Research Article

Bifurcations of Nontwisted Heteroclinic Loop with Resonant Eigenvalues

Yinlai Jin,¹ Xiaowei Zhu,^{1,2} Zheng Guo,¹ Han Xu,¹ Liqun Zhang,^{1,2} and Benyan Ding^{1,2}

¹ School of Science, Linyi University, Linyi, Shandong 276005, China

² School of Mathematics Science, Shandong Normal University, Jinan 250014, China

Correspondence should be addressed to Xiaowei Zhu; xiaoweizhu01@sina.com

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By using the foundational solutions of the linear variational equation of the unperturbed system along the heteroclinic orbits to establish the local coordinate systems in the small tubular neighborhoods of the heteroclinic orbits, we study the bifurcation problems of nontwisted heteroclinic loop with resonant eigenvalues. The existence, numbers, and existence regions of 1-heteroclinic loop, 1-homoclinic loop, 1-periodic orbit, 2-fold 1-periodic orbit, and two 1-periodic orbits are obtained. Meanwhile, we give the corresponding bifurcation surfaces.

1. Introduction and Hypotheses

With the development of nonlinear science and the deep study of chaotic phenomenon, many experts have been interested in the research on the bifurcation problems of homoclinic and heteroclinic loops for high-dimensional nonlinear dynamical systems (see [1–5] and references therein). Compared with the bifurcations of homoclinic and heteroclinic loops for planar systems, the obtained results have been not rich because of the complexity and the lack of methods. In 1990, Chow et al. studied the codimension 2 homoclinic bifurcation under nondegenerate condition at resonant eigenvalues in [6]. The traditional methods to construct Poincaré maps were adopted to later research. In 1998, Zhu studied the bifurcation problems of nondegenerated homoclinic loops for high-dimensional system $\dot{z} = f(z, \alpha) + \epsilon g(z, \mu, \epsilon)$ in [7] and the bifurcations of nondegenerated heteroclinic loops for high-dimensional system $\dot{z} = f(z) + g(z, \mu)$ in [8]. The methods used in [7, 8] were by generalizing the Floquet method to build the local coordinate systems and Poincaré map. The papers [7, 8] used the inherent characteristic values to describe the bifurcation surfaces and phenomena so that the results had the practicability and maneuverability.

In [9], the authors studied the bifurcations of rough heteroclinic loops with two saddle point. In [10], Tian and Zhu studied the bifurcations of fine heteroclinic loops with two saddle points. The heteroclinic loops studied in [9, 10] were both nontwisted and with no resonant eigenvalues.

In this paper, we study the bifurcation problems of heteroclinic loop with resonant eigenvalues for high-dimensional system $\dot{z} = f(z) + g(z, \mu)$. By simplifying the method in [7, 8], we use the foundational solutions of the linear variational equation of the unperturbed system along the heteroclinic orbits to establish the suitable local coordinate systems in the small tubular neighborhoods of the heteroclinic orbits. Then, we get the Poincaré maps and bifurcation equations by means of the improved method.

Consider the C^r system

$$\dot{z} = f(z), \quad (1)$$

where $r \geq 4$, $z \in \mathbb{R}^{m+n}$. We assume the following.

(H1) Hyperbolic assumption: $z = p_i$ are hyperbolic critical points of (1), $i = 1, 2$. The stable manifold $W_{p_i}^s$ and the unstable manifold $W_{p_i}^u$ of p_i are m -dimensional and n -dimensional, respectively. Moreover, $-\rho_i^1$ and λ_i^1 are the simple real eigenvalues of $D_z f(p_i)$ such that any other

eigenvalue σ of $D_z f(p_i)$ satisfies either $\operatorname{Re} \sigma < -\rho_i^0 < -\rho_i^1 < 0$ or $0 < \lambda_i^1 < \lambda_i^0 < \operatorname{Re} \sigma$, where ρ_i^0 and λ_i^0 are some positive constants.

(H2) Nondegeneration: system (1) has a heteroclinic loop $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i = \{z = r_i(t) : t \in R\}$, $r_i(+\infty) = r_{i+1}(-\infty) = p_{i+1}$, $r_3(t) = r_1(t)$, $p_3 = p_1$. $\dim(T_{r_i(t)} W_{p_i}^u \cap T_{r_i(t)} W_{p_{i+1}}^s) = 1$.

(H3) Strong inclination: $\lim_{x \rightarrow +\infty} (T_{r_i(t)} W_{p_i}^u + T_{r_i(t)} W_{p_{i+1}}^s) = T_{p_{i+1}} W_{p_{i+1}}^{uu} \oplus T_{p_{i+1}} W_{p_{i+1}}^s$, $\lim_{x \rightarrow -\infty} (T_{r_i(t)} W_{p_i}^u + T_{r_i(t)} W_{p_{i+1}}^s) = T_{p_i} W_{p_i}^u \oplus T_{p_i} W_{p_i}^{ss}$, where $W_{p_i}^{uu}$ and $W_{p_i}^{ss}$ are the strong unstable manifolds and the strong stable manifolds, respectively, $T_{p_i} W_{p_i}^{uu}$ is the generalized eigenspace corresponding to all the eigenvalues with larger real part than λ_i^0 , and $T_{p_i} W_{p_i}^{ss}$ is the generalized eigenspace corresponding to all the eigenvalues with smaller real part than $-\rho_i^0$. Denote $e_i^\pm = \lim_{t \rightarrow \mp\infty} \dot{r}_i(t)/|\dot{r}_i(t)|$, where $e_i^+ \in T_{p_i} W_{p_i}^u$ and $e_i^- \in T_{p_{i+1}} W_{p_{i+1}}^s$ are unit eigenvectors corresponding to λ_i^1 and $-\rho_{i+1}^1$, respectively. Furthermore, $\operatorname{span}(T_{p_i} W_{p_i}^{uu}, e_i^+) = T_{p_i} W_{p_i}^u$, $\operatorname{span}(T_{p_i} W_{p_i}^{ss}, e_{i+1}^-) = T_{p_i} W_{p_i}^s$.

Now, we consider the following C^r system:

$$\dot{z} = f(z) + g(z, \mu), \quad (2)$$

where $\mu \in R^l$, $l \geq 2$, $0 \leq |\mu| \ll 1$, $g(p_i, \mu) = g(z, 0) = 0$, $i = 1, 2$.

2. Local Coordinate Systems

Suppose that U_i is a sufficiently small neighborhood of $z = p_i$ and (H1)~(H3) hold; then, for $|\mu|$ small enough, there exists a C^r transformation such that system (2) has the following form in U_i , respectively:

$$\begin{aligned} \dot{x} &= [\lambda_i^1(\mu) + \cdots] x + \text{h.o.t.}, \\ \dot{y} &= [-\rho_i^1(\mu) + \cdots] y + \text{h.o.t.}, \\ \dot{u} &= [B_i^1(\mu) + \cdots] u + \text{h.o.t.}, \\ \dot{v} &= [-B_i^2(\mu) + \cdots] v + \text{h.o.t.}, \end{aligned} \quad (3)$$

where $z = (x, y, u^*, v^*)^*$, $x \in R^1$, $y \in R^1$, $u \in R^{n-1}$, $v \in R^{m-1}$, $*$ means transposition, $\lambda_i^1(0) = \lambda_i^1$, $\rho_i^1(0) = \rho_i^1$, $\operatorname{Re} \sigma(B_i^1(0)) > \lambda_i^0$, and $\operatorname{Re} \sigma(-B_i^2(0)) < -\rho_i^0$. Moreover, we suppose

$$\begin{aligned} W_{p_i}^u &= \{z : y = 0, v = 0\}, \\ W_{p_i}^s &= \{z : x = 0, u = 0\}, \\ W_{p_i}^{uu} &= \{z : x = x(u), y = 0, v = 0\}, \\ W_{p_i}^{ss} &= \{z : x = 0, u = 0, y = y(v)\}, \\ \Gamma \cap W_{p_i}^u &= \{z : u = u(x), y = 0, v = 0\}, \\ \Gamma \cap W_{p_i}^s &= \{z : x = 0, u = 0, v = v(y)\}, \end{aligned} \quad (4)$$

where $x(0) = \dot{x}(0) = 0$, $y(0) = \dot{y}(0) = 0$, $u(0) = \dot{u}(0) = 0$, $v(0) = \dot{v}(0) = 0$.

Let $r_i(t) = (r_i^x(t), r_i^y(t), (r_i^u(t))^*, (r_i^v(t))^*)^*$; we can select some times T_i^0 and T_i^1 such that $r_i(-T_i^0) = (\delta, 0, \delta_u^*, 0^*)^*$, $r_i(T_i^1) = (0, \delta, 0^*, \delta_v^*)^*$, where δ is small enough such that $\{(x, y, u^*, v^*)^* : |x|, |y|, |u|, |v| < 2\delta\} \subset U_i$, $|\delta_u| = O(\delta^\omega)$, $|\delta_v| = O(\delta^\omega)$, $\omega = \min\{\operatorname{Re} \sigma(B_i^2(\mu))/\rho_i^1(\mu), \operatorname{Re} \sigma(B_i^1(\mu))/\lambda_i^1(\mu)\} > 1$.

Consider the linear variational system and its adjoint system

$$\dot{z} = Df(r_i(t))z, \quad (5)$$

$$\dot{\phi} = -(Df(r_i(t)))^* \phi. \quad (6)$$

By [8, 9, 11–13], system (5) and (6) have exponential dichotomies in R^+ and R^- . It follows from [4] that system (5) has a fundamental solution matrix $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$, satisfying

$$z_i^1(t) \in (T_{r_i(t)} W_{p_i}^u \cup T_{r_i(t)} W_{p_{i+1}}^s)^c,$$

$$z_i^2(t) = -\frac{\dot{r}_i(t)}{|\dot{r}_i(T_i^1)|} \in T_{r_i(t)} W_{p_i}^u \cap T_{r_i(t)} W_{p_{i+1}}^s,$$

$$\begin{aligned} z_i^3(t) &= (z_i^{3,1}(t), \dots, z_i^{3,n-1}(t)) \in T_{r_i(t)} W_{p_i}^u \cap (T_{r_i(t)} W_{p_{i+1}}^s)^c \\ &= T_{r_i(t)} W_{p_i}^{uu}, \end{aligned}$$

$$\begin{aligned} z_i^4(t) &= (z_i^{4,1}(t), \dots, z_i^{4,m-1}(t)) \in (T_{r_i(t)} W_{p_i}^u)^c \cap T_{r_i(t)} W_{p_{i+1}}^s \\ &= T_{r_i(t)} W_{p_{i+1}}^{ss}, \end{aligned}$$

$$\begin{aligned} Z_i(-T_i^0) &= \begin{pmatrix} w_i^{11} & w_i^{21} & 0 & w_i^{41} \\ w_i^{12} & 0 & 0 & w_i^{42} \\ w_i^{13} & w_i^{23} & I & w_i^{43} \\ 0 & 0 & 0 & w_i^{44} \end{pmatrix}, \\ Z_i(T_i^1) &= \begin{pmatrix} 1 & 0 & w_i^{31} & 0 \\ 0 & 1 & w_i^{32} & 0 \\ 0 & 0 & w_i^{33} & 0 \\ w_i^{14} & w_i^{24} & w_i^{34} & I \end{pmatrix}, \end{aligned} \quad (7)$$

where $w_i^{21} < 0$, $w_i^{12} \neq 0$, $\|w_i^{33}\| \neq 0$, $\|w_i^{44}\| \neq 0$, and $|w_i^{1j}(w_i^{12})^{-1}| \ll 1$, $j \neq 2$; $|w_i^{2j}(w_i^{21})^{-1}| \ll 1$, $j = 3, 4$; $|w_i^{3j}(w_i^{33})^{-1}| \ll 1$, $j \neq 3$; $|w_i^{4j}(w_i^{44})^{-1}| \ll 1$, $j \neq 4$ for δ small enough.

Denote $\Phi_i(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t)) = (Z_i^{-1}(t))^*$; then $\Phi_i(t)$ is a fundamental solution matrix of (6). It is well known that $\phi_i^1(t)$ is bounded and tends to zero exponentially as $t \rightarrow \pm\infty$ [8, 9, 12].

Denote $\Delta_i = w_i^{12}/|w_i^{12}|$. Γ_i is nontwisted if $\Delta_i = 1$ and twisted if $\Delta_i = -1$. In this paper, we study the bifurcation problems under the nontwisted condition.

FIGURE 1

Let $z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t)$ be the local coordinate systems along Γ_i in the tubular neighborhood of Γ_i .

3. Poincaré Maps and Bifurcation Equations

Denote $h_i(t) = r_i(t) + Z_i(t)N_i$, $N_i = (n_i^1, 0, (n_i^3)^*, (n_i^4)^*)^*$, $n_i^3 = (n_i^{3,1}, \dots, n_i^{3,m-1})^*$, $n_i^4 = (n_i^{4,1}, \dots, n_i^{4,m-1})^*$. Define the Poincaré sections of Γ_i at $t = -T_i^0$ and $t = T_i^1$ by

$$\begin{aligned} S_i^0 &= \{z = h_i(-T_i^0) : |x|, |y|, |u|, |v| < 2\delta\}, \\ S_i^1 &= \{z = h_i(T_i^1) : |x|, |y|, |u|, |v| < 2\delta\}, \end{aligned} \quad (8)$$

respectively, where δ is small enough such that $S_i^0 \subset U_i$, $S_i^1 \subset U_{i+1}$, $U_3 = U_1$ (Figure 1).

Now, we set up the Poincaré map $F_i = F_i^1 \circ F_i^0 : S_{i-1}^1 \mapsto S_i^1$, where $F_i^0 : q_{i-1}^0 \in S_{i-1}^0 \mapsto q_i^0 \in S_i^0$, $F_i^1 : q_i^0 \in S_i^0 \mapsto q_i^1 \in S_i^1$, $S_0^0 = S_2^0$, and $q_0^0 = q_2^0$. Here, F_i^1 is constructed from the flow of (2) in the small tubular neighborhood of Γ_i ; F_i^0 is induced by the flow of the linear approximate system of (3) in the small neighborhood U_i of p_i .

Denote

$$\begin{aligned} q_i^0 &= (x_i^0, y_i^0, (u_i^0)^*, (v_i^0)^*)^* = r_i(-T_i^0) + Z_i(-T_i^0)N_i^0, \\ q_i^1 &= (x_i^1, y_i^1, (u_i^1)^*, (v_i^1)^*)^* = r_i(T_i^1) + Z_i(T_i^1)N_i^1, \end{aligned} \quad (9)$$

where $N_i^0 = (n_i^{0,1}, 0, (n_i^{0,3})^*, (n_i^{0,4})^*)^*$, $N_i^1 = (n_i^{1,1}, 0, (n_i^{1,3})^*, (n_i^{1,4})^*)^*$.

By the expressions of $Z_i(-T_i^0)$ and $Z_i(T_i^1)$, we get that the relations of the two kinds coordinates of q_i^j ($j = 0, 1$) are $x_i^0 \approx \delta$, $y_i^1 \approx \delta$, and

$$\begin{aligned} n_i^{0,1} &= (w_i^{12})^{-1} [y_i^0 - w_i^{42}(w_i^{44})^{-1}v_i^0], \\ n_i^{0,3} &= u_i^0 - \delta_u + [w_i^{11}w_i^{23}(w_i^{21})^{-1} - w_i^{13}](w_i^{12})^{-1}y_i^0 \\ &\quad + \{w_i^{13}(w_i^{12})^{-1}w_i^{42} - w_i^{43} - w_i^{23}(w_i^{21})^{-1} \\ &\quad \times [w_i^{11}(w_i^{12})^{-1}w_i^{42} - w_i^{41}]\}(w_i^{44})^{-1}v_i^0, \end{aligned}$$

$$n_i^{0,4} = (w_i^{44})^{-1}v_i^0,$$

$$n_i^{1,1} = x_i^1 - w_i^{31}(w_i^{33})^{-1}u_i^1,$$

$$n_i^{1,3} = (w_i^{33})^{-1}u_i^1,$$

$$\begin{aligned} n_i^{1,4} &= -w_i^{14}x_i^1 + (w_i^{14}w_i^{31} + w_i^{24}w_i^{32} - w_i^{34})(w_i^{33})^{-1}u_i^1 \\ &\quad + v_i^1 - \delta_v. \end{aligned} \quad (10)$$

First, we built the map $F_i^1 : S_i^0 \mapsto S_i^1, q_i^0 \mapsto q_i^1$. We make a coordinate transformation $z = h_i(t)$, $t \in [-T_i^0, T_i^1]$ and substitute it into (2). By $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$ and $\Phi_i^*(t)Z_i(t) = I$; (2) becomes

$$\dot{n}_i^j = \phi_i^{j*}(t)g_\mu(r_i(t), 0)\mu + \text{h.o.t.}, \quad j = 1, 3, 4. \quad (11)$$

Thus, we define $F_i^1 : S_i^0 \mapsto S_i^1, q_i^0 \mapsto q_i^1$ as

$$n_i^j(T_i^1) = n_i^j(-T_i^0) + M_i^j\mu + \text{h.o.t.}, \quad j = 1, 3, 4. \quad (12)$$

That is,

$$n_i^{1,j} = n_i^{0,j} + M_i^j\mu + \text{h.o.t.}, \quad j = 1, 3, 4, \quad (13)$$

where $M_i^j = \int_{-T_i^0}^{T_i^1} \phi_i^{j*}(t)g_\mu(r_i(t), 0)dt$, $i = 1, 2$, $j = 1, 3, 4$, are called the Melnikov vectors.

By [7-9, 14], $M_i^j = \int_{-T_i^0}^{T_i^1} \phi_i^{j*}(t)g_\mu(r_i(t), 0)dt = \int_{-\infty}^{+\infty} \phi_i^{j*}(t)g_\mu(r_i(t), 0)dt$, $i = 1, 2$, $j = 1, 3, 4$.

Second, we built the map $F_i^0 : S_{i-1}^1 \mapsto S_i^0, q_{i-1}^1 \mapsto q_i^0$, $S_0^0 = S_2^0$, $q_0^0 = q_2^0$. Without loss of generality, we assume $\rho_i^1 \geq \lambda_i^1$. Let τ_i be the flying time from q_{i-1}^1 to q_i^0 ; we say $s_i = e^{-\lambda_i^1(\mu)\tau_i}$ is Silnikov time. By (3) and neglecting the higher order terms, we have

$$\begin{aligned} x_{i-1}^1 &\approx s_i x_i^0, & y_i^0 &\approx s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} y_{i-1}^1, \\ u_{i-1}^1 &\approx s_i^{B_i^1(\mu)/\lambda_i^1(\mu)} u_i^0, & v_i^0 &\approx s_i^{B_i^2(\mu)/\lambda_i^1(\mu)} v_{i-1}^1. \end{aligned} \quad (14)$$

Here, (s_i, u_i^0, v_{i-1}^1) is called the Silnikov coordinate.

Last, it follows from (10)~(14) that we get the expression of Poincaré map $F_i = F_i^1 \circ F_i^0$ as follows:

$$\begin{aligned} n_i^{1,1} &= (w_i^{12})^{-1} \delta s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} + M_i^1\mu + \text{h.o.t.}, \\ n_i^{1,3} &= u_i^0 - \delta_u + [w_i^{11}w_i^{23}(w_i^{21})^{-1} - w_i^{13}](w_i^{12})^{-1} \delta s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} \\ &\quad + M_i^3\mu + \text{h.o.t.}, \\ n_i^{1,4} &= (w_i^{44})^{-1} s_i^{B_i^2(\mu)/\lambda_i^1(\mu)} v_{i-1}^1 + M_i^4\mu + \text{h.o.t.} \end{aligned} \quad (15)$$

Denote $G_i(q_{i-1}^1) = (G_i^1, G_i^3, G_i^4) = F_i(q_{i-1}^1) - q_i^1$. By (10) and (15), we get the successor function G_i as follows:

$$\begin{aligned} G_i^1 &= \delta \left[(w_i^{12})^{-1} s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} - s_{i+1} \right] + M_i^1 \mu + \text{h.o.t.}, \\ G_i^3 &= u_i^0 - \delta_u + \left[w_i^{11} w_i^{23} (w_i^{21})^{-1} - w_i^{13} \right] (w_i^{12})^{-1} \delta s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} \\ &\quad - (w_i^{33})^{-1} s_{i+1}^{B_{i+1}^1(\mu)/\lambda_{i+1}^1(\mu)} u_{i+1}^0 + M_i^3 \mu + \text{h.o.t.}, \\ G_i^4 &= -v_i^1 + \delta_v + w_i^{14} \delta s_{i+1} + (w_i^{44})^{-1} s_i^{B_i^2(\mu)/\lambda_i^1(\mu)} v_{i-1}^1 \\ &\quad + M_i^4 \mu + \text{h.o.t.} \end{aligned} \quad (16)$$

Let

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0. \quad (17)$$

Equation (17) is called the bifurcation equation.

4. The Preservations of Heteroclinic Orbits and Bifurcations of Homoclinic Loops

In this section, we discuss the nontwisted heteroclinic loop with resonant eigenvalues. Without loss of generality, we assume the following.

(H4) Resonant condition: $\beta_1 = \rho_1^1/\lambda_1^1 > 1$, $\beta_2 = \rho_2^1/\lambda_2^1 = 1$.

Let $\beta_i(\mu) = \rho_i^1(\mu)/\lambda_i^1(\mu)$, $\beta_i = \beta_i(0)$, $\rho_2^1(\mu) = (1 + \alpha(\mu))\lambda_2^1(\mu)$. By (H4) and the continuity of function, we get $\beta_1(\mu) > 1$, $\beta_2(\mu) = 1 + \alpha(\mu)$, $|\alpha(\mu)| \ll 1$, $|\alpha(0)| = 0$, and $\beta_1(\mu)\beta_2(\mu) > 1$, for $|\mu| = 1$.

Consider the solution of (17). Clearly, by the implicit function theorem, the equation $(G_1^3, G_1^4, G_2^3, G_2^4) = 0$ has a unique solution $u_i^0 = u_i^0(s_1, s_2, \mu)$, $v_i^1 = v_i^1(s_1, s_2, \mu)$, $i = 1, 2$. Substituting it into $(G_1^1, G_2^1) = 0$, we have

$$\begin{aligned} \delta \left[(w_1^{12})^{-1} s_1^{\beta_1(\mu)} - s_2 \right] + M_1^1 \mu + \text{h.o.t.} &= 0, \\ \delta \left[(w_2^{12})^{-1} s_2^{1+\alpha(\mu)} - s_1 \right] + M_2^1 \mu + \text{h.o.t.} &= 0. \end{aligned} \quad (18)$$

Equation (18) is equivalent to

$$\begin{aligned} s_1 &= (w_2^{12})^{-1} s_2^{1+\alpha(\mu)} + \delta^{-1} M_2^1 \mu + \text{h.o.t.}, \\ s_2 &= (w_1^{12})^{-1} s_1^{\beta_1(\mu)} + \delta^{-1} M_1^1 \mu + \text{h.o.t.} \end{aligned} \quad (19)$$

Now, we study the existence of heteroclinic orbits and the 1-homoclinic bifurcations. Denote

$$\begin{aligned} R_1^2 &= \{ \mu : M_1^1 \mu > 0, w_2^{12} M_2^1 \mu < 0, |\mu| \ll 1 \}, \\ R_2^1 &= \{ \mu : M_2^1 \mu > 0, w_1^{12} M_1^1 \mu < 0, |\mu| \ll 1 \}. \end{aligned} \quad (20)$$

Theorem 1. Suppose that hypotheses (H1)~(H4) are valid, and rank $(M_1^1, M_2^1) = 2$; then one has the following.

(i) There exists a $(l-1)$ -dimensional surface L_i with a normal vector M_i^1 at $\mu = 0$ such that (2) has a heteroclinic orbit joining p_1 and p_2 near Γ_i if and only if $\mu \in L_i$, $|\mu| \ll 1$, where $i = 1, 2$. Moreover, (2) has a 1-heteroclinic loop near Γ if and only if $\mu \in L_{12} = L_1 \cap L_2$ and $|\mu| \ll 1$, where L_{12} is $(l-2)$ -dimensional surface and $0 \in L_{12}$. That is, heteroclinic loop Γ is persistent.

(ii) There exists a $(l-1)$ -dimensional surface $L_1^2 \subset R_1^2$ which is tangent to $L_2(L_1)$ at $\mu = 0$ as $\alpha(\mu) > 0$ ($\alpha(\mu) < 0$), such that (2) has a unique 1-homoclinic loop Γ_1^2 homoclinic to p_1 near Γ as $\mu \in R_1^2 \cap L_1^2$ and $|\mu| \ll 1$. Meanwhile, there also exists a $(l-1)$ -dimensional surface $L_2^2 \subset R_2^2$ which is tangent to L_1 at $\mu = 0$ such that (2) has a unique 1-homoclinic loop Γ_2^2 homoclinic to p_2 near Γ as $\mu \in R_2^2 \cap L_2^2$ and $|\mu| \ll 1$.

Proof. (i) If (19) has a solution $s_1 = s_2 = 0$, then, we have

$$M_i^1 \mu + \text{h.o.t.} = 0, \quad i = 1, 2. \quad (21)$$

If $M_i^1 \neq 0$, then, there exists a $(l-1)$ -dimensional surface L_i defined by (21) with a normal vector M_i^1 at $\mu = 0$ such that the i th equation of (19) has a solution $s_1 = s_2 = 0$ as $\mu \in L_i$ and $|\mu| \ll 1$. That is, Γ_i is persistent.

If rank $(M_1^1, M_2^1) = 2$, then there exists a $(l-2)$ -dimensional surface $L_{12} = L_1 \cap L_2$ such that (19) has a solution $s_1 = s_2 = 0$ as $\mu \in L_{12}$ and $|\mu| \ll 1$. That is, Γ is persistent.

(ii) If (19) has a solution $s_1 = 0, s_2 > 0$, then, we have

$$\begin{aligned} s_2 &= \delta^{-1} M_1^1 \mu + \text{h.o.t.}, \\ (\delta^{-1} M_1^1 \mu + \text{h.o.t.})^{1+\alpha(\mu)} &= -\delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.} \end{aligned} \quad (22)$$

If $M_1^1 \mu > 0, w_2^{12} M_2^1 \mu < 0$, then, (22) defined a $(l-1)$ -dimensional surface L_1^2 with a normal vector $M_2^1(M_1^1)$ at $\mu = 0$ as $\alpha(\mu) > 0$ ($\alpha(\mu) < 0$), such that (19) has a solution $s_1 = 0, s_2 > 0$ as $\mu \in L_1^2$ and $|\mu| \ll 1$. That is, the system (2) has a 1-homoclinic orbit to p_1 near Γ for $\mu \in L_1^2$ and $|\mu| \ll 1$.

(iii) If (19) has a solution $s_1 > 0, s_2 = 0$, then, we have

$$\begin{aligned} s_1 &= \delta^{-1} M_2^1 \mu + \text{h.o.t.}, \\ (\delta^{-1} M_2^1 \mu + \text{h.o.t.})^{\beta_1(\mu)} &= -\delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.} \end{aligned} \quad (23)$$

If $M_2^1 \mu > 0, w_1^{12} M_1^1 \mu < 0$, then, (23) defined a $(l-1)$ -dimensional surface L_2^2 with a normal vector M_1^1 at $\mu = 0$ such that (19) has a solution $s_1 > 0, s_2 = 0$ as $\mu \in L_2^2$ and $|\mu| \ll 1$. That is, the system (2) has a 1-homoclinic orbit to p_2 near Γ for $\mu \in L_2^2$ and $|\mu| \ll 1$. $\square$

5. The Periodic Orbits Bifurcations

Next, we consider the bifurcation problems of 1-periodic orbits near Γ . In other words, we study the solutions of (19) satisfying $s_1 > 0, s_2 > 0$. We assume the following.

(H5) Nontwisted condition: $\Delta_1 = \Delta_2 = 1$.

Obviously, if (H5) holds, we have

$$\begin{aligned} R_1^2 &= \{\mu : M_1^1 \mu > 0, M_2^1 \mu < 0, |\mu| \ll 1\}, \\ R_2^1 &= \{\mu : M_1^1 \mu < 0, M_2^1 \mu > 0, |\mu| \ll 1\}. \end{aligned} \quad (24)$$

At first, we consider the bifurcations in R_1^2 . From (19), we get

$$\begin{aligned} &(s_1^{\beta_1(\mu)} + \delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.})^{1+\alpha(\mu)} \\ &= (w_1^{12})^{1+\alpha(\mu)} w_2^{12} (s_1 - \delta^{-1} M_2^1 \mu + \text{h.o.t.}). \end{aligned} \quad (25)$$

Denote $V_1(s_1)$ and $N_1(s_1)$ are the left and right hands of (25), respectively; then, we have the following.

Lemma 2. Suppose that hypotheses (H1)~(H5) are valid; then $V_1(s_1)$ is tangent to $N_1(s_1)$ at some point s_1 satisfying $0 < s_1$, $|\mu| \ll 1$ if and only if $\mu \in R_1^2$, $\alpha(\mu) > 0$, $|M_1^1 \mu| \ll |M_2^1 \mu|$, and

$$\begin{aligned} &\delta^{-1} M_1^1 \mu \\ &= \left(-\frac{M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{-1/\alpha(\mu)} - (w_1^{12})^{-1} \\ &\times \left(-\frac{w_1^{12} M_1^1 \mu}{(1 + \alpha(\mu)) \beta_1(\mu) M_2^1 \mu} \right)^{\beta_1(\mu)/(\beta_1(\mu)-1)} \\ &+ \text{h.o.t.} \end{aligned} \quad (26)$$

Proof. $V_1(s_1)$ is tangent to $N_1(s_1)$ at some point s_1 if and only if $V_1(s_1) = N_1(s_1)$ and $\dot{V}_1(s_1) = \dot{N}_1(s_1)$; that is,

$$\begin{aligned} &(s_1^{\beta_1(\mu)} + \delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.})^{1+\alpha(\mu)} \\ &= (w_1^{12})^{1+\alpha(\mu)} w_2^{12} (s_1 - \delta^{-1} M_2^1 \mu + \text{h.o.t.}), \\ &(1 + \alpha(\mu)) \beta_1(\mu) s_1^{\beta_1(\mu)-1} (s_1^{\beta_1(\mu)} + \delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.})^{\alpha(\mu)} \\ &= (w_1^{12})^{1+\alpha(\mu)} w_2^{12}. \end{aligned} \quad (27)$$

So

$$s_1 = \left(-\frac{w_1^{12} M_1^1 \mu}{(1 + \alpha(\mu)) \beta_1(\mu) M_2^1 \mu} \right)^{1/(\beta_1(\mu)-1)} + \text{h.o.t.} \quad (28)$$

Substituting (28) into the second expression of (27), we have that (26) holds.

Equation (28) means $|M_1^1 \mu| \ll \|M_2^1 \mu\|$, and (26) means $\mu \in R_1^2$ and $\alpha(\mu) > 0$. $\square$

If M_1^1 and M_2^1 are linearly independent, then there exists a $(l-1)$ -dimensional surface $\tilde{L}_1^2$ defined by (26) in the small

neighborhood of $\mu = 0$. It is easy to know that $\tilde{L}_1^2 \in R_1^2$ is tangent to L_2 for $\alpha(\mu) > 0$. By (21), (22), and (26), we get

$$\begin{aligned} \delta^{-1} M_1^1 \mu|_{\tilde{L}_1^2} &= \left(-\frac{M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{-1/\alpha(\mu)} + \text{h.o.t.} \\ &> \left(-\frac{M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{-1/\alpha(\mu)} - (w_1^{12})^{-1} \\ &\times \left(-\frac{w_1^{12} M_1^1 \mu}{(1 + \alpha(\mu)) \beta_1(\mu) M_2^1 \mu} \right)^{\beta_1(\mu)/(\beta_1(\mu)-1)} \\ &+ \text{h.o.t.} \\ &= \delta^{-1} M_1^1 \mu|_{\tilde{L}_1^2} > 0 = \delta^{-1} M_1^1 \mu|_{L_1}. \end{aligned} \quad (29)$$

That is to say, $\tilde{L}_1^2$ is located in the open region between L_1 and L_2 . Thus, we have the following.

(1) For $\alpha(\mu) > 0$, we define the following three open sections in R_1^2 : $(R_1^2)_1$ is bounded by L_2 and L_1^2 , $(R_1^2)_2$ is bounded by L_1^2 and $\tilde{L}_1^2$, and $(R_1^2)_0$ is bounded by $\tilde{L}_1^2$ and L_1 , and they have nonempty intersection with R_1^2 .

(2) For $\alpha(\mu) < 0$, we define the following two open sections in R_1^2 : $(R_1^2)_1$ is bounded by L_2 and L_1^2 and $(R_1^2)_0$ is bounded by L_1^2 and L_1 , and they have nonempty intersection with R_1^2 .

Now, we consider the nonnegative solutions of $V_1(s_1) = N_1(s_1)$ defined by (25). By $V_1(s_1) = (s_1^{\beta_1(\mu)} + \delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.})^{1+\alpha(\mu)}$, $N_1(s_1) = (w_1^{12})^{1+\alpha(\mu)} w_2^{12} (s_1 - \delta^{-1} M_2^1 \mu + \text{h.o.t.})$, we get $V_1(0) = (\delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.})^{1+\alpha(\mu)}$, $N_1(0) = (w_1^{12})^{1+\alpha(\mu)} w_2^{12} (-\delta^{-1} M_2^1 \mu + \text{h.o.t.})$. From (22), $V_1(0) = N_1(0)$ holds for $\mu \in L_1^2$, $|\mu| \ll 1$.

By Lemma 2, it is easy to get the following conclusions.

(1) For $\alpha(\mu) > 0$, we have the following (see Figure 2).

If $\mu \in (R_1^2)_1$, then $V_1(s_1) = N_1(s_1)$ has one small positive solution.

If $\mu \in L_1^2$, then $V_1(s_1) = N_1(s_1)$ has one small positive solution and one zero solution.

If $\mu \in (R_1^2)_2$, then $V_1(s_1) = N_1(s_1)$ has two small different positive solutions.

If $\mu \in \tilde{L}_1^2$, then $V_1(s_1) = N_1(s_1)$ has one small twofold positive solution.

If $\mu \in (R_1^2)_0$, then $V_1(s_1) = N_1(s_1)$ does not have any small nonnegative solutions.

(2) For $\alpha(\mu) < 0$, we have the following.

If $\mu \in (R_1^2)_1$, then $V_1(s_1) = N_1(s_1)$ has one small positive solution.

If $\mu \in L_1^2$, then $V_1(s_1) = N_1(s_1)$ has one zero solution.

If $\mu \in (R_1^2)_0$, then $V_1(s_1) = N_1(s_1)$ has not any small nonnegative solutions.

FIGURE 2

Thus, we have shown the following conclusions.

Theorem 3. Suppose that hypotheses (H1)~(H5) are valid, $\mu \in R_1^2$, $\alpha(\mu) > 0$; then the following conclusions are true.

- (i) System (2) has one simple 1-periodic orbit near Γ as $\mu \in (R_1^2)_1$.
- (ii) System (2) has one simple 1-periodic orbit and one 1-homoclinic loop homoclinic to p_1 near Γ as $\mu \in L_1^2$.
- (iii) System (2) has two simple 1-periodic orbits near Γ as $\mu \in (R_1^2)_2$.
- (iv) System (2) has a unique twofold 1-periodic orbit near Γ as $\mu \in \tilde{L}_1^2$.
- (v) System (2) has not any 1-periodic orbits and 1-homoclinic loops near Γ as $\mu \in (R_1^2)_0$.

Theorem 4. Suppose that hypotheses (H1)~(H5) are valid, $\mu \in R_1^2$, $\alpha(\mu) < 0$; then the following conclusions are true.

- (i) System (2) has one simple 1-periodic orbit near Γ as $\mu \in (R_1^2)_1$.
- (ii) System (2) has one 1-homoclinic loop homoclinic to p_1 near Γ as $\mu \in L_1^2$.
- (iii) System (2) does not have any 1-periodic orbits and 1-homoclinic loops near Γ as $\mu \in (R_1^2)_0$.

Next, we consider the bifurcations in R_1^1 . Similarly, from (19), we get

$$\begin{aligned} & (s_2^{1+\alpha(\mu)} + \delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1(\mu)} \\ &= w_1^{12} (w_2^{12})^{\beta_1(\mu)} (s_2 - \delta^{-1} M_1^1 \mu + \text{h.o.t.}). \end{aligned} \quad (30)$$

Denote $V_2(s_2)$ and $N_2(s_2)$ are the left and right hands of (30), respectively; then one has the following.

Lemma 5. Suppose that hypotheses (H1)~(H5) are valid, then $V_2(s_2)$ is tangent to $N_2(s_2)$ at some point s_2 satisfying $0 < s_2$, $|\mu| \ll 1$ if and only if $\alpha(\mu) < 0$, $\|M_1^1 \mu\| \ll \|M_2^1 \mu\|$, $\mu \in R_1^1$ and

$$\begin{aligned} & \delta^{-1} M_2^1 \mu \\ &= \left(-\frac{w_1^{12} M_1^1 \mu}{M_2^1 \mu} \right)^{1/(\beta_1(\mu)-1)} - (w_2^{12})^{-1} \\ & \times \left(-\frac{(1+\alpha(\mu))\beta_1(\mu)M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{-(1+\alpha(\mu))/\alpha(\mu)} \\ &+ \text{h.o.t.} \end{aligned} \quad (31)$$

Proof. $V_2(s_2)$ is tangent to $N_2(s_2)$ at some point s_2 if and only if $V_2(s_2) = N_2(s_2)$ and $\dot{V}_2(s_2) = \dot{N}_2(s_2)$; that is,

$$\begin{aligned} & (s_2^{1+\alpha(\mu)} + \delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1(\mu)} \\ &= w_1^{12} (w_2^{12})^{\beta_1(\mu)} (s_2 - \delta^{-1} M_1^1 \mu + \text{h.o.t.}), \\ & (1+\alpha(\mu))\beta_1(\mu) s_2^{\alpha(\mu)} (s_2^{1+\alpha(\mu)} + \delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1(\mu)-1} \\ &= w_1^{12} (w_2^{12})^{\beta_1(\mu)} + \text{h.o.t.} \end{aligned} \quad (32)$$

The solution of (32) is

$$s_2 = \left(-\frac{w_2^{12} M_2^1 \mu}{(1+\alpha(\mu))\beta_1(\mu)M_1^1 \mu} \right)^{1/\alpha(\mu)} + \text{h.o.t.} \quad (33)$$

Substituting (33) into the second expression of (32), we obtain (31).

Equation (33) holds means $|M_1^1 \mu| \ll \|M_2^1 \mu\|$, $\alpha(\mu) < 0$, or $|M_2^1 \mu| \ll \|M_1^1 \mu\|$, $\alpha(\mu) > 0$. But, if $|M_2^1 \mu| \ll \|M_1^1 \mu\|$, $\alpha(\mu) > 0$, then, by (31), we get that $\delta^{-1} M_2^1 \mu$ will be tented to ∞ . Last, $\mu \in R_1^1$ is obvious. $\square$

If M_1^1 and M_2^1 are linearly independent, then there exists a $(l-1)$ -dimensional surface $\tilde{L}_2^1$ defined by (31) in the small

neighborhood of $\mu = 0$. It is easy to know that $\tilde{L}_2^1$ is tangent to L_1 at $\mu = 0$. By (21), (23), and (31), we get

$$\begin{aligned} \delta^{-1} M_2^1 \mu|_{L_2^1} &= \left(-\frac{w_1^{12} M_1^1 \mu}{M_2^1 \mu} \right)^{1/(\beta_1(\mu)-1)} + \text{h.o.t.} \\ &> \left(-\frac{w_1^{12} M_1^1 \mu}{M_2^1 \mu} \right)^{1/(\beta_1(\mu)-1)} - (w_2^{12})^{-1} \\ &\quad \times \left(-\frac{(1 + \alpha(\mu))\beta_1(\mu)M_1^1 \mu}{w_2^{12} M_2^1 \mu} \right)^{-(1+\alpha(\mu))/\alpha(\mu)} \\ &\quad + \text{h.o.t.} \\ &= \delta^{-1} M_2^1 \mu|_{\tilde{L}_2^1} > 0 = \delta^{-1} M_2^1 \mu|_{L_2^1}. \end{aligned} \quad (34)$$

That is, $\tilde{L}_2^1$ is located in the open region between L_2 and L_2^1 . Thus, we have the following.

(1) For $\alpha(\mu) > 0$, we define the following two open sections: $(R_2^1)_1$ is bounded by L_1 and L_2^1 and $(R_2^1)_0$ is bounded by L_2^1 and L_2 , and they have nonempty intersection with R_2^1 .

(2) For $\alpha(\mu) < 0$, we define the following three open sections: $(R_2^1)_1$ is bounded by L_1 and L_2^1 , $(R_2^1)_2$ is bounded by L_2^1 and $\tilde{L}_2^1$, and $(R_2^1)_0$ is bounded by $\tilde{L}_2^1$ and L_2 , and they have nonempty intersection with R_2^1 .

Now, we consider the nonnegative solutions of $V_2(s_2) = N_2(s_2)$ defined by (30). By $V_2(s_2) = (s_2^{1+\alpha(\mu)} + \delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1(\mu)}$, $N_2(s_2) = w_1^{12} (w_2^{12})^{\beta_1(\mu)} (s_2 - \delta^{-1} M_1^1 \mu + \text{h.o.t.})$, we get $V_2(0) = (\delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1(\mu)}$, $N_2(0) = w_1^{12} (w_2^{12})^{\beta_1(\mu)} (-\delta^{-1} M_1^1 \mu + \text{h.o.t.})$. From (23), $V_2(0) = N_2(0)$ holds for $\mu \in L_2^1$, $|\mu| \ll 1$.

By Lemma 5, it is easy to get the following conclusions.

(1) For $\alpha(\mu) > 0$, we have the following.

If $\mu \in (R_2^1)_1$, then $V_2(s_2) = N_2(s_2)$ has one small positive solution.

If $\mu \in L_2^1$, then $V_2(s_2) = N_2(s_2)$ has one zero solution.

If $\mu \in (R_2^1)_0$, then $V_2(s_2) = N_2(s_2)$ does not have any small nonnegative solutions.

(2) For $\alpha(\mu) < 0$, we have the following.

If $\mu \in (R_2^1)_1$, then $V_2(s_2) = N_2(s_2)$ has one small positive solution.

If $\mu \in L_2^1$, then $V_2(s_2) = N_2(s_2)$ has one small positive solution and one zero solution.

If $\mu \in (R_2^1)_2$, then $V_2(s_2) = N_2(s_2)$ has two different positive solutions.

If $\mu \in \tilde{L}_2^1$, then $V_2(s_2) = N_2(s_2)$ has one twofold positive solution.

If $\mu \in (R_2^1)_0$, then $V_2(s_2) = N_2(s_2)$ does not have any small nonnegative solutions.

Thus, we have shown the following conclusions.

Theorem 6. Suppose that hypotheses (H1)~(H5) are valid, $\mu \in R_2^1$, $\alpha(\mu) > 0$; then the following conclusions are true.

- (i) System (2) has one simple 1-periodic orbit near Γ as $\mu \in (R_2^1)_1$.
- (ii) System (2) has one 1-homoclinic loop homoclinic to p_2 near Γ as $\mu \in L_2^1$.
- (iii) System (2) does not have any 1-periodic orbits and 1-homoclinic loops near Γ as $\mu \in (R_2^1)_0$.

Theorem 7. Suppose that hypotheses (H1)~(H5) are valid, $\mu \in R_2^1$, $\alpha(\mu) < 0$; then the following conclusions are true.

- (i) System (2) has one simple 1-periodic orbit near Γ as $\mu \in (R_2^1)_1$.
- (ii) System (2) has one simple 1-periodic orbit and one 1-homoclinic loop homoclinic to p_2 near Γ as $\mu \in L_2^1$.
- (iii) System (2) has two simple 1-periodic orbits near Γ as $\mu \in (R_2^1)_2$.
- (iv) System (2) has a unique twofold 1-periodic orbit near Γ as $\mu \in \tilde{L}_2^1$.
- (v) System (2) does not have any 1-periodic orbits and 1-homoclinic loops near Γ as $\mu \in (R_2^1)_0$.

Let D_1^2 be an open region that is bounded by L_1 and L_2 ; meanwhile, $D_1^2 \cap \{\mu : M_1^1 \mu > 0, M_2^1 \mu > 0, |\mu| \ll 1\} \neq \emptyset$. Let D_2^1 be an open region that is bounded by L_2 and L_1 ; meanwhile, $D_2^1 \cap \{\mu : M_1^1 \mu < 0, M_2^1 \mu < 0, |\mu| \ll 1\} \neq \emptyset$.

By (19), it is easy to see that (19) has a unique solution $s_1 > 0, s_2 > 0$ for $\mu \in D_1^2$ and does not have any nonnegative solutions for $\mu \in D_2^1$. So we get the following conclusions.

Theorem 8. Supposing that hypotheses (H1)~(H5) are valid, then

- (i) system (2) has one simple 1-periodic orbit near Γ as $\mu \in D_1^2$,
- (ii) system (2) has not any 1-periodic orbits, 1-homoclinic loops, and heteroclinic orbits near Γ as $\mu \in D_2^1$.

Combining Theorems 1~8, we get the bifurcation figures (see Figures 3 and 4).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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FIGURE 3

FIGURE 4

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