Research Article

The S-Transform of Distributions

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Parseval's formula and inversion formula for the S-transform are given. A relation between the S-transform and pseudodifferential operators is obtained. The S-transform is studied on the spaces $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

1. Introduction

The S-transform was first used by Stockwell et al. [1] in 1996. If $\omega(t, \xi)$ is a window function, then the continuous S-transform of $\phi(t)$ with respect to ω is defined as [2]

$$(S_\omega \phi)(\tau, \xi) = \int_{\mathbb{R}^n} \phi(t) \omega(\tau - t, \xi) e^{-i2\pi \langle t, \xi \rangle} dt, \quad (1)$$

for $x, \xi \in \mathbb{R}^n$.

In signal analysis, at least in dimension $n = 1$, $\mathbb{R}^{2n}$ is called the time-frequency plane, and in physics $\mathbb{R}^{2n}$ is called the phase space.

Equation (1) can be rewritten as a convolution as

$$(S_\omega \phi)(\tau, \xi) = (\phi(\cdot) e^{-i2\pi \langle \cdot, \xi \rangle} * \omega(\cdot, \xi))(\tau). \quad (2)$$

Now, we recall the definitions of Fourier transform on $\mathbb{R}^n$.

Definition 1. If $\phi(t)$ is defined on $\mathbb{R}^n$, then the Fourier transform of ϕ is given by

$$\mathcal{F}[\phi(t)](\xi) = \int_{\mathbb{R}^n} \phi(t) e^{-i2\pi \langle t, \xi \rangle} dt, \quad (3)$$

where $\langle t, \xi \rangle = \sum_{j=1}^n t_j \xi_j$ is the usual inner product on $\mathbb{R}^n$.

Definition 2. If $\phi(t, x)$ is defined on $\mathbb{R}^{2n}$, then the partial Fourier transform of $\phi(t, x)$ with respect to the first coordinate is given by

$$\mathcal{F}_1[\phi(t, x)](\xi, x) = \int_{\mathbb{R}^n} \phi(t, x) e^{-i2\pi \langle t, \xi \rangle} dt, \quad (4)$$

and the partial Fourier transform of $\phi(t, x)$ with respect to the second coordinate is given by

$$\mathcal{F}_2[\phi(t, x)](t, \eta) = \int_{\mathbb{R}^n} \phi(t, x) e^{-i2\pi \langle x, \eta \rangle} dx. \quad (5)$$

Applying the convolution property for the Fourier transform in (2), we obtain

$$(S_\omega \phi)(\tau, \xi) = \mathcal{F}_1^{-1}[\mathcal{F}_1(\phi)(\alpha + \xi) \mathcal{F}_1(\omega)(\alpha, \xi)](\tau, \xi), \quad (6)$$

where $\mathcal{F}_1^{-1}$ is the inverse Fourier transform.

Now, we define the translation, modulation, and involution operators, respectively, by

$$T_\tau \phi(t) = \phi(t - \tau) \quad (\text{translation})$$

$$M_\xi \phi(t) = e^{i2\pi \langle \xi, t \rangle} \phi(t) \quad (\text{modulation}) \quad (7)$$

$$\mathcal{I}\phi(t) = \overline{\phi(-t)} \quad (\text{involution}),$$

where $t, \tau, \xi \in \mathbb{R}^n$.

Definition 3 (the Dirac delta). The Dirac delta function is defined by

$$\iint_{\mathbb{R}^n} \phi(t, x) \delta(t, x) dt dx = \phi(0, 0). \quad (8)$$

Definition 4 (tempered distribution). A function $\phi \in C^\infty(\mathbb{R}^n)$ is said to be rapidly decreasing if

$$\gamma_{\alpha, \beta}(\phi) = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| < \infty, \quad (9)$$

for all pairs of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. The space of all rapidly decreasing functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$ or simply $\mathcal{S}$. Elements in the dual space $\mathcal{S}'$ of $\mathcal{S}$ are called tempered distribution.

2. Some Important Properties of S-Transform

Some properties of S-transform can be found in [3–8] and certain properties of S-transform are obtained in this section. By definition, we have

$$\begin{aligned} (S_\omega \phi)(\tau, \xi) &= \int_{\mathbb{R}^n} \omega(\tau - t, \xi) \phi(t) e^{-i2\pi \langle t, \xi \rangle} dt \\ &= \int_{\mathbb{R}^n} \omega(-x, \xi) \phi(\tau + x) e^{-i2\pi \langle \tau + x, \xi \rangle} dx \\ &= \int_{\mathbb{R}^n} \omega(-x, \xi) M_{-\xi} \phi(\tau + x) dx \\ &= \int_{\mathbb{R}^n} \omega(-x, \xi) T_{-\tau} M_{-\xi} \phi(x) dx. \end{aligned} \quad (10)$$

Thus, the S-transform S_ω appears as a superposition of time-frequency shifts as follows:

$$S_\omega := \int_{\mathbb{R}^n} \omega(-x, \xi) T_{-\tau} M_{-\xi} dx. \quad (11)$$

Example 5. If $\omega(t, \xi) = m(\xi)$, that is, independent of t , then

$$\begin{aligned} (S_\omega \phi)(\tau, \xi) &= \int_{\mathbb{R}^n} m(\xi) \phi(t) e^{-i2\pi \langle t, \xi \rangle} dt \\ &= m(\xi) (\mathcal{F}\phi)(\xi). \end{aligned} \quad (12)$$

So S_ω is a multiplication operator. In particular, if $\omega(t, \xi) = 1$, then $(S_\omega \phi)(\tau, \xi) = (\mathcal{F}\phi)(\xi)$.

Example 6. If $\omega(t, \xi) = m(t)$, then

$$\begin{aligned} (S_\omega \phi)(\tau, \xi) &= \int_{\mathbb{R}^n} m(\tau - t) \phi(t) e^{-i2\pi \langle t, \xi \rangle} dt \\ &= \int_{\mathbb{R}^n} T_{-\tau} m(-t) \phi(t) e^{-i2\pi \langle t, \xi \rangle} dt \\ &= \int_{\mathbb{R}^n} T_{-\tau} \overline{\mathcal{F}m}(t) \phi(t) e^{-i2\pi \langle t, \xi \rangle} dt \\ &= \mathcal{F}(\phi T_{-\tau} \overline{\mathcal{F}m})(\xi). \end{aligned} \quad (13)$$

Theorem 7 (Parseval's formula). Let ω_1 and ω_2 be the window functions such that

$$\int_{\mathbb{R}^n} \omega_1(\tau - t, \xi) \overline{\omega_2(\tau - x, \xi)} d\tau = \delta(x - t, \xi - \eta). \quad (14)$$

Let $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$ and let $(S_{\omega_1} \phi_1)$ and $(S_{\omega_2} \phi_2)$ be the S-transforms of ϕ_1 and ϕ_2 , respectively. Then

$$\begin{aligned} &\int_{\mathbb{R}^n} (S_{\omega_1} \phi_1)(\tau, \xi) \overline{(S_{\omega_2} \phi_2)(\tau, \xi)} d\tau d\xi \\ &= \int_{\mathbb{R}^n} \phi_1(t) \overline{\phi_2(t)} dt. \end{aligned} \quad (15)$$

This immediately implies the Plancherel formula

$$\|S_\omega \phi\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)}. \quad (16)$$

Proof. Consider

$$\begin{aligned} &\int_{\mathbb{R}^n} (S_{\omega_1} \phi_1)(\tau, \xi) \overline{(S_{\omega_2} \phi_2)(\tau, \xi)} d\tau d\xi \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi_1(t) \omega_1(\tau - t, \xi) e^{-i2\pi \langle t, \xi \rangle} dt \right) \\ &\quad \times \overline{\left(\int_{\mathbb{R}^n} \phi_2(x) \omega_2(\tau - x, \xi) e^{-i2\pi \langle x, \xi \rangle} dx \right)} d\tau d\xi \\ &= \iiint_{\mathbb{R}^n} \phi_1(t) \overline{\phi_2(x)} e^{i2\pi \langle (x-t), \xi \rangle} \\ &\quad \times \left(\int_{\mathbb{R}^n} \omega_1(\tau - t, \xi) \overline{\omega_2(\tau - x, \xi)} d\tau \right) d\xi dt dx \\ &= \iiint_{\mathbb{R}^n} \phi_1(t) \overline{\phi_2(x)} e^{i2\pi \langle (x-t), \xi \rangle} \delta(x - t, 0) d\xi dt dx \\ &= \int_{\mathbb{R}^n} \phi_1(t) \overline{\phi_2(t)} dt. \end{aligned} \quad (17)$$

□

Theorem 8 (inversion formula). If $\phi \in L^2(\mathbb{R}^n)$ and window function ω satisfy the condition (14) of the previous theorem, then

$$\phi(t) = \iint_{\mathbb{R}^n} (S_\omega \phi)(\tau, \xi) \overline{\omega(\tau - t, \xi)} e^{i2\pi \langle t, \xi \rangle} d\tau d\xi. \quad (18)$$

Proof. By the previous theorem we can write

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &= \iint_{\mathbb{R}^n} (S_\omega \phi_1)(\tau, \xi) \overline{(S_\omega \phi_2)(\tau, \xi)} d\tau d\xi \\ &= \iint_{\mathbb{R}^n} (S_\omega \phi_1)(\tau, \xi) \\ &\quad \times \overline{\left(\int_{\mathbb{R}^n} \phi_2(x) \omega(\tau - x, \xi) e^{-i2\pi \langle x, \xi \rangle} dx \right)} d\tau d\xi \\ &= \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^n} (S_\omega \phi_1)(\tau, \xi) \right. \\ &\quad \times \overline{\omega(\tau - x, \xi)} e^{i2\pi \langle x, \xi \rangle} d\tau d\xi \Big) \overline{\phi_2(x)} dx. \end{aligned} \quad (19)$$

Hence

$$\phi_1(t) = \iint_{\mathbb{R}^n} (S_\omega \phi_1)(\tau, \xi) \overline{\omega(\tau - t, \xi)} e^{i2\pi \langle t, \xi \rangle} d\tau d\xi. \quad (20)$$

□

Definition 9. Let ω be a window function and S_ω is the S-transform. Then the transform S_ω^* defined by

$$\langle S_\omega \phi, \psi \rangle = \langle \phi, S_\omega^* \psi \rangle \quad (21)$$

is called the adjoint of S_ω . If $\phi \in L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then (21) implies that

$$(S_\omega^* \psi)(t) = \iint_{\mathbb{R}^n} \psi(\tau, \xi) \overline{\omega(\tau - t, \xi)} e^{i2\pi\langle t, \xi \rangle} d\tau d\xi, \quad (22)$$

where $t, \tau, \xi \in \mathbb{R}^n$.

Theorem 10 (Parseval's formula for S_ω^*). *Let ω_1 and ω_2 be the window functions that satisfy the condition (14). If $\psi_1, \psi_2 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then*

$$\begin{aligned} \int_{\mathbb{R}^n} (S_{\omega_1}^* \psi_1)(t) \overline{(S_{\omega_2}^* \psi_2)(t)} dt \\ = \iint_{\mathbb{R}^n} \psi_1(\tau, \xi) \overline{\psi_2(\tau, \xi)} d\tau d\xi, \end{aligned} \quad (23)$$

and the Plancherel formula is

$$\|S_\omega^* \psi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}. \quad (24)$$

Proof. Consider

$$\begin{aligned} \int_{\mathbb{R}^n} (S_{\omega_1}^* \psi_1)(t) \overline{(S_{\omega_2}^* \psi_2)(t)} dt \\ = \int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^n} \psi_1(\tau, \xi) \overline{\omega_1(\tau - t, \xi)} e^{i2\pi\langle t, \xi \rangle} d\tau d\xi \right) \\ \times \overline{\left(\iint_{\mathbb{R}^n} \psi_2(\alpha, \eta) \overline{\omega_2(\alpha - t, \eta)} e^{i2\pi\langle t, \eta \rangle} d\alpha d\eta \right)} dt \\ = \iiint_{\mathbb{R}^n} \psi_1(\tau, \xi) \overline{\psi_2(\alpha, \eta)} e^{i2\pi\langle t, \xi - \eta \rangle} \\ \times \left(\int_{\mathbb{R}^n} \overline{\omega_1(\tau - t, \xi)} \omega_2(\alpha - t, \eta) dt \right) d\tau d\xi d\alpha d\eta \\ = \iiint_{\mathbb{R}^n} \psi_1(\tau, \xi) \overline{\psi_2(\alpha, \eta)} e^{i2\pi\langle t, \xi - \eta \rangle} \\ \times \delta(\tau - \alpha, \xi - \eta) d\tau d\xi d\alpha d\eta \\ = \iint_{\mathbb{R}^n} \psi_1(\tau, \xi) \overline{\psi_2(\tau, \xi)} d\tau d\xi. \end{aligned} \quad (25)$$

This proves the theorem. $\square$

Theorem 11. *If the window function ω satisfies the condition (14), then*

$$S_\omega S_\omega^* = I = S_\omega^* S_\omega, \quad (26)$$

where I is the identity operator.

Proof. By definition

$$\begin{aligned} S_\omega^* [S_{\omega_1} \psi](t) &= \iint_{\mathbb{R}^n} (S_{\omega_1} \psi)(\tau, \xi) \overline{\omega(\tau - t, \xi)} e^{i2\pi\langle t, \xi \rangle} d\tau d\xi \\ &= S_\omega^{-1} [S_{\omega_1} \psi](t). \end{aligned} \quad (27)$$

Thus

$$S_\omega^* = S_\omega^{-1}. \quad (28)$$

This proves the theorem. $\square$

Definition 12 (pseudodifferential operator). Let σ be a (measurable) function or a tempered distribution on $\mathbb{R}^n$. Then the operator

$$K_\sigma \phi(t) = \int_{\mathbb{R}^n} \sigma(t, \xi) \hat{\phi}(\xi) e^{i2\pi\langle t, \xi \rangle} d\xi \quad (29)$$

is called the pseudodifferential operator.

The pseudodifferential operator plays an important role in the theory of partial differential equations. The pseudodifferential operator has been studied on function and distribution spaces by many authors. Details of the concept can be found in [9, 10].

2.1. Relation between the S-Transform and Pseudodifferential Operator. Here we give a direct relation between S-transform and pseudodifferential operator which will may be very useful in the study of S-transform of distribution spaces. The continuous S-transform of a function ϕ with respect to a window function ω is given by

$$\begin{aligned} (S_\omega \phi)(\tau, \xi) &= \int_{\mathbb{R}^n} \phi(t) \omega(\tau - t, \xi) e^{-i2\pi\langle t, \xi \rangle} dt \\ &= \int_{\mathbb{R}^n} \phi(\tau - x) \omega(x, \xi) e^{-i2\pi\langle (\tau - x), \xi \rangle} dx \\ &= e^{-i2\pi\langle \tau, \xi \rangle} \int_{\mathbb{R}^n} T_{-\tau} \phi(-x) \omega(x, \xi) e^{i2\pi\langle x, \xi \rangle} dx \\ &= e^{-i2\pi\langle \tau, \xi \rangle} \int_{\mathbb{R}^n} \mathcal{F}[\mathcal{F}(T_{-\tau} \phi)](x) \\ &\quad \times \omega(x, \xi) e^{i2\pi\langle x, \xi \rangle} dx \\ &= e^{-i2\pi\langle \tau, \xi \rangle} K_\sigma[\mathcal{F}(T_{-\tau} \phi)](\xi), \end{aligned} \quad (30)$$

where $\sigma(\xi, x) = \omega(x, \xi)$.

3. The S-Transform of Distributions

In this section we will investigate the S-transform of tempered distribution by means of the Fourier transform.

Theorem 13. *If $\omega \in \mathcal{S}(\mathbb{R}^{2n})$, then S_ω maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$.*

Proof. By (6) we have

$$(S_\omega \phi)(\tau, \xi) = \mathcal{F}_1^{-1} [\mathcal{F}_1(\phi)(\alpha + \xi) \mathcal{F}_1(\omega)(\alpha, \xi)](\tau, \xi). \quad (31)$$

Thus, $(S_\omega \phi) \in \mathcal{S}(\mathbb{R}^{2n})$, since the Fourier transform is continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, and its inverse is also a continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ (see [11], page 66-67). $\square$

Theorem 14. If $\omega \in \mathcal{S}'(\mathbb{R}^{2n})$, then S_ω maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^{2n})$.

Proof. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}'(\mathbb{R}^{2n})$, we have

$$\begin{aligned} & \langle (S_\omega f)(\tau, \xi), \psi(\tau, \xi) \rangle \\ &= \iint_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(t) \omega(\tau - t, \xi) e^{-i2\pi\langle t, \xi \rangle} dt \right) \overline{\psi(\tau, \xi)} d\tau d\xi \\ & \quad \text{(in fact } \overline{\psi} = \psi, \text{ since } \psi \in \mathcal{S}) \\ &= \int_{\mathbb{R}^n} f(t) \left(\iint_{\mathbb{R}^n} \overline{\psi(\tau, \xi)} \omega(\tau - t, \xi) e^{-i2\pi\langle t, \xi \rangle} d\tau d\xi \right) dt \\ &= \langle f, \phi \rangle, \end{aligned} \quad (32)$$

where

$$\begin{aligned} \phi(t) &= \iint_{\mathbb{R}^n} \psi(\tau, \xi) \overline{\omega(\tau - t, \xi)} e^{i2\pi\langle t, \xi \rangle} d\tau d\xi \\ &= \int_{\mathbb{R}^n} (\psi * (\mathcal{F}\omega))(t, \xi) e^{i2\pi\langle t, \xi \rangle} d\xi \\ &= \mathcal{F}_2^{-1}(\psi * (\mathcal{F}\omega))(t) \in \mathcal{S}(\mathbb{R}^n). \end{aligned} \quad (33)$$

Thus, $(S_\omega f)(\tau, \xi) \in \mathcal{S}'(\mathbb{R}^{2n})$. $\square$

Theorem 15. If $\omega \in \mathcal{S}'(\mathbb{R}^{2n})$, then S_ω maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^{2n})$.

Proof. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}'(\mathbb{R}^{2n})$, then

$$U_{\phi, \psi}(\tau, \xi) := \overline{\mathcal{F}(\phi)(\alpha + \xi)} \psi(\alpha, \xi) \in \mathcal{S}(\mathbb{R}^{2n}). \quad (34)$$

Thus for any $\omega \in \mathcal{S}'(\mathbb{R}^{2n})$, we have

$$\begin{aligned} & \langle (\mathcal{F}_1 \omega)(\alpha, \xi), U_{\phi, \psi}(\alpha, \xi) \rangle \\ &= \iint_{\mathbb{R}^n} (\mathcal{F}_1 \omega)(\alpha, \xi) (\mathcal{F}\phi)(\alpha + \xi) \overline{\psi(\alpha, \xi)} d\alpha d\xi \\ &= \iint_{\mathbb{R}^n} (\mathcal{F}_1 (S_\omega \phi))(\alpha, \xi) \overline{\psi(\alpha, \xi)} d\alpha d\xi \\ &= \langle (\mathcal{F}_1 (S_\omega \phi))(\alpha, \xi), \psi(\alpha, \xi) \rangle. \end{aligned} \quad (35)$$

Thus $\mathcal{F}_1(S_\omega \phi) \in \mathcal{S}'(\mathbb{R}^{2n})$ and hence $(S_\omega \phi) \in \mathcal{S}'(\mathbb{R}^{2n})$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests.

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