_{n \rightarrow \infty} \sum_i a_{ni} = 1$, we have

$$\begin{aligned} \sigma_C\text{-}\lim(Ax) &= \sigma_C\text{-}\lim \left(\sum_i a_{ni} (x_i - \ell) + \ell \sum_i a_{ni} \right) \\ &= \sigma_C\text{-}\lim \left(\sum_i a_{ni} (x_i - \ell) + \ell \right) \\ &= \lim_{n \rightarrow \infty} \sup_{p \in \mathbb{N}} \sum_i \sum_{k=0}^n \sum_{j=0}^k \frac{a_{\sigma^j(p),i}}{(n+1)(k+1)} \\ &\quad \times (x_i - \ell) + \ell. \end{aligned} \quad (39)$$

On the other hand, since

$$\begin{aligned} &\left| \sum_i \sum_{k=0}^n \sum_{j=0}^k \frac{a_{\sigma^j(p),i}}{(n+1)(k+1)} (x_i - \ell) \right| \\ &\leq \|x\|_\infty \sum_{i \in E} \frac{1}{n+1} \left| \sum_{k=0}^n \sum_{j=0}^k \frac{a_{\sigma^j(p),i}}{(k+1)} \right| + \varepsilon \|A\|, \end{aligned} \quad (40)$$

the condition (36) implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_i \sum_{k=0}^n \sum_{j=0}^k \frac{a_{\sigma^j(p),i}}{(n+1)(k+1)} (x_k - \ell) &= 0 \\ &\text{uniformly in } p. \end{aligned} \quad (41)$$

Hence, $\sigma_C\text{-}\lim(Ax) = st\text{-}\lim x$; that is, $A \in (st \cap m : \tilde{V}_\sigma)_{\text{reg}}$, which completes the proof. $\square$

In the special case $\sigma(p) = p+1$, we also have the following theorem.

Theorem 23 (see [6]). Consider that $A \in (S \cap \ell_\infty : \tilde{f})_{\text{reg}}$ if and only if $A \in (c : \tilde{f})_{\text{reg}}$ and

$$\lim_{n \rightarrow \infty} \sum_{i \in E} \frac{1}{n+1} \left| \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k a_{j+p,i} \right| = 0 \quad \text{uniformly in } p, \quad (42)$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

Corollary 24. Consider that $\tilde{q}_\sigma(Ax) \leq \beta(x)$ for all $x \in \ell_\infty$ if and only if $A \in (st \cap \ell_\infty : \tilde{V}_\sigma)_{\text{reg}}$ and (26) holds.

Proof. Consider the following.

Necessity. Firstly, assume that $\tilde{q}_\sigma(Ax) \leq \beta(x)$ for all $x \in \ell_\infty$ where $\beta(x) = st\text{-}\lim \sup x$. Hence, since $\beta(x) = st\text{-}\lim \sup x \leq L(x)$ for all $x \in \ell_\infty$ (see [12]), we have (26)

from Theorem 18. Furthermore, one can also easily see that $-\beta(-x) \leq -\tilde{q}_\sigma(-Ax) \leq \tilde{q}_\sigma(Ax) \leq \beta(x)$; that is, $st\text{-}\lim \inf x \leq -\tilde{q}_\sigma(-Ax) \leq \tilde{q}_\sigma(Ax) \leq st\text{-}\lim \sup x$. If $x \in st \cap \ell_\infty$, then $st\text{-}\lim \inf x = st\text{-}\lim \sup x = st\text{-}\lim x$. Thus, the last inequality implies that $st\text{-}\lim x = -\tilde{q}_\sigma(-Ax) = \tilde{q}_\sigma(Ax) = \sigma_C\text{-}\lim(Ax)$; that is, $A \in (st \cap \ell_\infty : \tilde{V}_\sigma)_{\text{reg}}$.

Sufficiency. Let $A \in (st \cap \ell_\infty : \tilde{V}_\sigma)_{\text{reg}}$ and let (26) holds. If $x \in \ell_\infty$, then $\beta(x)$ is finite. Let E be a subset of $\mathbb{N}$ defined by $E = \{i : x_i > \beta(x) + \varepsilon\}$ for a given $\varepsilon > 0$. Then, it is obvious that $\delta(E) = 0$ and $x_i \leq \beta(x) + \varepsilon$ if $i \notin E$. For any real number λ , we write $\lambda^+ = \max\{\lambda, 0\}$ and $\lambda^- = \max\{-\lambda, 0\}$ whence $|\lambda| = \lambda^+ + \lambda^-$, $\lambda = \lambda^+ - \lambda^-$ and $|\lambda| - \lambda = 2\lambda^-$. Now, we can write

$$\begin{aligned} \sum_i c_{ni}(p) x_i &= \sum_{i < i_0} c_{ni}(p) x_i + \sum_{i \geq i_0} c_{ni}(p) x_i \\ &= \sum_{i < i_0} c_{ni}(p) x_i + \sum_{i \geq i_0} c_{ni}^+(p) x_i - \sum_{i \geq i_0} c_{ni}^-(p) x_i \\ &\leq \|x\| \sum_{i < i_0} |c_{ni}(p)| + \sum_{\substack{i \geq i_0 \\ i \notin E}} c_{ni}^+(p) x_i \\ &\quad + \sum_{\substack{i \geq i_0 \\ i \in E}} c_{ni}^+(p) x_i + \|x\| \sum_{i \geq i_0} [|c_{ni}(p)| - c_{ni}(p)] \\ &\leq \|x\| \sum_{i < i_0} |c_{ni}(p)| \\ &\quad + [\beta(x) + \varepsilon] \sum_{\substack{i \geq i_0 \\ i \notin E}} |c_{ni}(p)| + \|x\| \sum_{\substack{i \geq i_0 \\ i \in E}} |c_{ni}(p)| \\ &\quad + \|x\| \sum_{i \geq i_0} [|c_{ni}(p)| - c_{ni}(p)]. \end{aligned} \quad (43)$$

Applying the operator $\limsup_{n \rightarrow \infty} \sup_{p \in \mathbb{N}}$, we obtained from hypothesis that $\tilde{q}_\sigma(Ax) \leq \beta(x) + \varepsilon$. This completes the proof since ε is arbitrary. $\square$

In the special case $\sigma(p) = p+1$, we also have the following theorem.

Theorem 25 (see [6]). Consider that $B_{C\text{-core}}(Ax) \subseteq st\text{-core}(x)$ for all $x \in \ell_\infty$ if and only if $A \in (S \cap \ell_\infty : \tilde{f})_{\text{reg}}$ and (33) holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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