Research Article

Gap Phenomenon of an Abstract Willmore Type Functional of Hypersurface in Unit Sphere

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For an n -dimensional hypersurface in unit sphere, we introduce an abstract Willmore type called $W_{(n,F)}$ -Willmore functional, which generalizes the well-known classic Willmore functional. Its critical point is called the $W_{(n,F)}$ -Willmore hypersurface, for which the variational equation and Simons' type integral equalities are obtained. Moreover, we construct a few examples of $W_{(n,F)}$ -Willmore hypersurface and give a gap phenomenon characterization by use of our integral formula.

1. Introduction

Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be an n -dimensional compact without boundary hypersurface in unit sphere. Choose an orthonormal frames field $\{e_1, \dots, e_n, e_{n+1}\}$ along M such that $\{e_1, \dots, e_n\}$ are tangent to M and $\{e_{n+1}\}$ is normal to M . Their dual frames are $\{\theta^1, \dots, \theta^n\}$ and $\{\theta^{n+1}\}$, respectively; obviously, $\theta^{n+1} = 0$ when it is restricted over M . Let h denote the second fundamental form of the immersion $\varphi : M \rightarrow S^{n+1}(1)$. Write

$$h = h_{ij}\theta^i \otimes \theta^j \quad (1)$$

and define

$$\begin{aligned} H &= \frac{1}{n} \sum_{i=1}^n h_{ii}, & S &= \sum_{ij} (h_{ij})^2, \\ \rho &= S - nH^2, & P_k &= \text{tr}(h^k). \end{aligned} \quad (2)$$

One calls H, S, P_k to be mean curvature, square of the length, and k th power polynomial of second fundamental form, respectively. It is well known that hypersurface $\varphi : M^n \rightarrow S^{n+1}(1)$ is minimal when $H \equiv 0$ and is totally geodesic when $S \equiv 0$. For invariant ρ , it is obvious that $\rho(Q) \geq 0$ for all $Q \in M$, $\rho(Q) = 0$ if and only if Q is an umbilical point of M , and $0 \leq \rho(Q) \leq C$ for all $Q \in M$ and some positive constant C since M is compact.

In differential geometry, there is a famous classical Willmore functional of hypersurface which is defined as

$$W_{(n,n/2)}(\varphi) = \int_M \rho^{n/2} dv. \quad (3)$$

See [1–4] for the details. It is well known that this functional is invariant under conformal transformation. Its critical point is called classical Willmore or $W_{(n,n/2)}$ -Willmore hypersurface. The famous Willmore conjecture says that $W_{(2,1)}(\varphi) \geq 4\pi^2$ holds for all immersed tori $\varphi : M \rightarrow S^3$; recently, it has been completely solved by Marques and Neves in article [5] using Min-Max theory for two dimensional case, but higher dimensional case is still an open problem. When $n \geq 3$, the functional $W_{(n,n/2)}(\varphi)$ was studied extensively in [6–11].

Due to the importance of Willmore conjecture, many geometric experts generalized the classic Willmore functional to a wide range and some interesting results have been obtained.

In [12], Cai studied the $W_{(2,p)}$ -Willmore functional for surfaces in 3-dimensional unit sphere $S^3(1)$:

$$W_{(2,p)}(\varphi) = \int_M (k_1 - k_2)^p dv, \quad (4)$$

where k_1 and k_2 denote the principal curvatures of M and $k_1 \geq k_2$. Under some proper conditions, Cai obtained some interesting inequalities.

In [13], Guo and Li considered the $W_{(n,1)}$ -Willmore functional for submanifolds in unit sphere $S^{n+1}(1)$:

$$W_{(n,1)}(\varphi) = \int_M \rho dv. \quad (5)$$

They calculated the first variation and obtained Simons' type inequalities and finally classified the point-wise gap phenomenon. In [14], Xu and Yang derived the global but not point-wise gap phenomenon for the same functional.

In [15], Wu researched the $W_{(n,r)}$ -Willmore functional for submanifolds in unit sphere $S^{n+1}(1)$:

$$W_{(n,r)}(\varphi) = \int_M \rho^r dv. \quad (6)$$

The first variation formula, Simons' type inequalities, and point-wise gap phenomenon are obtained.

Motivated by the work mentioned above, in this paper, we study an abstract Willmore type called $W_{(n,F)}$ -Willmore functional:

$$W_{(n,F)}(\varphi) = \int_M F(\rho) dv, \quad (7)$$

where we always assume that the abstract function satisfies

$$F: u \longrightarrow F(u), \quad F \in C^3([0, +\infty) \longrightarrow \mathbb{R}). \quad (8)$$

Obviously, functionals $W_{(n,n/2)}$, $W_{(2,p)}$, $W_{(n,1)}$, and $W_{(n,r)}$ are special cases of $W_{(n,F)}$, so the construction and investigation of $W_{(n,F)}$ are meaningful. As usual, the critical point of functional $W_{(n,F)}(\varphi)$ is called $W_{(n,F)}$ -Willmore hypersurface. In Section 3, we calculate the 1st variation formula of functional $W_{(n,F)}$.

Theorem 1. Let $\varphi: M^n \rightarrow S^{n+1}(1)$ be an n -dimensional hypersurface in an $(n+1)$ -dimensional unit sphere $S^{n+1}(1)$; then M is a $W_{(n,F)}$ -Willmore hypersurface if and only if

$$\sum_{ij} (F'(\rho) h_{ij})_{,ji} - \Delta(F'(\rho) H) + (P_3 - SH) F'(\rho) - \frac{n}{2} F(\rho) H = 0. \quad (9)$$

In particular, if M is an isoparametric (all principal curvatures are constant) hypersurface, then M is a $W_{(n,F)}$ -Willmore hypersurface if and only if

$$(P_3 - SH) F'(\rho) - \frac{n}{2} F(\rho) H = 0. \quad (10)$$

Remark 2. when $F(\rho) = \rho^{n/2}, \rho, \rho^r$, Theorem 1 coincides with corresponding results in [6, 13, 15], respectively.

It is well known that Simons' integral inequality plays an important role in the study of minimal hypersurface. It says that if M is an n -dimensional compact minimal hypersurface in $(n+1)$ -dimensional unit sphere $S^{n+1}(1)$, then

$$\int_M S(n-S) dv \leq 0. \quad (11)$$

For classic Willmore hypersurface, Li [6] proved a Simons' type integral result which says if M is an n -dimensional compact classical Willmore hypersurface in unit sphere $S^{n+1}(1)$, then

$$\int_M \rho^{n/2} (n-\rho) dv \leq 0. \quad (12)$$

Similarly, for functionals $W_{(n,1)}$ and $W_{(n,r)}$, [13, 15] obtained integral inequalities, respectively. A natural question is that whether a Simons' type inequality or equality can be established for $W_{(n,F)}$ -Willmore functional. We give a confirm answer in this study.

Theorem 3. Let M be an n -dimensional compact $W_{(n,F)}$ -Willmore hypersurface in sphere $S^{n+1}(1)$; then we have Simons' type equality and can give a discussion according to the sign of F, F' , and F'' :

$$0 = \int_M F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) - 2\rho(\rho-n)F'(\rho) + nH^2(nF(\rho) - 2\rho F'(\rho)) dv. \quad (13)$$

(1) When $\rho \equiv n$, there holds

$$0 = \int_M 2F'(n) (|\nabla h|^2 - n|\nabla H|^2) + nH^2(nF(n) - 2nF'(n)) dv. \quad (14)$$

(2) When $\rho \equiv 0$, there holds

$$0 = \int_M 2F'(0) (|\nabla h|^2 - n|\nabla H|^2) + n^2 H^2 F(0) dv. \quad (15)$$

(3) When $F' \equiv 0, F \equiv c \neq 0$, there holds

$$0 = \int_M cn^2 H^2 dv. \quad (16)$$

(4) When $nF - 2uF' \geq 0, F' \geq 0, F'' \geq 0$, there holds

$$\begin{aligned} & \int_M 2\rho(\rho-n)F'(\rho) \\ &= \int_M F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) \\ & \quad + nH^2(nF(\rho) - 2\rho F'(\rho)) dv. \end{aligned} \quad (17)$$

(5) When $nF - 2uF' \leq 0, F' \geq 0, F'' \geq 0$, there holds

$$\begin{aligned} & \int_M 2\rho(\rho-n)F'(\rho) - nH^2(nF(\rho) - 2\rho F'(\rho)) \\ &= \int_M F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) dv. \end{aligned} \quad (18)$$

Remark 4. When $F(\rho) = \rho^{n/2}, \rho, \rho^r$, Theorem 3 coincides with corresponding results in [6, 13, 15], respectively. Similar equalities also hold for the other sign cases of F, F' , and F'' .

Using the integral equalities in Theorem 3, together with two famous Chern et al. results in [16], we can obtain some conclusions of gap phenomenon.

Theorem 5. Let M be an n -dimensional closed $W_{(n,F)}$ -Willmore hypersurface in unit sphere $S^{n+1}(1)$, we have the following.

- (1) When $F' \equiv 0$, $F \equiv c \neq 0$ over $(0, +\infty)$, if $0 \leq \rho \leq n$, then $\rho = 0$ or $\rho = n$.
 - (a) For $\rho = 0$, then $H = 0$, $\rho = 0$, $S = 0$, and M is totally geodesic;
 - (b) For $\rho = n$, then $H = 0$, $\rho = n$, $S = n$, and $M = C_{m,n-m} = S^m(\sqrt{m/n}) \times S^{n-m}(\sqrt{(n-m)/n})$ for some m with $1 \leq m \leq n-1$.
- (2) When $nF - 2uF' > 0$, $F' > 0$, $F'' \geq 0$ over $(0, \infty)$, if $0 \leq \rho \leq n$, then $\rho = 0$ or $\rho = n$.
 - (a) For $\rho = 0$, then M is totally umbilical;
 - (b) For $\rho = n$, then $n \equiv 0 \pmod{2}$, $H = 0$, $\rho = n$, $S = n$, and $M = C_{n/2,n/2} = S^{n/2}(\sqrt{1/2}) \times S^{n/2}(\sqrt{1/2})$.
- (3) When $nF - 2uF' = 0$, $F' > 0$, $F'' \geq 0$ over $(0, +\infty)$, if $0 \leq \rho \leq n$, then $F(u) = cu^{n/2}$, $c > 0$, $\rho = 0$, or $\rho = n$.
 - (a) For $\rho = 0$, M is totally umbilical;
 - (b) For $\rho = n$, $M = W_{m,n-m} = S^m(\sqrt{(n-m)/n}) \times S^{n-m}(\sqrt{m/n})$ for some m with $1 \leq m \leq n-1$.
- (4) When $nF - 2uF' < 0$, $F' > 0$, $F'' \geq 0$ over $(0, +\infty)$, if $H = 0$, $0 \leq \rho \leq n$, then $H = 0$, $\rho = 0$ or $H = 0$, $\rho = n$.
 - (a) For $H = 0$, $\rho = 0$, then M is totally geodesic;
 - (b) For $H = 0$, $\rho = n$, then $n \equiv 0 \pmod{2}$, $H = 0$, $\rho = n$, $S = n$, $M = C_{n/2,n/2} = S^{n/2}(\sqrt{1/2}) \times S^{n/2}(\sqrt{1/2})$.

Remark 6. When $F(\rho) = \rho^{n/2}$, ρ, ρ' , Theorem 5 coincides with corresponding results in [6, 13, 15], respectively. Similar conclusions also hold for the other sign cases of F, F' , and F'' . The examples appeared in Theorem 5 would be constructed in Section 3.

2. Preliminaries

Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be an n -dimensional closed hypersurface in an $(n+1)$ -dimensional unit sphere and let $\Phi(\cdot, \cdot) : M \times (-\epsilon, \epsilon) \rightarrow S^{n+1}(1)$ be a variation of φ ; it means that

$$\varphi_t := \Phi(\cdot, t) : M \times \{t\} \rightarrow S^{n+1}(1), \quad \forall t \in (-\epsilon, \epsilon) \quad (19)$$

is still an isometric immersion with $\varphi_0 \equiv \varphi$.

Let $s = (s_1, \dots, s_n, s_{n+1})$ and $\sigma = (\sigma^1, \dots, \sigma^n, \sigma^{n+1})$ be the orthonormal local frames of $TS^{n+1}(1)$ and $T^*S^{n+1}(1)$, respectively; then $e = (e_1, \dots, e_n, e_{n+1}) = \varphi_t^{-1}s$ is the orthonormal local frames of the pullback vector bundle

$\varphi_t^{-1}TS^{n+1}(1)$ over $M \times \{t\}$, such that $\{e_1, \dots, e_n\}$ are tangent to M and $\{e_{n+1}\}$ is normal to M due to

$$\varphi_t^{-1}TS^{n+1}(1) = TM_t \oplus T^\perp M_t. \quad (20)$$

Use ω to denote the connection form over $TS^{n+1}(1)$; by the pullback operation, we have the following decomposition:

$$\begin{aligned} \Phi^* \sigma &= \theta + V dt, & \Phi^* \sigma^i &= \theta^i + V^i dt, \\ \Phi^* \sigma^{n+1} &= V^{n+1} dt =: f dt, \\ \Phi^* \omega &= \phi + L dt, & \Phi^* \omega_i^j &= \phi_i^j + L_i^j dt, \\ \Phi^* \omega_i^{n+1} &= \phi_i^{n+1} + b_i dt. \end{aligned} \quad (21)$$

Thus, one can derive that $\{\theta^i\}$ are the orthonormal frames of T^*M and $\theta^{n+1} \equiv 0$ when it is restricted over M , $\{V^i, V^{n+1} =: f\}$ are the variation vector fields of Φ , $\{\phi_i^j\}$ is the connection form of TM , and $\{\phi_i^{n+1} = h_{ij}\theta^j\}$ is the second fundamental form. In particular, we have

$$\left. \frac{\partial}{\partial t} \varphi_t \right|_{t=0} = \sum_i V^i e_i + f e_{n+1}. \quad (22)$$

Use Ω and $\Omega^\top$ to denote curvature forms of $TS^{n+1}(1)$ and TM , respectively and write their components:

$$\begin{aligned} \Omega_{AB} &= \frac{1}{2} \bar{R}_{ABCD} \sigma^C \wedge \sigma^D, \quad A, B, C, D \in \{1, \dots, n+1\}, \\ \Omega_{ij}^\top &= \frac{1}{2} R_{ijkl} \theta^k \wedge \theta^l, \quad i, j, k, l \in \{1, 2, \dots, n\}. \end{aligned} \quad (23)$$

From any one standard differential geometry book, we know that the curvature of unit sphere is

$$\bar{R}_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \quad (24)$$

Using connection form $\{\phi_i^j\}$, the covariant derivatives of h_{ij}, H, ρ, V^i, f , and b_i can be defined as

$$\begin{aligned} h_{ij,k} \theta^k &= d_M h_{ij} - \sum_p h_{pj} \phi_i^p - \sum_p h_{ip} \phi_j^p, \\ h_{ij,kl} \theta^l &= d_M h_{ij,k} - \sum_p h_{pj,k} \phi_i^p - \sum_p h_{ip,k} \phi_j^p - \sum_p h_{ij,p} \phi_k^p, \\ H_{,i} \theta^i &= d_M H, & H_{,ij} \theta^j &= d_M H_{,i} - \sum_p H_{,ip} \phi_i^p, \end{aligned}$$

$$\begin{aligned}
\Delta H &= \sum_i H_{,ii}, \\
\rho_{,i}\theta^i &= d_M \rho, \quad \rho_{,ij}\theta^j = d_M \rho_{,i} - \sum_p \rho_{,p}\phi_i^p, \quad \Delta \rho = \sum_i \rho_{,ii}, \\
V_{,j}\theta^j &= d_M V^i + \sum_p V^p \phi_p^i, \\
V_{,jk}^i \theta^k &= d_M V_{,j}^i - \sum_p V_{,p}^i \phi_j^p + \sum_p V_{,j}^p \phi_p^i, \\
f_{,i}\theta^i &= d_M f, \quad f_{,ij}\theta^j = d_M f_{,i} - \sum_p f_{,p}\phi_i^p, \quad \Delta f = \sum_i f_{,ii}, \\
b_{i,j}\theta^j &= d_M b_i - \sum_p b_p \phi_i^p, \\
b_{i,jk}\theta^k &= d_M b_{i,j} - \sum_p b_{p,j}\phi_i^p - \sum_p b_{i,p}\phi_j^p.
\end{aligned} \tag{25}$$

We use d_M , $d_{M \times (-\epsilon, \epsilon)} = d_M + dt(\partial/\partial t)$ and d to denote the differential operators on M , $M \times (-\epsilon, \epsilon)$, and $S^{n+1}(1)$, respectively. By calculating directly and comparing both sides of the following equations:

$$\Omega = d\omega - \omega \wedge \omega, \quad 0 = d\sigma - \sigma \wedge \omega, \tag{26}$$

$$\Phi^*(\Omega) = d_{M \times (-\epsilon, \epsilon)} \Phi^*(\omega) - \Phi^*(\omega) \wedge \Phi^*(\omega), \tag{27}$$

$$0 = d_{M \times (-\epsilon, \epsilon)} \Phi^*(\sigma) - \Phi^*(\sigma) \wedge \Phi^*(\omega), \tag{28}$$

we would derive out the lemmas below; also see article [10]. In fact, the right hand side of (27) is

$$\begin{aligned}
\Phi^*(\Omega_A^B) &= \Phi^*\left(\frac{1}{2}\bar{R}_{ABCD}\sigma^C \wedge \sigma^D\right) \\
&= \frac{1}{2}\bar{R}_{ABCD}\Phi^*(\sigma^C) \wedge \Phi^*(\sigma^D) \\
&= \frac{1}{2}\bar{R}_{ABCD}\theta^C \wedge \theta^D + dt \wedge (\bar{R}_{ABCD}V^C\theta^D),
\end{aligned} \tag{29}$$

while the left hand side of (27) is

$$\begin{aligned}
d_{M \times (-\epsilon, \epsilon)} \Phi^*(\omega_A^B) - \Phi^*(\omega_A^C) \wedge \Phi^*(\omega_C^B) \\
&= \left(d_M + dt \frac{\partial}{\partial t}\right)(\phi_A^B + dt L_A^B) - (\phi_A^C + dt L_A^C)(\phi_C^B + dt L_C^B) \\
&= (d_M \phi_A^B - \phi_A^C \wedge \phi_C^B) \\
&\quad + dt \left(\frac{\partial}{\partial t} \phi_A^B - (d_M L_A^B + L_A^C \phi_C^B - \phi_A^C L_C^B)\right);
\end{aligned} \tag{30}$$

comparing (29) with (30), we obtain

$$\begin{aligned}
(d_M \phi_A^B - \phi_A^C \wedge \phi_C^B) &= \frac{1}{2}\bar{R}_{ABCD}\theta^C \wedge \theta^D, \\
\frac{\partial}{\partial t} \phi_A^B - (d_M L_A^B + L_A^C \phi_C^B - \phi_A^C L_C^B) &= \bar{R}_{ABCD}V^C\theta^D.
\end{aligned} \tag{31}$$

The equation of (28) is

$$\begin{aligned}
0 &= d_{M \times (-\epsilon, \epsilon)} \Phi^*(\sigma) - \Phi^*(\sigma) \wedge \Phi^*(\omega) \\
&= \left(d_M + dt \frac{\partial}{\partial t}\right)(\theta + dt V) - (\theta + dt V) \wedge (\phi + dt L) \\
&= d_M \theta - \theta \wedge \phi + dt \left(\frac{\partial \theta}{\partial t} - d_M V - V\phi + \theta L\right).
\end{aligned} \tag{32}$$

Comparing both sides, we obtain

$$d_M \theta - \theta \wedge \phi = 0, \quad \frac{\partial \theta}{\partial t} - d_M V - V\phi + \theta L = 0. \tag{33}$$

From (31) and (33), we can easily conclude the following lemmas which are useful in the variation calculation and establishment of Simons' type integral equalities.

Lemma 7. Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be a hypersurface; one has structure equations:

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \quad h_{ij,k} = h_{ik,j}. \tag{34}$$

Lemma 8. Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be a hypersurface; one has Ricci identity:

$$\begin{aligned}
h_{ij,kl} - h_{ij,lk} &= -(\delta_{ik}h_{jl} - \delta_{il}h_{jk}) - (\delta_{jk}h_{il} - \delta_{jl}h_{ik}) \\
&\quad - (h_{ik}h_{pl}h_{pj} - h_{il}h_{pk}h_{pj}) \\
&\quad - (h_{jk}h_{pl}h_{ip} - h_{jl}h_{pk}h_{ip}).
\end{aligned} \tag{35}$$

Lemma 9. Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be a hypersurface and let $V = V^\top + V^\perp = V^i e_i + f e_{n+1}$ be a variation vector field; one has

$$\begin{aligned}
\frac{\partial \theta^i}{\partial t} &= (V_{,j}^i - L_j^i - h_{ij}f)\theta^j, \\
\frac{\partial h_{ij}}{\partial t} &= f_{,ij} + \sum_p h_{ij,p}V^p + \sum_p h_{pj}L_i^p + \sum_p h_{ip}L_j^p \\
&\quad + \sum_p h_{ip}h_{pj}f + \delta_{ij}f.
\end{aligned} \tag{36}$$

Lemma 10. With the same notations as above, one has

$$\sum_{ijk} h_{ik}h_{kj}L_j^i = 0, \quad \sum_{ij} h_{ij}L_j^i = 0. \tag{37}$$

3. Variation Calculation and Examples

To calculate the first variation of $W_{(n,F)}$ -Willmore functional, we need two lemmas.

Lemma 11. Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be a hypersurface and let $V = V^\top + V^\perp = V^i e_i + f e_{n+1}$ be a variation vector field. Suppose that $dv = \theta^1 \wedge \cdots \wedge \theta^n$ denotes the volume element; one has

$$\frac{\partial dv}{\partial t} = \sum_i V_{,i}^i - nHf. \tag{38}$$

Proof. By Lemma 9, we have $(\partial/\partial t)(\theta^i) = (V_j^i - L_j^i - h_{ij}f)\theta^j$. Hence,

$$\frac{\partial dv}{\partial t} = \sum_i \theta^1 \wedge \cdots \frac{\partial}{\partial t} (\theta^i) \cdots \wedge \theta^n = \sum_i V_{,i}^i - nHf. \quad (39)$$

With this, we complete the proof of Lemma 11. $\square$

Lemma 12. Let $\varphi : M^n \rightarrow S^{n+1}(1)$ be a hypersurface and let $V = V^\top + V^\perp = V^i e_i + f e_{n+1}$ be a variation vector field; one has

$$\frac{\partial \rho}{\partial t} = \sum_{ij} 2h_{ij}f_{,ij} - 2H\Delta f + 2P_3f - 2SHf + \sum_i \rho_{,i}V^i. \quad (40)$$

Proof. By the definition of ρ , S and H , one has

$$\frac{\partial}{\partial t}(\rho) = \frac{\partial}{\partial t}(S) - \frac{\partial}{\partial t}(nH^2). \quad (41)$$

It follows Lemmas 9 and 10 that

$$\begin{aligned} \frac{\partial}{\partial t}(S) &= 2 \sum_{ij} h_{ij} \frac{\partial}{\partial t}(h_{ij}) \\ &= 2 \sum_{ij} h_{ij} \left[f_{,ij} + \sum_p h_{ij,p} V^p + \sum_p h_{pj} L_i^p \right. \\ &\quad \left. + \sum_p h_{ip} L_j^p + \sum_p h_{ip} h_{pj} f + \delta_{ij} f \right] \\ &= 2 \sum_{ij} h_{ij} f_{,ij} + \sum_i S_{,i} V^i + 2P_3f + 2nHf. \end{aligned} \quad (42)$$

By Lemmas 9 and 10 again, one has

$$\begin{aligned} \frac{\partial}{\partial t}(nH^2) &= 2nH \frac{\partial}{\partial t} H \\ &= 2H \sum_i \left[f_{,ii} + \sum_p h_{ii,p} V^p + \sum_p h_{pi} L_i^p \right. \\ &\quad \left. + \sum_p h_{pi} L_i^p + \sum_p h_{ip} h_{pi} f + \delta_{ii} f \right] \\ &= 2H\Delta f + \sum_i (nH^2)_{,i} V^i + 2HSf + 2nHf. \end{aligned} \quad (43)$$

Hence,

$$\frac{\partial \rho}{\partial t} = \sum_{ij} 2h_{ij}f_{,ij} - 2H\Delta f + 2P_3f - 2HSf + \sum_i \rho_{,i}V^i. \quad (44)$$

With this, we complete the proof of Lemma 12. $\square$

Proof of Theorem 1. By Lemmas 11 and 12, we can calculate

$$\begin{aligned} \frac{\partial}{\partial t} W_{(n,F)}(\varphi_t) &= \int_{M_t} F'(\rho) \frac{\partial}{\partial t}(\rho) + F(\rho) \left(\sum_i V_{,i}^i - nHf \right) dv \\ &= \int_{M_t} F'(\rho) \left(\sum_{ij} 2h_{ij}f_{,ij} - 2H\Delta f + 2P_3f - 2SHf + \sum_i \rho_{,i}V^i \right) \\ &\quad + F(\rho) \left(\sum_i V_{,i}^i - nHf \right) dv \\ &= \int_{M_t} 2 \left[\sum_{ij} (F'(\rho) h_{ij})_{,ji} - \Delta(F'(\rho) H) \right. \\ &\quad \left. + (P_3 - SH)F'(\rho) - \frac{n}{2}HF(\rho) \right] f dv. \end{aligned} \quad (45)$$

With this, we complete the proof of Theorem 1. $\square$

From Theorem 1, we can find a few examples in unit sphere $S^{n+1}(1)$, in particular, the isoparametric hypersurfaces. Since all principal curvatures $k_1, \dots, k_n$ are constant, then ρ , H , S are constant and $H = (1/n)P_1$, $S = P_2$, $\rho = P_2 - (1/n)(P_1)^2$. Thus, $W_{(n,F)}$ -Willmore hypersurface equation becomes

$$\left(P_3 - \frac{1}{n}P_1P_2 \right) F'(\rho) - \frac{1}{2}P_1F(\rho) = 0. \quad (46)$$

Example 13. Obviously, totally geodesic hypersurfaces are $W_{(n,F)}$ -Willmore for any abstract function F , totally umbilical but not totally geodesic hypersurfaces are $W_{(n,F)}$ -Willmore if and only if $F(0) = 0$.

Example 14. For a particular hypersurface with $n \equiv 0 \pmod{2}$,

$$C_{n/2,n/2} = S^{n/2} \left(\frac{1}{\sqrt{2}} \right) \times S^{n/2} \left(\frac{1}{\sqrt{2}} \right) \longrightarrow S^{n+1}(1). \quad (47)$$

All principal curvatures are

$$k_1 = \cdots = k_{n/2} = 1, \quad k_{(n/2)+1} = \cdots = k_n = -1. \quad (48)$$

We can derive the quantities P_1 , P_2 , P_3 , ρ , respectively, $P_1 = 0$, $P_2 = n$, $P_3 = 0$, $\rho = n$. Obviously $C_{n/2,n/2}$ always is a $W_{(n,F)}$ -Willmore hypersurface of $S^{n+1}(1)$ for any function F .

Example 15. For a family hypersurfaces with parameters λ , μ , $0 < \lambda, \mu < 1$, $\lambda^2 + \mu^2 = 1$.

$$S^m(\lambda) \times S^{n-m}(\mu) \longrightarrow S^{n+1}(1), \quad 1 \leq m \leq n-1. \quad (49)$$

Obviously, all principal curvatures are

$$k_1 = \cdots = k_m = \frac{\mu}{\lambda}, \quad k_{m+1} = \cdots = k_n = -\frac{\lambda}{\mu}. \quad (50)$$

Then the quantities P_1, P_2, P_3, ρ are, respectively,

$$\begin{aligned} P_1 &= m \frac{\mu}{\lambda} - (n-m) \frac{\lambda}{\mu}, & P_2 &= m \frac{\mu^2}{\lambda^2} + (n-m) \frac{\lambda^2}{\mu^2}, \\ P_3 &= m \frac{\mu^3}{\lambda^3} - (n-m) \frac{\lambda^3}{\mu^3}, \\ \rho &= \frac{m(n-m)}{n} \left[\frac{\mu^2}{\lambda^2} + \frac{\lambda^2}{\mu^2} + 2 \right]. \end{aligned} \quad (51)$$

Setting $\mu/\lambda = x > 0$, $W_{(n,F)}$ -Willmore hypersurface equation becomes

$$\begin{aligned} &2m(n-m)F'(\rho)x^6 - m[nF(\rho) - 2(n-m)F'(\rho)]x^4 \\ &+ (n-m)[nF(\rho) - 2mF'(\rho)]x^2 \\ &- 2m(n-m)F'(\rho) = 0. \end{aligned} \quad (52)$$

Example 16 (Clifford torus [16]; $F(\rho) \equiv 1$). Consider that

$$C_{m,n-m} = S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right), \quad 1 \leq m \leq n-1 \quad (53)$$

are minimal hypersurfaces with $H \equiv 0$, $S \equiv n$, $\rho \equiv n$.

Example 17 (classic Willmore torus [6]; $F(\rho) = \rho^{n/2}$). Consider that

$$W_{m,n-m} = S^m \left(\sqrt{\frac{n-m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{m}{n}} \right), \quad 1 \leq m \leq n-1 \quad (54)$$

are $W_{(n,n/2)}$ -Willmore hypersurfaces with $\rho = n$. When some $W_{m,n-m}$ is minimal, then $n \equiv 0 \pmod{2}$, $m = n/2$.

4. Simons' Type Equalities and Gap Phenomenon

Some lemmas are needed for the establishment of Simons' type integral equalities and the discussion of gap phenomenon.

Lemma 18. For ρ , one has

$$\begin{aligned} \Delta \rho &= \sum_{ij} 2nh_{ij}H_{,ij} - 2nH\Delta H + 2|\nabla h|^2 \\ &- 2n|\nabla H|^2 + 2n\rho + 2nHP_3 - 2S^2, \end{aligned} \quad (55)$$

where $|\nabla h|^2 = \sum_{ijk} h_{ij,k}^2$.

Proof. By the definition of ρ and Laplacian operator, we have

$$\begin{aligned} \Delta \rho &= \sum_k \left(\sum_{ij} h_{ij}^2 - nH^2 \right)_{,kk} \\ &= \sum_k \left(\sum_{ij} 2h_{ij}h_{ij,k} - 2nHH_{,k} \right)_{,k} \\ &= \sum_k \left(\sum_{ij} 2h_{ij,k}^2 + \sum_{ij} 2h_{ij}h_{ij,kk} - 2nH_{,k}^2 - 2nHH_{,kk} \right) \\ &= 2|\nabla h|^2 - 2nH\Delta H - 2n|\nabla H|^2 + 2\sum_{ijk} h_{ij}h_{ij,kk} \\ &= 2 \left(|\nabla h|^2 - nH\Delta H - n|\nabla H|^2 + (T_1) \sum_{ijk} h_{ij}h_{ij,kk} \right), \end{aligned} \quad (56)$$

for term $T_1 = \sum_{ijk} h_{ij}h_{ij,kk}$; by Lemmas 7 and 8, easily we obtain

$$\begin{aligned} T_1 &= \sum_{ijk} h_{ij}h_{ij,kk} = \sum_{ijk} h_{ij}h_{ik,jk} \\ &= \sum_{ijk} h_{ij} (h_{ik,jk} - h_{ik,kj} + h_{ik,kj}) \\ &= \sum_{ijk} h_{ij}h_{kk,ij} + \sum_{ijk} h_{ij} (h_{ik,jk} - h_{ik,kj}) \\ &= \sum_{ij} nh_{ij}H_{,ij} + (T_2) \sum_{ijk} h_{ij} (h_{ik,jk} - h_{ik,kj}), \end{aligned} \quad (57)$$

for term $T_2 = \sum_{ijk} h_{ij} (h_{ik,jk} - h_{ik,kj})$; by Lemma 8, we have

$$\begin{aligned} T_2 &= \sum_{ijk} h_{ij} (h_{ik,jk} - h_{ik,kj}) \\ &= \sum_{ijk} h_{ij} \{ [-(\delta_{jk}h_{ik} - \delta_{kk}h_{ij}) - (h_{jk}h_{ip}h_{pk} - h_{kk}h_{ip}h_{pj})] \\ &\quad + [-(\delta_{ij}h_{kk} - \delta_{ik}h_{jk}) - (h_{ij}h_{kp}h_{pk} - h_{ik}h_{kp}h_{pj})] \} \\ &= -S + nS - \sum_{ijkp} h_{ij}h_{jk}h_{ip}h_{pk} + nHP_3 - n^2H^2 \\ &\quad + S - S^2 + \sum_{ijkp} h_{ij}h_{ik}h_{kp}h_{pj} \\ &= n\rho - S^2 + nHP_3. \end{aligned} \quad (58)$$

Substituting (58) into (57), we have

$$T_1 = \sum_{ij} nh_{ij}H_{,ij} + n\rho - S^2 + nHP_3. \quad (59)$$

Substituting (59) into (56), we have

$$\Delta \rho = \sum_{ij} 2nh_{ij}H_{,ij} - 2nH\Delta H + 2|\nabla h|^2 - 2n|\nabla H|^2 + 2n\rho + 2nHP_3 - 2S^2. \quad (60)$$

With this, we complete the proof of Lemma 18. $\square$

Lemma 19 (Huisken's estimate [17]). *Consider that*

$$|\nabla h|^2 \geq \frac{3n^2}{n+2} |\nabla H|^2 \geq n|\nabla H|^2, \quad (61)$$

and $|\nabla h|^2 = n|\nabla H|^2$ if and only if $\nabla h = 0$.

Proof. We decompose the tensor ∇h as follows:

$$h_{ij,k} = E_{ijk} + F_{ijk}, \quad (62)$$

where

$$E_{ijk} = \frac{n}{n+2} (H_{,i}\delta_{jk} + H_{,j}\delta_{ik} + H_{,k}\delta_{ij}), \quad (63)$$

$$F_{ijk} = h_{ij,k} - E_{ijk}.$$

Through a computation, we obtain

$$|E|^2 = \frac{3n^2}{n+2} |\nabla H|^2, \quad \langle E, F \rangle = 0. \quad (64)$$

Finally, by the triangle inequality, we have

$$|\nabla h|^2 \geq |E|^2 = \frac{3n^2}{n+2} |\nabla H|^2 \geq n|\nabla H|^2. \quad (65)$$

If $|\nabla h|^2 = n|\nabla H|^2$, then all inequalities become equalities, $F_{ijk} = 0$, $\nabla H = 0$, and $E_{ijk} = 0$; thus, $\nabla h = 0$. With this, we complete the proof of Lemma 19. $\square$

Lemma 20. *For $F(\rho)$, one has*

$$\begin{aligned} \Delta F(\rho) &= F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) \\ &\quad - 2\rho(\rho - n)F'(\rho) + nH^2(nF(\rho) - 2\rho F'(\rho)) \\ &\quad + 2n \left[\sum_{ij} F'(\rho) h_{ij}H_{,ij} - F'(\rho) H\Delta H \right. \\ &\quad \left. + (P_3 - SH)F'(\rho)H - \frac{n}{2}F(\rho)H^2 \right]. \end{aligned} \quad (66)$$

Proof. We know that

$$\Delta F(\rho) = F''(\rho) |\nabla \rho|^2 + F'(\rho) \Delta \rho. \quad (67)$$

By Lemmas 18 and 19, we can easily obtain the identity. $\square$

Proof of Theorem 3. Integrating the equality in Lemma 20 over M , we have

$$\begin{aligned} 0 &= \int_M F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) \\ &\quad - 2\rho(\rho - n)F'(\rho) + nH^2(nF(\rho) - 2\rho F'(\rho)) \\ &\quad + 2n \left[\sum_{ij} F'(\rho) h_{ij}H_{,ij} - F'(\rho) H\Delta H \right. \\ &\quad \left. + (P_3 - SH)F'(\rho)H - \frac{n}{2}F(\rho)H^2 \right]. \end{aligned} \quad (68)$$

By Theorem 1, we know that

$$\begin{aligned} \sum_{ij} (F'(\rho) h_{ij})_{,ji} - \Delta(F'(\rho)H) \\ + (P_3 - SH)F'(\rho) - \frac{n}{2}F(\rho)H = 0. \end{aligned} \quad (69)$$

Using Stokes theorem, we obtain

$$\begin{aligned} 0 &= \int_M F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) \\ &\quad - 2\rho(\rho - n)F'(\rho) + nH^2(nF(\rho) - 2\rho F'(\rho)). \end{aligned} \quad (70)$$

We complete the proof of Theorem 3. $\square$

In order to prove Theorem 5, we need two very important conclusions which are treated as a lemma and a main Theorem in Chern et al's article [16]. For a hypersurface, we choose frame fields in such a way that $h_{ij} = 0$, for all $i \neq j$, $h_i = h_{ii}$.

Lemma 21 (Lemma 3 of [16]). *Let $x : M^n \rightarrow S^{n+1}(1)$ be a compact hypersurface with $\nabla h \equiv 0$; then there are two cases.*

- (1) $h_1 = \dots = h_n = \lambda = \text{constant}$, and M is a totally umbilic ($\lambda > 0$) or totally geodesic ($\lambda = 0$);
- (2) $h_1 = \dots = h_m = \lambda = \text{constant} > 0$, $h_{m+1} = \dots = h_n = -(1/\lambda)$, $1 \leq m \leq n-1$, and M is a Riemannian product of $M_1 \times M_2$, where $M_1 = S^m(1/\sqrt{1+\lambda^2})$, $M_2 = S^{n-m}(\lambda/\sqrt{1+\lambda^2})$.

Lemma 22 (main Theorem of [16]). *Clifford torus $C_{m,n-m}$, $1 \leq m \leq n-1$ are the only compact minimal ($H = 0$) hypersurfaces of dimension n in unit sphere $S^{n+1}(1)$ satisfying $S = n$.*

Proof of Theorem 5. Obviously, case (1) of Theorem 5 is a corollary of Lemma 22; thus, we just focus cases (2) and (3); the rest can be proved by similar argument.

(2) When $nF - 2uF' > 0$, $F' > 0$, $F'' \geq 0$ over $(0, +\infty)$, by the case (4) of Theorem 3, we have

$$\begin{aligned} & \int_M 2\rho(\rho - n) F'(\rho) \\ &= \int_M F''(\rho) |\nabla \rho|^2 + 2F'(\rho) (|\nabla h|^2 - n|\nabla H|^2) \\ & \quad + nH^2 (nF(\rho) - 2\rho F'(\rho)) \geq 0. \end{aligned} \quad (71)$$

If $0 \leq \rho \leq n$, the right hand side of identity (71) is nonpositive, while the left hand side of (71) is nonnegative; thus, we can conclude that $\rho = 0$ or $\rho = n$. For $\rho = 0$, M is totally umbilical; for $\rho = n$, substituting it into the above equality together with Lemma 19, we can derive $\nabla h = 0$, $H = 0$, $\rho = n$, and $S = n$, by Lemma 22, we know that $M = C_{m,n-m}$; moreover, by $W_{(n,F)}$ -Willmore equation, we conclude that $n \equiv 0 \pmod{2}$, $H = 0$, $\rho = n$, $S = n$, and $M = C_{n/2,n/2}$.

(3) When $nF - 2uF' = 0$, $F' > 0$, $F'' \geq 0$ $(0, +\infty)$, we know that $F(u) = cu^{n/2}$, $c = \text{const} > 0$. By the case (4) of Theorem 3, we have

$$\begin{aligned} & \int_M \rho^{n/2} (\rho - n) \\ &= \int_M \frac{n-2}{4} |\nabla \rho|^2 + \rho^{(n/2)-1} (|\nabla h|^2 - n|\nabla H|^2) \geq 0. \end{aligned} \quad (72)$$

Moreover, if $0 \leq \rho \leq n$, then $\rho = 0$ or $\rho = n$. For $\rho = 0$, M is totally umbilical; for $\rho = n$, substituting it into the above equality together with Lemma 19, we obtain $\nabla h = 0$; by Lemma 22, we know that M is a torus; then, by Example 17, we can conclude that M is a classic Willmore torus $W_{m,n-m}$ for some m . $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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