Research Article

Generalizations on Some Hermite-Hadamard Type Inequalities for Differentiable Convex Functions with Applications to Weighted Means

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Some new Hermite-Hadamard type inequalities for differentiable convex functions were presented by Xi and Qi. In this paper, we present new generalizations on the Xi-Qi inequalities.

1. Introduction

The Hermite-Hadamard inequality [1–3] states that if f is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Below we recall some Hermite-Hadamard type inequalities.

In 1998, Dragomir and Agarwal [4] showed that (i) if $|f'(x)|$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a) \left(|f'(a)| + |f'(b)| \right)}{8}, \quad (2)$$

and (ii) if $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$ with $p > 1$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2(p+1)^{1/p}} \\ & \quad \times \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right)^{(p-1)/p}. \end{aligned} \quad (3)$$

In 2000, Pearce and Pečarić [5] showed that if $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q},$$

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}. \quad (4)$$

In 2004, Kirmaci [6] showed that if $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$ with $p > 1$, then

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left(\frac{4}{p+1} \right)^{1/p} (|f'(a)| + |f'(b)|), \\ & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} \\ & \quad \times \left(|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)} \right)^{(p-1)/p} \\ & \quad + \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} \\ & \quad \times \left(3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right)^{(p-1)/p}. \end{aligned} \quad (5)$$

In 2010, Sarikaya et al. [7] showed that if $f' \in L[a, b]$ and $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{12} \left(\frac{2^{q+1} + 1}{3(q+1)} \right)^{1/q} \\ & \quad \times \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{1/q} \right], \end{aligned}$$

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{5(b-a)}{72} \\ & \quad \times \left[\left(\frac{61|f'(a)|^q + 29|f'(b)|^q}{90} \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{29|f'(a)|^q + 61|f'(b)|^q}{90} \right)^{1/q} \right]. \end{aligned} \quad (6)$$

In 2012, Xi and Qi [8] showed that if $\lambda, \mu \in [0, 1]$ and if $f' \in L[a, b]$ and $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} (1-2\lambda+2\lambda^2)^{1-1/q} \\ & \quad \times \left[((4-9\lambda+12\lambda^2-2\lambda^3)|f'(a)|^q \right. \\ & \quad \left. + (2-3\lambda+2\lambda^3)|f'(b)|^q) \times (6)^{-1} \right]^{1/q} \\ & \quad + \frac{b-a}{8} (1-2\mu+2\mu^2)^{1-1/q} \\ & \quad \times \left[((2-3\mu+2\mu^3)|f'(a)|^q \right. \\ & \quad \left. + (4-9\mu+12\mu^2-2\mu^3)|f'(b)|^q) \times (6)^{-1} \right]^{1/q}, \\ & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{2(q+1)(q+2)} \right)^{1/q} \\ & \quad \times \left[((q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1})|f'(a)|^q \right. \\ & \quad \left. + ((q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2})|f'(b)|^q \right]^{1/q} \\ & \quad + \frac{b-a}{4} \left(\frac{1}{2(q+1)(q+2)} \right)^{1/q} \\ & \quad \times \left[((q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2})|f'(a)|^q \right. \\ & \quad \left. + ((q+3-\mu)(1-\mu)^{q+1} + (2q+4-\mu)\mu^{q+1}) \right. \\ & \quad \left. \times |f'(b)|^q \right]^{1/q}. \end{aligned} \quad (7)$$

Moreover, for other results involving the Hermite-Hadamard type inequalities, we also refer to [9–23].

In this paper, we generalize the Xi-Qi inequalities.

2. Preliminaries

Lemma 1. Let $\lambda, \mu \in \mathbb{R}$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. Then

$$\begin{aligned} & \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \\ & + [\epsilon(1 - \lambda) + (b - a - \epsilon)(1 - \mu)] f(a + \epsilon) - \int_a^b f(x) dx \\ & = \int_0^1 [\epsilon^2(1 - \lambda - t) f'(ta + (1 - t)(a + \epsilon)) \\ & \quad + (b - a - \epsilon)^2(\mu - t)] \\ & \quad \times f'(t(a + \epsilon) + (1 - t)b) dt. \end{aligned} \quad (8)$$

Proof. Integrating by part and changing variable, we have

$$\begin{aligned} & \int_0^1 \epsilon^2(1 - \lambda - t) f'(ta + (1 - t)(a + \epsilon)) dt \\ & = \int_{t=0}^{t=1} \epsilon(\lambda + t - 1) df(ta + (1 - t)(a + \epsilon)) \\ & = [\epsilon(\lambda + t - 1) f(ta + (1 - t)(a + \epsilon))]_{t=0}^{t=1} \\ & \quad - \epsilon \int_0^1 f(ta + (1 - t)(a + \epsilon)) dt \\ & = \epsilon \lambda f(a) + \epsilon(1 - \lambda) f(a + \epsilon) - \int_a^{a+\epsilon} f(x) dx, \\ & \int_0^1 (b - a - \epsilon)^2(\mu - t) f'(t(a + \epsilon) + (1 - t)b) dt \\ & = \int_{t=0}^{t=1} (b - a - \epsilon)(t - \mu) df(t(a + \epsilon) + (1 - t)b) \\ & = [(b - a - \epsilon)(t - \mu) f(t(a + \epsilon) + (1 - t)b)]_{t=0}^{t=1} \\ & \quad - (b - a - \epsilon) \int_0^1 f(t(a + \epsilon) + (1 - t)b) dt \\ & = (b - a - \epsilon)(1 - \mu) f(a + \epsilon) \\ & \quad + (b - a - \epsilon) \mu f(b) - \int_{a+\epsilon}^b f(x) dx. \end{aligned} \quad (9)$$

Thus,

$$\begin{aligned} & \int_0^1 [\epsilon^2(1 - \lambda - t) f'(ta + (1 - t)(a + \epsilon)) \\ & \quad + (b - a - \epsilon)^2(\mu - t)] f'(t(a + \epsilon) + (1 - t)b) dt \\ & = \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \\ & \quad + [\epsilon(1 - \lambda) + (b - a - \epsilon)(1 - \mu)] \\ & \quad \times f(a + \epsilon) - \int_a^b f(x) dx. \end{aligned} \quad (10)$$

Lemma 2 (see [8]). Let $s > 0$ and $0 \leq \xi \leq 1$. Then

$$\begin{aligned} \int_0^1 |\xi - t|^s dt &= \frac{\xi^{s+1} + (1 - \xi)^{s+1}}{s + 1}, \\ \int_0^1 t |\xi - t|^s dt &= \frac{\xi^{s+2} + (s + 1 + \xi)(1 - \xi)^{s+1}}{(s + 1)(s + 2)}. \end{aligned} \quad (11)$$

3. Main Results

Theorem 3. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \right. \\ & \quad + [\epsilon(1 - \lambda) + (b - a - \epsilon)(1 - \mu)] \\ & \quad \times f(a + \epsilon) - \int_a^b f(x) dx \left| \leq \epsilon^2 \left(\frac{(1 - \lambda)^2 + \lambda^2}{2} \right)^{1-(1/q)} \right. \\ & \quad \times \left\{ \frac{1}{6} \left[3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b - a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)|^q \right. \\ & \quad \quad \left. + \frac{1}{6} \left[\frac{\epsilon}{b - a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)|^q \right\}^{1/q} \\ & \quad + (b - a - \epsilon)^2 \left(\frac{\mu^2 + (1 - \mu)^2}{2} \right)^{1-(1/q)} \\ & \quad \times \left\{ \frac{1}{6} \left[\left(1 - \frac{\epsilon}{b - a} \right) (2 - 3\mu + 2\mu^3) \right] |f'(a)|^q \right. \\ & \quad \quad \left. + \frac{1}{6} \left[1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b - a} (2 - 3\mu + 2\mu^3) \right] \right. \\ & \quad \quad \left. \times |f'(b)|^q \right\}^{1/q}. \end{aligned} \quad (12)$$

Proof. Suppose that $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$. By Lemma 1, we have

$$\begin{aligned} & \left| \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \right. \\ & \quad + [\epsilon(1 - \lambda) + (b - a - \epsilon)(1 - \mu)] \\ & \quad \times f(a + \epsilon) - \int_a^b f(x) dx \left| \leq \epsilon^2 \int_0^1 |1 - \lambda - t| \right. \\ & \quad \quad \times |f'(ta + (1 - t)(a + \epsilon))| dt \\ & \quad + (b - a - \epsilon)^2 \int_0^1 |\mu - t| \\ & \quad \quad \times |f'(t(a + \epsilon) + (1 - t)b)| dt. \end{aligned} \quad (13)$$

Case ($q = 1$). By the convexity of $|f'(x)|$ and Lemma 2, we have

$$\begin{aligned}
 & \int_0^1 |1 - \lambda - t| \left| f'(ta + (1-t)(a + \epsilon)) \right| dt \\
 &= \int_0^1 |1 - \lambda - t| \\
 & \quad \times \left| f' \left(\left(1 - \frac{\epsilon(1-t)}{b-a}\right)a + \frac{\epsilon(1-t)}{b-a}b \right) \right| dt \\
 &\leq \int_0^1 |1 - \lambda - t| \\
 & \quad \times \left[\left(1 - \frac{\epsilon(1-t)}{b-a}\right) |f'(a)| + \frac{\epsilon(1-t)}{b-a} |f'(b)| \right] dt \\
 &= \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)| + \frac{\epsilon}{b-a} |f'(b)| \right] \\
 & \quad \times \int_0^1 |1 - \lambda - t| dt \\
 & \quad + \left[\frac{\epsilon}{b-a} |f'(a)| - \frac{\epsilon}{b-a} |f'(b)| \right] \\
 & \quad \times \int_0^1 t |1 - \lambda - t| dt \\
 &= \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)| + \frac{\epsilon}{b-a} |f'(b)| \right] \\
 & \quad \times \frac{(1-\lambda)^2 + \lambda^2}{2} \\
 & \quad + \left[\frac{\epsilon}{b-a} |f'(a)| - \frac{\epsilon}{b-a} |f'(b)| \right] \\
 & \quad \times \frac{(1-\lambda)^3 + (3-\lambda)\lambda^2}{6} \\
 &= \frac{1}{6} \left[3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)| \\
 & \quad + \frac{1}{6} \left[\frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)|, \\
 & \int_0^1 |\mu - t| \left| f'(t(a + \epsilon) + (1-t)b) \right| dt \\
 &= \int_0^1 |\mu - t| \\
 & \quad \times \left| f' \left(\left(t - \frac{\epsilon t}{b-a}\right)a + \left(1 - \left(t - \frac{\epsilon t}{b-a}\right)\right)b \right) \right| dt \\
 &\leq \int_0^1 |\mu - t| \left[\left(t - \frac{\epsilon t}{b-a}\right) |f'(a)| \right. \\
 & \quad \left. + \left(1 - \left(t - \frac{\epsilon t}{b-a}\right)\right) |f'(b)| \right] dt
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)| - \left(1 - \frac{\epsilon}{b-a}\right) |f'(b)| \right] \\
 & \quad \times \int_0^1 t |\mu - t| dt \\
 & \quad + |f'(b)| \int_0^1 |\mu - t| dt \\
 &= \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)| - \left(1 - \frac{\epsilon}{b-a}\right) |f'(b)| \right] \\
 & \quad \times \frac{\mu^3 + (2 + \mu)(1 - \mu)^2}{6} \\
 & \quad + |f'(b)| \frac{\mu^2 + (1 - \mu)^2}{2} \\
 &= \frac{1}{6} \left[\left(1 - \frac{\epsilon}{b-a}\right) (2 - 3\mu + 2\mu^3) \right] |f'(a)| \\
 & \quad + \frac{1}{6} \left[1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] \\
 & \quad \times |f'(b)|.
 \end{aligned} \tag{14}$$

Thus,

$$\begin{aligned}
 & \left| \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \right. \\
 & \quad + [\epsilon(1 - \lambda) + (b - a - \epsilon)(1 - \mu)] \\
 & \quad \times f(a + \epsilon) - \int_a^b f(x) dx \Big| \\
 &\leq \frac{\epsilon^2}{6} \left[3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)| \\
 & \quad + \frac{\epsilon^2}{6} \left[\frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)| \\
 & \quad + \frac{(b - a - \epsilon)^2}{6} \\
 & \quad \times \left[\left(1 - \frac{\epsilon}{b-a}\right) (2 - 3\mu + 2\mu^3) \right] |f'(a)| \\
 & \quad + \frac{(b - a - \epsilon)^2}{6} \\
 & \quad \times \left[1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] |f'(b)|.
 \end{aligned} \tag{15}$$

Case ($q > 1$). By Hölder's inequality, we have

$$\begin{aligned}
 & \left| \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \right. \\
 & \quad + [\epsilon(1 - \lambda) + (b - a - \epsilon)(1 - \mu)] \\
 & \quad \times f(a + \epsilon) - \int_a^b f(x) dx \Big| \leq \epsilon^2 \left(\int_0^1 |1 - \lambda - t| dt \right)^{1-(1/q)}
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^1 |1 - \lambda - t| \left| f'(ta + (1-t)(a + \epsilon)) \right|^q dt \right)^{1/q} \\
& + (b - a - \epsilon)^2 \left(\int_0^1 |\mu - t| dt \right)^{1-(1/q)} \\
& \times \left(\int_0^1 |\mu - t| \left| f'(t(a + \epsilon) + (1-t)b) \right|^q dt \right)^{1/q} \\
& = \epsilon^2 \left(\frac{(1-\lambda)^2 + \lambda^2}{2} \right)^{1-(1/q)} \\
& \times \left(\int_0^1 |1 - \lambda - t| \left| f'(ta + (1-t)(a + \epsilon)) \right|^q dt \right)^{1/q} \\
& + (b - a - \epsilon)^2 \left(\frac{\mu^2 + (1-\mu)^2}{2} \right)^{1-(1/q)} \\
& \times \left(\int_0^1 |\mu - t| \left| f'(t(a + \epsilon) + (1-t)b) \right|^q dt \right)^{1/q}. \tag{16}
\end{aligned}$$

By the convexity of $|f'(x)|^q$ and Lemma 2, we have

$$\begin{aligned}
& \int_0^1 |1 - \lambda - t| \left| f'(ta + (1-t)(a + \epsilon)) \right|^q dt \\
& = \int_0^1 |1 - \lambda - t| \left| f' \left(\left(1 - \frac{\epsilon(1-t)}{b-a} \right) a + \frac{\epsilon(1-t)}{b-a} b \right) \right|^q dt \\
& \leq \int_0^1 |1 - \lambda - t| \\
& \quad \times \left[\left(1 - \frac{\epsilon(1-t)}{b-a} \right) |f'(a)|^q + \frac{\epsilon(1-t)}{b-a} |f'(b)|^q \right] dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a} \right) |f'(a)|^q + \frac{\epsilon}{b-a} |f'(b)|^q \right] \int_0^1 |1 - \lambda - t| dt \\
& \quad + \left[\frac{\epsilon}{b-a} |f'(a)|^q - \frac{\epsilon}{b-a} |f'(b)|^q \right] \int_0^1 t |1 - \lambda - t| dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a} \right) |f'(a)|^q + \frac{\epsilon}{b-a} |f'(b)|^q \right] \frac{(1-\lambda)^2 + \lambda^2}{2} \\
& \quad + \left[\frac{\epsilon}{b-a} |f'(a)|^q - \frac{\epsilon}{b-a} |f'(b)|^q \right] \frac{(1-\lambda)^3 + (3-\lambda)\lambda^2}{6} \\
& = \frac{1}{6} \left[3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)|^q \\
& \quad + \frac{1}{6} \left[\frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)|^q,
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 |\mu - t| \left| f'(t(a + \epsilon) + (1-t)b) \right|^q dt \\
& = \int_0^1 |\mu - t| \left| f' \left(\left(t - \frac{\epsilon t}{b-a} \right) a + \left(1 - \left(t - \frac{\epsilon t}{b-a} \right) \right) b \right) \right|^q dt \\
& \leq \int_0^1 |\mu - t| \\
& \quad \times \left[\left(t - \frac{\epsilon t}{b-a} \right) |f'(a)|^q \right. \\
& \quad \left. + \left(1 - \left(t - \frac{\epsilon t}{b-a} \right) \right) |f'(b)|^q \right] dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a} \right) |f'(a)|^q - \left(1 - \frac{\epsilon}{b-a} \right) |f'(b)|^q \right] \\
& \quad \times \int_0^1 t |\mu - t| dt \\
& \quad + |f'(b)|^q \int_0^1 |\mu - t| dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a} \right) |f'(a)|^q - \left(1 - \frac{\epsilon}{b-a} \right) |f'(b)|^q \right] \\
& \quad \times \frac{\mu^3 + (2+\mu)(1-\mu)^2}{6} \\
& \quad + |f'(b)|^q \left(\frac{\mu^2 + (1-\mu)^2}{2} \right) \\
& = \frac{1}{6} \left[\left(1 - \frac{\epsilon}{b-a} \right) (2 - 3\mu + 2\mu^3) \right] |f'(a)|^q \\
& \quad + \frac{1}{6} \left[1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] \\
& \quad \times |f'(b)|^q. \tag{17}
\end{aligned}$$

Thus,

$$\begin{aligned}
& \left| \epsilon \lambda f(a) + (b - a - \epsilon) \mu f(b) \right. \\
& \quad \left. + [\epsilon(1-\lambda) + (b - a - \epsilon)(1-\mu)] \right. \\
& \quad \left. \times f(a + \epsilon) - \int_a^b f(x) dx \right| \\
& \leq \epsilon^2 \left(\frac{(1-\lambda)^2 + \lambda^2}{2} \right)^{1-(1/q)} \\
& \quad \times \left\{ \frac{1}{6} \left[3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] \right. \\
& \quad \times |f'(a)|^q + \frac{1}{6} \left[\frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)|^q \left. \right\}^{1/q} \\
& \quad + (b - a - \epsilon)^2 \left(\frac{\mu^2 + (1-\mu)^2}{2} \right)^{1-(1/q)}
\end{aligned}$$

$$\begin{aligned} & \times \left\{ \frac{1}{6} \left[\left(1 - \frac{\epsilon}{b-a} \right) (2 - 3\mu + 2\mu^3) \right] |f'(a)|^q \right. \\ & \quad + \frac{1}{6} \left[1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] \\ & \quad \left. \times |f'(b)|^q \right\}^{1/q}. \end{aligned} \quad (18)$$

This proof is completed. $\square$

It is easy to notice that if we put $\epsilon = (b-a)/2$ in Theorem 3 then we get the following.

Corollary 4 (see [8]). Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} (1 - 2\lambda + 2\lambda^2)^{1-1/q} \\ & \quad \times \left[\left((4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (2 - 3\lambda + 2\lambda^3) |f'(b)|^q \right) \times (6)^{-1} \right]^{1/q} \\ & + \frac{b-a}{8} (1 - 2\mu + 2\mu^2)^{1-1/q} \\ & \quad \times \left[\left((2 - 3\mu + 2\mu^3) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (4 - 9\mu + 12\mu^2 - 2\mu^3) |f'(b)|^q \right) \times (6)^{-1} \right]^{1/q}. \end{aligned} \quad (19)$$

One can easily check that if we put $\epsilon = (b-a)/3$ in Theorem 3, then we get the following.

Corollary 5. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \frac{\lambda f(a) + 2\mu f(b)}{3} + \left(1 - \frac{\lambda}{3} - \frac{2\mu}{3} \right) f\left(\frac{2a+b}{3}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{9} \left(\frac{(1-\lambda)^2 + \lambda^2}{2} \right)^{1-(1/q)} \\ & \quad \times \left(\frac{1}{18} (7 - 15\lambda + 18\lambda^2 - 2\lambda^3) |f'(a)|^q \right. \\ & \quad \left. + \frac{1}{18} (2 - 3\lambda + 2\lambda^3) |f'(b)|^q \right)^{1/q} \end{aligned}$$

$$\begin{aligned} & + \frac{4(b-a)}{9} \left(\frac{\mu^2 + (1-\mu)^2}{2} \right)^{1-(1/q)} \\ & \times \left(\frac{1}{9} (2 - 3\mu + 2\mu^3) |f'(a)|^q \right. \\ & \quad \left. + \frac{1}{18} (5 - 12\mu + 18\mu^2 - 4\mu^3) |f'(b)|^q \right)^{1/q}. \end{aligned} \quad (20)$$

One can easily check that if we put $\epsilon = 2(b-a)/3$ in Theorem 3 then we get the following.

Corollary 6. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \frac{2\lambda f(a) + \mu f(b)}{3} + \left(1 - \frac{2\lambda}{3} - \frac{\mu}{3} \right) f\left(\frac{a+2b}{3}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{4(b-a)}{9} \left(\frac{(1-\lambda)^2 + \lambda^2}{2} \right)^{1-(1/q)} \\ & \quad \times \left(\frac{1}{18} (5 - 12\lambda + 18\lambda^2 - 4\lambda^3) |f'(a)|^q \right. \\ & \quad \left. + \frac{1}{9} (2 - 3\lambda + 2\lambda^3) |f'(b)|^q \right)^{1/q} \\ & + \frac{b-a}{9} \left(\frac{\mu^2 + (1-\mu)^2}{2} \right)^{1-(1/q)} \\ & \quad \times \left(\frac{1}{18} (2 - 3\mu + 2\mu^3) |f'(a)|^q \right. \\ & \quad \left. + \frac{1}{18} (7 - 15\mu + 18\mu^2 - 2\mu^3) |f'(b)|^q \right)^{1/q}. \end{aligned} \quad (21)$$

It is easy to notice that if we put $\lambda = \mu = 0$ in Theorem 3 then we get the following.

Corollary 7. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b-a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & 2^{1-(1/q)} (b-a) \left| f(a+\epsilon) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \epsilon^2 \left(\left(\frac{1}{2} - \frac{\epsilon}{3(b-a)} \right) |f'(a)|^q + \frac{\epsilon}{3(b-a)} |f'(b)|^q \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left(\left(\frac{1}{3} - \frac{\epsilon}{3(b-a)} \right) |f'(a)|^q \right. \\ & \quad \left. + \left(\frac{1}{6} + \frac{\epsilon}{3(b-a)} \right) |f'(b)|^q \right)^{1/q}. \end{aligned} \quad (22)$$

It is easy to notice that if we put $\lambda = \mu = 1/2$ in Theorem 3 then we get the following.

Corollary 8. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & 4^{1-(1/q)} \left| \frac{\epsilon f(a) + (b-a-\epsilon)f(b) + (b-a)f(a+\epsilon)}{2} \right. \\ & \quad \left. - \int_a^b f(x) dx \right| \\ & \leq \epsilon^2 \left(\left(\frac{1}{4} - \frac{\epsilon}{8(b-a)} \right) |f'(a)|^q + \frac{\epsilon}{8(b-a)} |f'(b)|^q \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left(\left(\frac{1}{8} - \frac{\epsilon}{8(b-a)} \right) |f'(a)|^q \right. \\ & \quad \left. + \left(\frac{1}{8} + \frac{\epsilon}{8(b-a)} \right) |f'(b)|^q \right)^{1/q}. \end{aligned} \quad (23)$$

It is easy to notice that if we put $\lambda = \mu = 1$ in Theorem 3 then we get the following.

Corollary 9. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & 2^{1-(1/q)} \left| \epsilon f(a) + (b-a-\epsilon)f(b) - \int_a^b f(x) dx \right| \\ & \leq \epsilon^2 \left(\left(\frac{1}{3} - \frac{\epsilon}{6(b-a)} \right) |f'(a)|^q + \frac{\epsilon}{6(b-a)} |f'(b)|^q \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left(\left(\frac{1}{6} - \frac{\epsilon}{6(b-a)} \right) |f'(a)|^q \right. \\ & \quad \left. + \left(\frac{1}{3} + \frac{\epsilon}{6(b-a)} \right) |f'(b)|^q \right)^{1/q}. \end{aligned} \quad (24)$$

Theorem 10. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \epsilon \lambda f(a) + (b-a-\epsilon) \mu f(b) \right. \\ & \quad \left. + [\epsilon(1-\lambda) + (b-a-\epsilon)(1-\mu)] \right. \\ & \quad \left. \times f(a+\epsilon) - \int_a^b f(x) dx \right| \end{aligned}$$

$$\begin{aligned} & \leq \epsilon^2 \left\{ \left[\frac{(1-\lambda)^{q+1} + \lambda^{q+1}}{q+1} - \left(\frac{\epsilon}{b-a} \right) \right. \right. \\ & \quad \times \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{(q+1)(q+2)} \Big] |f'(a)|^q \\ & \quad + \left[\left(\frac{\epsilon}{b-a} \right) \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{(q+1)(q+2)} \right] \\ & \quad \times |f'(b)|^q \Big\}^{1/q} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left\{ \left[\left(1 - \frac{\epsilon}{b-a} \right) \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \right] |f'(a)|^q \right. \\ & \quad + \left[\frac{\mu^{q+1} + (1-\mu)^{q+1}}{q+1} - \left(1 - \frac{\epsilon}{b-a} \right) \right. \\ & \quad \times \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \Big] \\ & \quad \times |f'(b)|^q \Big\}^{1/q}. \end{aligned} \quad (25)$$

Proof. Suppose that $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$. If $q = 1$, then, by Theorem 3, we have

$$\begin{aligned} & \left| \epsilon \lambda f(a) + (b-a-\epsilon) \mu f(b) \right. \\ & \quad \left. + [\epsilon(1-\lambda) + (b-a-\epsilon)(1-\mu)] \right. \\ & \quad \times f(a+\epsilon) - \int_a^b f(x) dx \Big| \\ & \leq \epsilon^2 \left\{ \frac{1}{6} \left[3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)| \right. \\ & \quad + \frac{1}{6} \left[\frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)| \Big\} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left\{ \frac{1}{6} \left[\left(1 - \frac{\epsilon}{b-a} \right) (2 - 3\mu + 2\mu^3) \right] |f'(a)| \right. \\ & \quad + \frac{1}{6} \left[1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] \\ & \quad \times |f'(b)| \Big\} \\ & = \epsilon^2 \left\{ \left[\frac{(1-\lambda)^2 + \lambda^2}{2} - \left(\frac{\epsilon}{b-a} \right) \frac{(2+\lambda)(1-\lambda)^2 + \lambda^3}{6} \right] \right. \\ & \quad \times |f'(a)| + \left[\left(\frac{\epsilon}{b-a} \right) \frac{(2+\lambda)(1-\lambda)^2 + \lambda^3}{6} \right] \\ & \quad \times |f'(b)| \Big\} \end{aligned}$$

$$\begin{aligned}
& + (b-a-\epsilon)^2 \\
& \times \left\{ \left[\left(1 - \frac{\epsilon}{b-a}\right) \frac{\mu^3 + (2+\mu)(1-\mu)^2}{6} \right] |f'(a)| \right. \\
& \quad + \left[\frac{\mu^2 + (1-\mu)^2}{2} - \left(1 - \frac{\epsilon}{b-a}\right) \right. \\
& \quad \left. \left. \times \frac{\mu^3 + (2+\mu)(1-\mu)^2}{6} \right] |f'(b)| \right\}. \quad (26)
\end{aligned}$$

Next, we suppose that $q > 1$. By Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
& \left| \epsilon \lambda f(a) + (b-a-\epsilon) \mu f(b) \right. \\
& \quad \left. + [\epsilon(1-\lambda) + (b-a-\epsilon)(1-\mu)] \right. \\
& \quad \left. \times f(a+\epsilon) - \int_a^b f(x) dx \right| \\
& \leq \epsilon^2 \int_0^1 |1-\lambda-t| |f'(ta+(1-t)(a+\epsilon))| dt \\
& \quad + (b-a-\epsilon)^2 \int_0^1 |\mu-t| |f'(t(a+\epsilon)+(1-t)b)| dt \\
& \leq \epsilon^2 \left(\int_0^1 dt \right)^{1-(1/q)} \\
& \quad \times \left(\int_0^1 |1-\lambda-t|^q |f'(ta+(1-t)(a+\epsilon))|^q dt \right)^{1/q} \quad (27) \\
& \quad + (b-a-\epsilon)^2 \left(\int_0^1 dt \right)^{1-(1/q)} \\
& \quad \times \left(\int_0^1 |\mu-t|^q |f'(t(a+\epsilon)+(1-t)b)|^q dt \right)^{1/q} \\
& = \epsilon^2 \left(\int_0^1 |1-\lambda-t|^q |f'(ta+(1-t)(a+\epsilon))|^q dt \right)^{1/q} \\
& \quad + (b-a-\epsilon)^2 \\
& \quad \times \left(\int_0^1 |\mu-t|^q |f'(t(a+\epsilon)+(1-t)b)|^q dt \right)^{1/q}.
\end{aligned}$$

By the convexity of $|f'(x)|^q$ and Lemma 2, we have

$$\begin{aligned}
& \int_0^1 |1-\lambda-t|^q |f'(ta+(1-t)(a+\epsilon))|^q dt \\
& = \int_0^1 |1-\lambda-t|^q \left| f' \left(\left(1 - \frac{\epsilon(1-t)}{b-a}\right) a + \frac{\epsilon(1-t)}{b-a} b \right) \right|^q dt \\
& \leq \int_0^1 |1-\lambda-t|^q \\
& \quad \times \left[\left(1 - \frac{\epsilon(1-t)}{b-a}\right) |f'(a)|^q + \frac{\epsilon(1-t)}{b-a} |f'(b)|^q \right] dt
\end{aligned}$$

$$\begin{aligned}
& = \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)|^q + \frac{\epsilon}{b-a} |f'(b)|^q \right] \\
& \quad \times \int_0^1 |1-\lambda-t|^q dt \\
& \quad + \left[\frac{\epsilon}{b-a} |f'(a)|^q - \frac{\epsilon}{b-a} |f'(b)|^q \right] \\
& \quad \times \int_0^1 t |1-\lambda-t|^q dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)|^q + \frac{\epsilon}{b-a} |f'(b)|^q \right] \\
& \quad \times \frac{(1-\lambda)^{q+1} + \lambda^{q+1}}{q+1} \\
& \quad + \left[\frac{\epsilon}{b-a} |f'(a)|^q - \frac{\epsilon}{b-a} |f'(b)|^q \right] \\
& \quad \times \frac{(1-\lambda)^{q+2} + (q+2-\lambda)\lambda^{q+1}}{(q+1)(q+2)} \\
& = \left[\frac{(1-\lambda)^{q+1} + \lambda^{q+1}}{q+1} - \left(\frac{\epsilon}{b-a}\right) \right. \\
& \quad \times \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{(q+1)(q+2)} \left. \right] |f'(a)|^q \\
& \quad + \left[\left(\frac{\epsilon}{b-a}\right) \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{(q+1)(q+2)} \right] |f'(b)|^q, \\
& \int_0^1 |\mu-t|^q |f'(t(a+\epsilon)+(1-t)b)|^q dt \\
& = \int_0^1 |\mu-t|^q \\
& \quad \times \left| f' \left(\left(t - \frac{\epsilon t}{b-a}\right) a + \left(1 - \left(t - \frac{\epsilon t}{b-a}\right)\right) b \right) \right|^q dt \\
& \leq \int_0^1 |\mu-t|^q \\
& \quad \times \left[\left(t - \frac{\epsilon t}{b-a}\right) |f'(a)|^q \right. \\
& \quad \left. + \left(1 - \left(t - \frac{\epsilon t}{b-a}\right)\right) |f'(b)|^q \right] dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)|^q - \left(1 - \frac{\epsilon}{b-a}\right) |f'(b)|^q \right] \\
& \quad \times \int_0^1 t |\mu-t|^q dt \\
& \quad + |f'(b)|^q \int_0^1 |\mu-t|^q dt \\
& = \left[\left(1 - \frac{\epsilon}{b-a}\right) |f'(a)|^q - \left(1 - \frac{\epsilon}{b-a}\right) |f'(b)|^q \right] \\
& \quad \times \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \\
& \quad + |f'(b)|^q \left(\frac{\mu^{q+1} + (1-\mu)^{q+1}}{q+1} \right)
\end{aligned}$$

$$\begin{aligned}
&= \left[\left(1 - \frac{\epsilon}{b-a} \right) \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \right] |f'(a)|^q \\
&+ \left[\frac{\mu^{q+1} + (1-\mu)^{q+1}}{q+1} - \left(1 - \frac{\epsilon}{b-a} \right) \right. \\
&\quad \times \left. \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \right] \\
&\times |f'(b)|^q.
\end{aligned} \tag{28}$$

Thus,

$$\begin{aligned}
&\left| \epsilon \lambda f(a) + (b-a-\epsilon) \mu f(b) \right. \\
&\quad + [\epsilon(1-\lambda) + (b-a-\epsilon)(1-\mu)] \\
&\quad \times \left. f(a+\epsilon) - \int_a^b f(x) dx \right| \\
&\leq \epsilon^2 \left\{ \left[\frac{(1-\lambda)^{q+1} + \lambda^{q+1}}{q+1} - \left(\frac{\epsilon}{b-a} \right) \right. \right. \\
&\quad \times \left. \left. \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{(q+1)(q+2)} \right] |f'(a)|^q \right. \\
&\quad + \left[\left(\frac{\epsilon}{b-a} \right) \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{(q+1)(q+2)} \right] \\
&\quad \times \left. |f'(b)|^q \right\}^{1/q} + (b-a-\epsilon)^2 \\
&\times \left\{ \left[\left(1 - \frac{\epsilon}{b-a} \right) \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \right] \right. \\
&\quad \times |f'(a)|^q \\
&\quad + \left[\frac{\mu^{q+1} + (1-\mu)^{q+1}}{q+1} - \left(1 - \frac{\epsilon}{b-a} \right) \right. \\
&\quad \times \left. \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \right] \\
&\quad \times \left. |f'(b)|^q \right\}^{1/q}.
\end{aligned} \tag{29}$$

This proof is completed. $\square$

It is easy to notice that if we put $\epsilon = (b-a)/2$ in Theorem 10 then we get the following.

Corollary 11 (see [8]). Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned}
&\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq \frac{b-a}{4} \left(\frac{1}{2(q+1)(q+2)} \right)^{1/q} \\
&\times \left[((q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1}) |f'(a)|^q \right. \\
&\quad + ((q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}) |f'(b)|^q \Big]^{1/q} \\
&+ \frac{b-a}{4} \left(\frac{1}{2(q+1)(q+2)} \right)^{1/q} \\
&\times \left[((q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2}) |f'(a)|^q \right. \\
&\quad + ((q+3-\mu)(1-\mu)^{q+1} + (2q+4-\mu)\mu^{q+1}) \\
&\quad \times \left. |f'(b)|^q \right]^{1/q}.
\end{aligned} \tag{30}$$

One can easily check that if we put $\epsilon = (b-a)/3$ in Theorem 10 then we get the following.

Corollary 12. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned}
&\left| \frac{\lambda f(a) + 2\mu f(b)}{3} + \left(1 - \frac{\lambda}{3} - \frac{2\mu}{3} \right) f\left(\frac{2a+b}{3}\right) \right. \\
&\quad \times \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq \frac{b-a}{9} \\
&\times \left\{ \frac{(5+2q-\lambda)(1-\lambda)^{q+1} + (6+3q-\lambda)\lambda^{q+1}}{3(q+1)(q+2)} |f'(a)|^q \right. \\
&\quad + \left. \frac{(1+\lambda+q)(1-\lambda)^{q+1} + \lambda^{q+2}}{3(q+1)(q+2)} |f'(b)|^q \right\}^{1/q}
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(b-a)}{9} \\
& \times \left\{ \frac{2\mu^{q+2} + 2(q+1+\mu)(1-\mu)^{q+1}}{3(q+1)(q+2)} |f'(a)|^q \right. \\
& \quad + \frac{(6+3q-2\mu)\mu^{q+1} + (4+q-2\mu)(1-\mu)^{q+1}}{3(q+1)(q+2)} \\
& \quad \left. \times |f'(b)|^q \right\}^{1/q}.
\end{aligned} \tag{31}$$

One can easily check that if we put $\epsilon = 2(b-a)/3$ in Theorem 10 then we get the following.

Corollary 13. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned}
& \left| \frac{2\lambda f(a) + \mu f(b)}{3} + \left(1 - \frac{2\lambda}{3} - \frac{\mu}{3}\right) f\left(\frac{a+2b}{3}\right) \right. \\
& \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{4(b-a)}{9} \\
& \times \left\{ \frac{(4-q-2\lambda)(1-\lambda)^{q+1} + (6+3q-2\lambda)\lambda^{q+1}}{3(q+1)(q+2)} \right. \\
& \quad \times |f'(a)|^q + \frac{2(1+\lambda+q)(1-\lambda)^{q+1} + 2\lambda^{q+2}}{3(q+1)(q+2)} \\
& \quad \left. \times |f'(b)|^q \right\}^{1/q} \\
& + \frac{b-a}{9} \\
& \times \left\{ \frac{\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1}}{3(q+1)(q+2)} |f'(a)|^q \right. \\
& \quad + \frac{(6+3q-\mu)\mu^{q+1} + (5+2q-\mu)(1-\mu)^{q+1}}{3(q+1)(q+2)} \\
& \quad \left. \times |f'(b)|^q \right\}^{1/q}.
\end{aligned} \tag{32}$$

It is easy to notice that if we put $\lambda = \mu = 0$ in Theorem 10 then we get the following.

Corollary 14. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b-a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned}
& \left| (b-a)f(a+\epsilon) - \int_a^b f(x) dx \right| \\
& \leq \epsilon^2 \left\{ \left(\frac{1}{q+1} - \frac{\epsilon}{(b-a)(q+2)} \right) |f'(a)|^q \right. \\
& \quad \left. + \frac{\epsilon}{(b-a)(q+2)} |f'(b)|^q \right\}^{1/q} \\
& + (b-a-\epsilon)^2 \\
& \times \left\{ \left(1 - \frac{\epsilon}{b-a} \right) \left(\frac{1}{q+2} \right) |f'(a)|^q \right. \\
& \quad \left. + \left(\frac{1}{q+1} - \left(1 - \frac{\epsilon}{b-a} \right) \left(\frac{1}{q+2} \right) \right) |f'(b)|^q \right\}^{1/q}.
\end{aligned} \tag{33}$$

It is easy to notice that if we put $\lambda = \mu = 1/2$ in Theorem 10 then we get the following.

Corollary 15. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b-a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned}
& \left| \frac{\epsilon f(a) + (b-a-\epsilon)f(b) + (b-a)f(a+\epsilon)}{2} - \int_a^b f(x) dx \right| \\
& \leq \frac{\epsilon^2}{2} \left(\frac{1}{q+1} \right)^{1/q} \left\{ \left(1 - \frac{\epsilon}{2(b-a)} \right) |f'(a)|^q \right. \\
& \quad \left. + \frac{\epsilon}{2(b-a)} |f'(b)|^q \right\}^{1/q} \\
& + \frac{(b-a-\epsilon)^2}{2} \left(\frac{1}{q+1} \right)^{1/q} \\
& \times \left\{ \left(\frac{1}{2} - \frac{\epsilon}{2(b-a)} \right) |f'(a)|^q \right. \\
& \quad \left. + \left(\frac{1}{2} + \frac{\epsilon}{2(b-a)} \right) |f'(b)|^q \right\}^{1/q}.
\end{aligned} \tag{34}$$

It is easy to notice that if we put $\lambda = \mu = 1$ in Theorem 10 then we get the following.

Corollary 16. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\begin{aligned} & \left| \epsilon f(a) + (b - a - \epsilon) f(b) - \int_a^b f(x) dx \right| \\ & \leq \epsilon^2 \left(\frac{1}{q+1} \right)^{1/q} \left\{ \left(1 - \frac{\epsilon}{(b-a)(q+2)} \right) |f'(a)|^q \right. \\ & \quad \left. + \frac{\epsilon}{(b-a)(q+2)} |f'(b)|^q \right\}^{1/q} \\ & \quad + (b - a - \epsilon)^2 \left(\frac{1}{q+1} \right)^{1/q} \\ & \quad \times \left\{ \left(1 - \frac{\epsilon}{b-a} \right) \left(\frac{1}{q+2} \right) |f'(a)|^q \right. \\ & \quad \left. + \left(1 - \left(1 - \frac{\epsilon}{b-a} \right) \left(\frac{1}{q+2} \right) \right) |f'(b)|^q \right\}^{1/q}. \end{aligned} \quad (35)$$

4. Applications

In this section, we suppose that $\{s, q\} \subseteq [1, \infty)$ and $\{a, b, w_a, w_b\} \subseteq (0, \infty)$ with $a < b$. Let $0 < \epsilon < b - a$.

The weighted arithmetic mean of data $\{a, b\}$ with weight $\{w_a, w_b\}$ is defined by

$$A(a, b; w_a, w_b) = \frac{w_a a + w_b b}{w_a + w_b}. \quad (36)$$

The weighted geometric mean of data $\{a, b\}$ with weight $\{w_a, w_b\}$ is defined by

$$G(a, b; w_a, w_b) = a^{w_a/(w_a+w_b)} b^{w_b/(w_a+w_b)}. \quad (37)$$

The generalized logarithmic mean of data $\{a, b\}$ is defined by

$$L(a, b) = \left(\frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} \right)^{1/s}. \quad (38)$$

The identric mean of data $\{a, b\}$ is defined by

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}. \quad (39)$$

Applying Corollary 7 with $f(x) = x^s$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & \frac{2^{1-(1/q)}(b-a)}{s} |A^s(a, b; b-a-\epsilon, \epsilon) - L^s(a, b)| \\ & \leq \epsilon^2 \left(\frac{a^{(s-1)q}}{2} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{3(b-a)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \left(\frac{2a^{(s-1)q} + b^{(s-1)q}}{6} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{3(b-a)} \right)^{1/q}. \end{aligned} \quad (40)$$

Applying Corollary 9 with $f(x) = x^s$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & \frac{2^{1-(1/q)}(b-a)}{s} |A(a^s, b^s; \epsilon, b-a-\epsilon) - L^s(a, b)| \\ & \leq \epsilon^2 \left(\frac{a^{(s-1)q}}{3} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{6(b-a)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \left(\frac{a^{(s-1)q} + 2b^{(s-1)q}}{6} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{6(b-a)} \right)^{1/q}. \end{aligned} \quad (41)$$

Applying Corollary 7 with $f(x) = \ln x$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & 2^{1-(1/q)}(b-a) \left| \ln \left(\frac{A(a, b; b-a-\epsilon, \epsilon)}{I(a, b)} \right) \right| \\ & \leq \epsilon^2 \left(\frac{a^{-q}}{2} + \frac{\epsilon(b^{-q} - a^{-q})}{3(b-a)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \left(\frac{2a^{-q} + b^{-q}}{6} + \frac{\epsilon(b^{-q} - a^{-q})}{3(b-a)} \right)^{1/q}. \end{aligned} \quad (42)$$

Applying Corollary 9 with $f(x) = \ln x$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & 2^{1-(1/q)}(b-a) \left| \ln \left(\frac{G(a, b; \epsilon, b-a-\epsilon)}{I(a, b)} \right) \right| \\ & \leq \epsilon^2 \left(\frac{a^{-q}}{3} + \frac{\epsilon(b^{-q} - a^{-q})}{6(b-a)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \left(\frac{a^{-q} + 2b^{-q}}{6} + \frac{\epsilon(b^{-q} - a^{-q})}{6(b-a)} \right)^{1/q}. \end{aligned} \quad (43)$$

Applying Corollary 14 with $f(x) = x^s$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & \frac{b-a}{s} |A^s(a, b; b-a-\epsilon, \epsilon) - L^s(a, b)| \\ & \leq \epsilon^2 \left(\frac{a^{(s-1)q}}{q+1} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{(b-a)(q+2)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left(\frac{b^{(s-1)q}}{q+1} + \frac{a^{(s-1)q} - b^{(s-1)q}}{q+2} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{(b-a)(q+2)} \right)^{1/q}. \end{aligned} \quad (44)$$

Applying Corollary 16 with $f(x) = x^s$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & \frac{(q+1)^{1/q}(b-a)}{s} \left| A(a^s, b^s; \epsilon, b-a-\epsilon) - L^s(a, b) \right| \\ & \leq \epsilon^2 \left(a^{(s-1)q} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{(b-a)(q+2)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \\ & \quad \times \left(b^{(s-1)q} + \frac{a^{(s-1)q} - b^{(s-1)q}}{q+2} + \frac{\epsilon(b^{(s-1)q} - a^{(s-1)q})}{(b-a)(q+2)} \right)^{1/q}. \end{aligned} \quad (45)$$

Applying Corollary 14 with $f(x) = \ln x$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & (b-a) \left| \ln \left(\frac{A(a, b; b-a-\epsilon, \epsilon)}{I(a, b)} \right) \right| \\ & \leq \epsilon^2 \left(\frac{a^{-q}}{q+1} + \frac{\epsilon(b^{-q} - a^{-q})}{(b-a)(q+2)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \left(\frac{b^{-q}}{q+1} + \frac{a^{-q} - b^{-q}}{q+2} + \frac{\epsilon(b^{-q} - a^{-q})}{(b-a)(q+2)} \right)^{1/q}. \end{aligned} \quad (46)$$

Applying Corollary 16 with $f(x) = \ln x$ on $(0, \infty)$, we get the following:

$$\begin{aligned} & (q+1)^{1/q}(b-a) \left| \ln \left(\frac{G(a, b; \epsilon, b-a-\epsilon)}{I(a, b)} \right) \right| \\ & \leq \epsilon^2 \left(a^{-q} + \frac{\epsilon(b^{-q} - a^{-q})}{(b-a)(q+2)} \right)^{1/q} \\ & \quad + (b-a-\epsilon)^2 \left(b^{-q} + \frac{a^{-q} - b^{-q}}{q+2} + \frac{\epsilon(b^{-q} - a^{-q})}{(b-a)(q+2)} \right)^{1/q}. \end{aligned} \quad (47)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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