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Research Article

$L(2,1)$-Labeling of the Strong Product of Paths and Cycles

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An $L(2,1)$-labeling of a graph $G = (V, E)$ is a function f from the vertex set $V(G)$ to the set of nonnegative integers such that the labels on adjacent vertices differ by at least two and the labels on vertices at distance two differ by at least one. The span of f is the difference between the largest and the smallest numbers in $f(V)$. The λ-number of G, denoted by $\lambda(G)$, is the minimum span over all $L(2,1)$-labelings of G. We consider the λ-number of $P_n \boxtimes C_m$ and for $n \leq 11$ the λ-number of $C_n \boxtimes C_m$. We determine λ-numbers of graphs of interest with the exception of a finite number of graphs and we improve the bounds on the λ-number of $C_n \boxtimes C_m$, $m \geq 24$ and $n \geq 26$.

1. Introduction

The Frequency Assignment Problem (FAP) requires assigning frequencies to transmitters in a wireless network. In a broadcasting network, each transmitter is assigned a frequency channel for its transmissions. Two transmissions can interfere if their channels are too close. That means that even if two transmitters use different channels, there still may be interference if the two transmitters are located close to each other [1, 2].

The spectrum of frequencies gets more and more scarce because of increasing demands, both civil and military. Thus the task is to minimize the span of frequencies while avoiding interference. One of the graph-theoretical models of FAP which is well elaborated is the concept of distance-constrained labeling of graphs [1]. Many variants of this concept have been proposed; however, the $L(2,1)$-labeling problem where adjacent vertices must be assigned colors of distance at least two apart and vertices of distance two must be assigned different colors has attracted the most of interest [3, 4].

An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of nonnegative integers C (called *labels* or *colors*) such that for any two vertices u and v

$$\begin{aligned} |f(u) - f(v)| &\geq 2 \quad \text{if } d(u,v) = 1, \\ |f(u) - f(v)| &\geq 1 \quad \text{if } d(u,v) = 2. \end{aligned} \tag{1}$$

A k-$L(2,1)$-*labeling* is a $L(2,1)$-labeling of G such that $C = \{0, \ldots, k\}$. An optimal $L(2,1)$-labeling of G is a k-$L(2,1)$-labeling with k smallest possible. The largest label used by an optimal $L(2,1)$-labeling is called the λ-*number* of G and denoted by $\lambda(G)$.

There is a number of studies on the algorithms for $L(2,1)$-labeling problem [1, 5, 6]. It is known to be **NP**-hard for general graphs [4]. Even for some relatively simple families of graphs such as planar graphs, bipartite graphs, chordal graphs [5], and graphs of treewidth two [7], the problem is also **NP**-hard.

Product graphs are considered in order to gain global information from the factor graphs [8]. Many interesting wireless networks are based on product graphs with simple factors, such as paths and cycles. In particular, any square

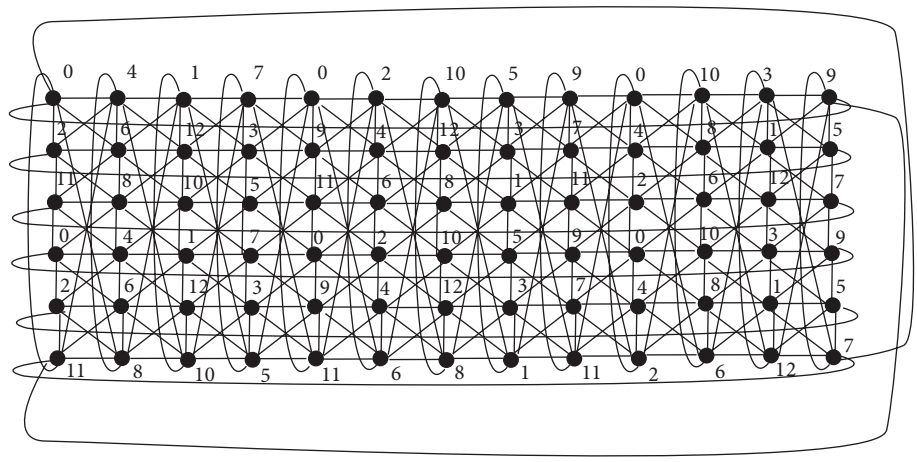

Figure 1: Strong product $C_6 \boxtimes C_{13}$ with a 12-$L(2,1)$-labeling.

grid (resp., torus) is the Cartesian product of two paths (resp., cycles) and any octagonal grid (resp., torus) is the strong product of two paths (resp., cycles) [9]. For the Cartesian product of these factors the λ numbers have been completely determined [10–12], while for the strong and the direct product only partial results have been found [13–15].

The paper is organized as follows. In Section 2, we give definitions and concepts needed in this paper. We also report on the known results for the λ numbers of the graphs of interest. In Section 3, two main computer search methods applied in the paper are described: the dynamic algorithm and the SAT reduction. Finally, in Section 4, we present the results on the λ-number of $P_n \boxtimes C_m$ and the λ-number of $C_n \boxtimes C_m$.

2. Preliminaries and Previous Results

For a graph $G = (V, E)$, $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. A directed graph D consists of vertices $V(D)$ together with a set of arcs $A(D) \subseteq V(D) \times V(D)$. We write G also to stand for the vertex set of the graph G. In this paper, only directed and undirected graphs without multiple edges or loops are considered.

The *strong product* of graphs G and H is the graph $G \boxtimes H$ with vertex set $G \times H$ and $(x_1, x_2)(y_1, y_2) \in E(G \boxtimes H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$, or $x_2 y_2 \in E(H)$ and $x_1 = y_1$, or $x_1 y_1 \in E(G)$ and $x_2 y_2 \in E(H)$. The strong product is commutative and associative, having the trivial graph as a unit (cf. [8]). The subgraph of $G \boxtimes H$ induced by $u \times V(H)$ is isomorphic to H. It is called an H-fiber and denoted by H^u.

The *path* P_n is the graph whose vertices are 0, 1, …, $n-1$ and for which two vertices are adjacent precisely if their difference is ± 1. For an integer $n \geq 3$, the *cycle of length* n is the graph C_n whose vertices are $0, 1, \ldots, n-1$ and whose edges are the pairs $i, i+1$, where the arithmetic is done modulo n. Note that the strong product $C_6 \boxtimes C_{13}$, depicted in Figure 1, can be regarded as a graph composed of six copies of C_{13} (denoted by $C_{13}^0, \ldots, C_{13}^5$) or a graph composed of 13 copies of C_6 (denoted by $C_6^0, \ldots, C_6^{12}$).

A *walk* in a directed graph D is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in A(D)$ for $1, 2, \ldots, n-1$. If $v_1 = v_n$, we say it is a *closed walk*.

If P is a path (resp., walk), then its *length* is its number of edges (resp., arcs).

The following simple lemma is well known.

Lemma 1. *If H is a subgraph of G, then $\lambda(H) \leq \lambda(G)$.*

Let f denote a k-$L(2,1)$-labeling of $C_n \boxtimes G$. We denote by $f_{i,p}$ the restriction of f to $G^i, G^{i+1}, \ldots, G^{i+p-1}$, $i = 0, 1, \ldots, n-1$ and $p = 1, \ldots, n-p$. Note that G^i is isomorphic to G; that is, G^i is the subgraph of $G \boxtimes C_n$ induced by $V(G) \times i$. We will also write f_i for $f_{i,1}$.

The following lemma provides an upper bound for the λ-number of the strong product of a graph with a cycle.

Lemma 2. *Let $t \geq 1$ and f be a k-$L(2,1)$-labeling of $C_n \boxtimes G$. If $f_{0,p}$ is a k-$L(2,1)$-labeling of $C_p \boxtimes G$, then $\lambda(C_{n+(t-1)p} \boxtimes G) \leq k$.*

Proof. Let f' be a function from $V(C_{n+(t-1)p} \boxtimes G)$ onto the set $\{0, 1, \ldots, k\}$ and f'_i the restriction of f' to $V(G^i)$. The function f' is defined as follows:

$$f'_i = \begin{cases} f_i, & i < n \\ f_{(i-n) \bmod p} & i \geq n. \end{cases} \tag{2}$$

It is not difficult to see that f' is a k-$L(2,1)$-labeling of $C_{n+(t-1)p} \boxtimes G$. □

Given two integers r and s, let $S(r, s)$ denote the set of all nonnegative integer combinations of r and s:

$$S(r, s) = \{\alpha r + \beta s : \alpha, \ \beta \in Z^+\}. \tag{3}$$

We will need the result of Sylvester [16].

Lemma 3. *If $r, s > 1$ are relatively prime integers, then $t \in S(r, s)$ for all $t \geq (s-1)(r-1)$.*

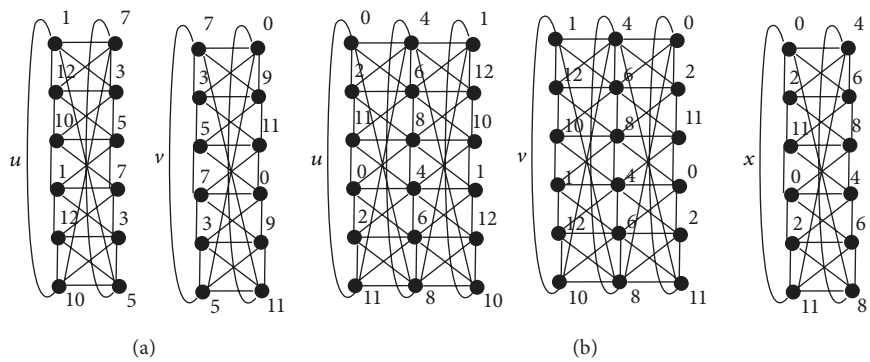

Figure 2: Two vertices of $D_{6,12}$ (a) and two vertices of $V^3_{6,12}$ with a vertex of $D^2_{6,12}$ (b).

Some partial results on the λ-number for the strong products of two cycles are given in [15].

Theorem 4. *Let $m \geq 3$. Then*

$$\lambda(C_3 \boxtimes C_m) = \begin{cases} 16, & m = 3, 6 \\ 14, & m = 5, 7, 10, 11, 15 \\ 13, & m = 9, 14, 18, 19, 22, 23, 27, 31, 35 \\ 12, & \textit{otherwise}. \end{cases} \tag{4}$$

Theorem 5. *Let $m \geq 3$. Then*

$$\lambda(C_4 \boxtimes C_m) = \begin{cases} 19, & m = 5 \\ 15, & m = 4, 8 \\ 14, & m = 11 \\ 13, & m = 7, 10, 14, 17, 20, 23 \\ 11, & m \equiv 0 \pmod 6 \\ 12, & \textit{otherwise}. \end{cases} \tag{5}$$

Theorem 6. *If $m \geq 24$ and $n \geq 36$, then $\lambda(C_n \boxtimes C_m) \leq 12$.*

For the strong product of more than two cycles the following result presented in [14] is known.

Theorem 7. *If $k \geq 1$ and $m_0, \ldots, m_{k-1}$ are each multiple of $3^k + 2$, then $\lambda(C_{m_0} \boxtimes \cdots \boxtimes C_{m_{k-1}}) = 3^k + 1$.*

3. Computer Search

3.1. Dynamic Algorithm. The idea is introduced in [10] in a more general framework and later used several times, for example, [13, 15]. In order to make the paper self-contained we first describe its basic definitions and results.

We define a digraph $D_{n,k}$ as follows. Its vertices are the k-$L(2,1)$-labelings of $C_n \boxtimes P_2$. Let $u = u_1u_2$ be a vertex of $D_{n,k}$. Then u_1 and u_2 represent the k-$L(2,1)$-labeling of $C_n \boxtimes P_2$ restricted to the first and second copies of C_n, respectively.

Let u and v be two vertices of $D_{n,k}$. Then $\widetilde{uv}$ denotes the labeling of $C_n \boxtimes P_3$ obtained by applying u_1, u_2, and v_2 to the consecutive copies of C_n. (Note that $\widetilde{uv}$ is not always a k-$L(2,1)$-labeling of $C_n \boxtimes P_3$.) We make an arc from u to v in $D_{n,k}$ if and only if the following two conditions are fulfilled:

(i) u_2 equals v_1; (*ii*) $\widetilde{uv}$ is a k-$L(2,1)$-labeling of $C_n \boxtimes P_3$.

Analogously, we define a digraph $D'_{n,k}$ with the vertex set composed by k-$L(2,1)$-labelings of $P_n \boxtimes P_2$. In other words, if $u = u_1u_2$ is a vertex of $D'_{n,k}$, then u_1 and u_2 represent the k-$L(2,1)$-labeling of $P_n \boxtimes P_2$ restricted to the first and second copies of P_n, respectively. The set of arcs of $D'_{n,k}$ is formed analogously as the set of arcs of $D_{n,k}$.

Figure 2(a) shows two vertices of $D_{6,12}$ denoted by u and v. We can see that the labeling of the second copy of C_6 in u equals the labeling of the first copy of C_6 in v. Moreover, the labeling of u and the labeling of the second copy of C_6 in v induce a 12-$L(2,1)$-labeling of $C_6 \boxtimes P_3$. It follows that $A(D_{6,12})$ admits an arc from u to v.

The next theorem follows from the results presented in [10].

Theorem 8. *$C_i \boxtimes C_\ell$ (resp., $P_i \boxtimes C_\ell$) admits a k-$L(2,1)$-labeling if and only if $D_{i,k}$ (resp. $D'_{i,k}$) contains a closed directed walk of length ℓ.*

The dynamic algorithm first generates all k-$L(2,1)$-labelings of $C_i \boxtimes P_2$ which are the vertices of $D_{i,k}$. Since a main building block $C_i \boxtimes P_2$ is usually relatively small, a simple method, for example, backtracking, can be applied for this step. In the next step, the set of edges of $D_{i,k}$ has to be generated. The procedure for this step is described in [15]. The described algorithm however has the time complexity $O(n^2)$, where n denotes the number of vertices in $D_{i,k}$. Note that n can be very large even for i and k of a moderate size. Some examples for i and k of interest are $|D_{5,14}| = 114984000$, $|D_{6,11}| = 1925760$, and $|D_{8,11}| = 1072523264$. The complexity of the algorithm does not allow a computation of $A(D_{i,k})$ in a reasonable time for these cases. We have therefore improved this method as described in the sequel.

Let $V^3_{i,k}$ denote the set of all k-$L(2,1)$-labelings of $C_i \boxtimes P_3$. If u is an element of $V^3_{i,k}$, then u_1, u_2, u_3 denote the restriction of u to the first, second, and third copies of C_i in $C_i \boxtimes P_3$;

```
Procedure CREATE GRAPH(i, k, N);
begin
V^3_{i,k} := set of all k-L(2,1)-labelings of C_n ⊠ P_3;
for all v = v_1 v_2 v_3 ∈ V^3_{i,k} do Insert v_1 v_2 in V^2_{i,k};
Sort vertices u_1 u_2 of V^2_{i,k} with respect to u_1, vertices with the same value of u_1 sort with respect to u_2;
for all v = v_1 v_2 v_3 ∈ V^3_{i,k} do
    Find u = u_1 u_2 in V^2_{i,k} such that u_1 = v_1 and u_2 = v_2;
    Find w = w_1 w_2 in V^2_{i,k} such that w_1 = v_2 and w_2 = v_3;
    Insert w in N_u;
end.
```

Algorithm 1

respectively. Note that $V^3_{i,k}$ contains symmetric labelings of $C_i \boxtimes P_3$, that is, if $u \in V^3_{i,k}$, then $v \in V^3_{i,k}$ exists, such that $u_1 = v_3$, $u_3 = v_1$, and $u_2 = v_2$.

We now define the digraph $D^2_{i,k}$ as follows. Let $V(D^2_{i,k})$ denote the set of k-$L(2,1)$-labelings of $C_i \boxtimes P_2$ obtained from $V^3_{i,k}$ in the following way: $x = x_1x_2$ belongs to $V(D^2_{i,k})$ if and only if there exist $u, v \in V^3_{i,k}$ such that $u_1 = x_1$, $u_2 = x_2$, and $v_3 = x_1$, $v_2 = x_2$.

Figure 2(b) shows two vertices of $V^3_{6,12}$ denoted by u and v. We can see that the labelings of the first and the second copies of C_6 in u equal the third and the second copies of C_6 in v, respectively. It follows that $V(D^2_{6,12})$ possesses the vertex x comprising these two labelings.

Let $x, y \in V(D^2_{i,k})$. We make an arc from $x = x_1x_2$ to $y = y_1y_2$ if and only if $x_2 = y_1$ and $x_1x_2y_2$ belongs to $V^3_{i,k}$.

Note that analogous as above we can improve the method for $P_i \boxtimes C_\ell$. The graph obtained with this procedure (a subgraph of $D'_{i,k}$) will be denoted by $D'^2_{i,k}$ in the sequel.

For a vertex v of a directed graph D, the number of inward (resp., outward) directed arcs from v in D is called an *indegree* (resp., *outdegree*) and denoted by $indeg_{D(v)}$ (resp., $outdeg_{D(v)}$).

We obtain the main result of this section.

Theorem 9. *$C_i \boxtimes C_\ell$ (resp., $P_i \boxtimes C_\ell$) admits a k-$L(2,1)$-labeling if and only if $D^2_{i,k}$ (resp., $D'^2_{i,k}$) contains a closed directed walk of length ℓ.*

Proof. It is easy to see that $V(D^2_{i,k}) \subseteq V(D_{i,k})$ and $A(D^2_{i,k}) \subseteq A(D_{i,k})$. From the definition of $V^2_{i,k}$ for $x \in V(D^2_{i,k})$ it follows that $indeg_{D_{i,k}}(x) > 0$ and $outdeg_{D_{i,k}}(x) > 0$.

Suppose now for $y \in V(D_{i,k})$ that $indeg_{D_{i,k}}(y) > 0$ and $outdeg_{D_{i,k}}(y) > 0$. Since $V^3_{i,k}$ contains all k-$L(2,1)$-labelings of $C_i \boxtimes P_3$ and $outdeg_{D_{i,k}}(y) > 0$, there has to be a vertex $v \in V^3_{i,k}$ such that $v_1 = y_1$ and $v_2 = y_2$. Moreover, since $indeg_{D_{i,k}}(y) > 0$, there has to be a vertex $u \in V^3_{i,k}$ such that $u_2 = y_1$ and $u_3 = y_2$. It follows that y belongs to $V(D^2_{i,k})$. We have therefore proven that $x \in V(D^2_{i,k})$ if and only if $indeg_{D_{i,k}}(x) > 0$ and $outdeg_{D_{i,k}}(x) > 0$.

Analogously as above we can show that xy is an arc in $D^2_{i,k}$ if and only if xy is an arc in $D_{i,k}$. In other words, we can show that $D^2_{i,k}$ is isomorphic to subgraph of $D_{i,k}$ induced by $V^2_{i,k}$. It follows that $D^2_{i,k}$ contains a closed directed walk of length ℓ if and only if $D_{i,k}$ contains a closed directed walk of the same size and the proof for $D_{i,k}$ is settled.

Since the proof for $D'^2_{i,k}$ is analogous, the proof of the theorem is complete. □

The algorithm for generating the graph $D^2_{i,k}$ is depicted in Algorithm 1 (Procedure CREATE GRAPH).

Note that the number of vertices of $D^2_{i,k}$ can be much smaller than in $D_{i,k}$. Some examples for i and k of interest are: $|D^2_{5,14}| = 386345700$, $|D^2_{6,11}| = 12336$, and $|D^2_{8,11}| = 8157632$. However, this reduction is not the only positive effect of the new approach. It is also of a great importance that the running time of CREATE GRAPH is $O(n \log n)$, where n, the number of vertices in $V^3_{i,k}$. In order to see this, note that the running time of an efficient sorting algorithm is also within this time bound. Moreover, this is also the running time of the duration of loop, since a single search in an ordered list with n elements requires $O(\log n)$ time.

The final step of the approach is the search for closed direct walks in $D^2_{i,k}$. We can find these walks by applying a matrix multiplication of the adjacency matrix of $D^2_{i,k}$ or a breadth (depth) first search in $D^2_{i,k}$. Since graphs $D^2_{i,k}$ are relatively sparse for i and k of interest, the later approach has been applied in order to compute the results of this paper.

3.2. SAT Reduction for k-$L(2,1)$-Labeling. The approach is proposed in [17] for the distance-constrained labeling problem. Here we present this method adapted for $L(2,1)$-labeling.

Let $G = (V, E)$ be a graph and k a positive integer. For every $v \in V$ and every $i \in \{0, 1, 2, \ldots, k\}$ introduce an atom $x_{v,i}$. Intuitively, this atom shows that the vertex v is assigned the color i. Consider the following propositional formulas:

(1) for all $v \in V$, $\vee_{i=0}^{k} x_{v,i}$;

(2) for all $v \in V$, $0 \le i < j \le k$, $\neg x_{v,i} \vee \neg x_{v,j}$;

(3) for all $v, u \in V$, $0 \le i, j \le k$ with $d(v,u) = 1$ and $|i - j| < 2$ or $d(v,u) = 2$ and $|i - j| < 1$, $\neg x_{v,i} \vee \neg x_{u,j}$.

Clauses (1) and (2) ensure that each vertex is labeled with exactly one label. Clause (3) guarantees that an obtained

Table 1: Values of $\lambda(P_n \boxtimes C_m)$.

$n \setminus m$	3	4	5	6	7	8	9	10	11	12
3	11	11	14	11	11	11	11	11	10	10
4	11	11	14	11	11	11	11	11	10	11
5	12	11	14	11	11	11	12	12	10	11
6	12	11	14	11	11	11	12	12	10	11
7	12	11	14	11	12	11	12	12	10	11

Table 2: Values of $\lambda(C_n \boxtimes C_m)$.

$n \setminus m$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
5	24[a]	14[a]	17	19	14	16	18	14	16	17	14	15	16	14	15	16	14	15	16	14	15	16
6		16[a]	14[a]	11	13	13	13	11	12	13	12	11	12	13	12	11	12	12	12	11	12	12
7			13	13	13	12	12	12	12	13	13	13	13	13	13	12	12	12	12	12	12	12
8				13	12	13	12	11	12	12	12	12	12	11	12	12	12	12	12	11	12	12
9					13	13/14	13	12	12	13	13	12	12	13	13	12	12	13	13	12	12	12
10						13	13	12	13	12	13	13	13	12	12	12	12	13	12	12	12	13
11							10	12	12	12	12	12	11	12	12	12	12	10	12	11	12	12
12								11	12	12	12	11	12	11	12	11	12	12	12	11	12	12

labeling is a k-$L(2,1)$-labeling of G. Therefore, the above propositional formulas transform an $L(2,1)$-labeling problem into a propositional satisfiability test (SAT). We can see that an obtained SAT instance is satisfiable if and only if G admits a k-$L(2,1)$-labeling.

4. Results

4.1. SAT Reduction. We solve the SAT instances transformed from $L(2,1)$-labeling problems described in Section 4.2 by using the software MiniSat [18]. As a result, we have obtained the λ-numbers of $P_n \boxtimes C_m$ presented in Table 1 and the λ-numbers of $C_n \boxtimes C_m$ presented in Table 2.

The values in Table 2 marked with a denote the results already obtained in [15], while the entry with 13/14 means that the corresponding value is either 13 or 14.

4.2. λ-Labeling of $C_n \boxtimes C_m$

Proposition 10. $\lambda(C_5 \boxtimes C_m) = 14$ *only if* $m \equiv 0 \pmod 3$.

Proof. Note that $\lambda(C_5 \boxtimes P_3) = 14$. We can see that $\lambda(C_5 \boxtimes P_3) \leq 14$ from the fact that every pair of vertices $u, v \in V(C_5 \boxtimes P_3)$ is at distance at most two. Let f denote a 14-$L(2,1)$-labeling of $C_5 \boxtimes C_m$ and f_k its restriction to C_5^k. Let also L_k denote the set of labels used in f_k. Since $\lambda(C_5 \boxtimes P_3) = 14$, we have $|L_k| = 5$ and $|L_k| + |L_{k+1}| = 10$. Therefore, the restriction of f to C_5^{k+3} has to comprise the same set of labels as the restriction of f to C_5^k or, more formally, $L_k = L_{k+3}$. It is straightforward to see that this equality can be satisfied in $C_5 \boxtimes C_m$ only if $m \equiv 0 \pmod 3$. □

Theorem 11. *Let $m \geq 5$. Then*

$$\lambda(C_5 \boxtimes C_m) = \begin{cases} 14, & m \equiv 0 \pmod 3, \\ 17, & m = 7, 14, \\ 19, & m = 8, \\ 18, & m = 11, \\ 16, & m = 10, 13, 17, 20, 23, 26, 29, \\ 15, & \textit{otherwise}. \end{cases} \tag{6}$$

Proof. Note that the values for $m \leq 26$ are given in Table 2. We can also show by using the SAT reduction that $\lambda(C_5 \boxtimes C_{29}) = 16$. Since $\lambda(C_5 \boxtimes C_j) = 14$ only if $j \equiv 0 \pmod 3$, we construct below a 14-$L(2,1)$-labeling for $C_5 \boxtimes C_{3j}$, $j \geq 1$, a 15-$L(2,1)$-labeling for $C_5 \boxtimes C_{3j+1}$, $j \geq 5$, and a 15-$L(2,1)$-labeling for $C_5 \boxtimes C_{3j+2}$, $j \geq 10$.

Let $[a, b]$ for $b \geq a$ denote the set $\{a, a+1, \ldots, b\}$.

Let f denote a function from $V(C_5 \boxtimes C_m)$ to $[0, 14]$ and f_k its restriction to C_5^k, $k \leq m-1$. Let also L_k denote the set of labels used in f_k. If we set for $i \geq 0$ and $s \in \{0, 1, 2\}$: $L_{3i+s} := [5s, 5s+4]$, then f is a 14-$L(2,1)$-labeling of $C_5 \boxtimes C_{3j}$ for $j \geq 1$.

Let f' denote a function from $V(C_5 \boxtimes C_m)$ to $[0, 15]$ and f'_k its restriction to C_5^k, $k \leq m-1$. Let also L'_k denote the set of labels used in f'_k. If we set for for $0 \leq i \leq 3$: $L'_{3i} := [15-i, 15] \cup [0, 3-i]$, $L'_{3i+1} := [4-i, 8-i]$, $L'_{3i+2} := [9-i, 13-i]$ and for $i \geq 4$ and $s \in \{0, 1, 2\}$: $L_{3i+s} := [5s, 5s+4]$, then f' is a 15-$L(2,1)$-labeling of $C_5 \boxtimes C_{3j+1}$ for $j \geq 5$. As an example, observe the following pattern representing a 15-$L(2,1)$-labeling of $C_5 \boxtimes C_{22}$:

0	2	4	6	10	12	1	5	7	9	11	0	4	6	8	10	1	12	5	7	9
5	7	9	0	2	4	8	10	12	3	5	7	9	11	0	4	6	8	10	1	3
10	12	3	5	7	11	0	2	6	8	10	12	3	5	7	9	11	0	4	6	8
2	6	8	10	1	3	5	9	11	0	2	6	8	10	12	1	3	7	9	11	0
9	11	0	4	6	8	12	1	3	5	9	11	0	2	4	6	10	12	1	3	7
1	3	7	9	11	0	4	6	8	12	1	3	5	9	11	0	2	4	6	10	12
6	10	12	1	3	7	9	11	2	4	6	8	12	1	3	5	7	9	0	2	4

Figure 3: 12-$L(2,1)$-labeling of $C_7 \boxtimes C_{21}$.

15	4	9	14	3	8	13	2	7	12	1	6	11	0	5	10	0	5	10	0	5	10
0	5	10	15	4	9	14	3	8	13	2	7	12	1	6	11	1	6	11	1	6	11
1	6	11	0	5	10	15	4	9	14	3	8	13	2	7	12	2	7	12	2	7	12
2	7	12	1	6	11	0	5	10	15	4	9	14	3	8	13	3	8	13	3	8	13
3	8	13	2	7	12	1	6	11	0	5	10	15	4	9	14	4	9	14	4	9	14

Let f'' denote a function from $V(C_5 \boxtimes C_m)$ to $[0, 15]$ and f''_k its restriction to C_5^k, $k \le m-1$. Let also L''_k denote the set of labels used in f''_k. If we set for for $0 \le k \le 31$: $L''_k := L'_k$ and for $i \ge 11$ and $s \in \{0,1,2\}$: $L''_{3i+s} := [5s, 5s+4]$, then f'' is a 15-$L(2,1)$-labeling of $C_5 \boxtimes C_{3j+2}$ for $j \ge 10$. This assertion concludes the proof. □

The following results partially depend on comprehensive constructions which provide labelings of interest. These constructions are mostly not included in this paper and can be obtained by the authors.

Proposition 12. $\lambda(C_6 \boxtimes C_m) = 11$ *only if* $m \equiv 0 \pmod 4$.

Proof. The graph $D^2_{6,11}$ with 12336 vertices and the largest outdegree six has been computed. Since breadth first search algorithm has found only cycles of length four, Theorem 9 yields the proof. □

Theorem 13. *If* $m \ge 6$, *then*

$$\lambda(C_6 \boxtimes C_m) = \begin{cases} 11, & m \equiv 0 \pmod 4 \\ 16, & m = 6 \\ 14, & m = 7 \\ 13, & m = 9, 10, 11, 14 \\ 12, & \textit{otherwise.} \end{cases} \tag{7}$$

Proof. The results for $m = 6, 7, 9, 10, 11, 14$ follow from Table 2. We can also see in Table 1 that $\lambda(C_6 \boxtimes P_3) = 11$; thus, from Lemma 1 it follows that $\lambda(C_6 \boxtimes C_m) \ge 11$. Moreover, Proposition 12 says that $\lambda(C_6 \boxtimes C_m) = 11$ only if $m \equiv 0 \pmod 4$. In order to see that $\lambda(C_6 \boxtimes C_m) \le 12$ for other m of interest, see as an example a 12-$L(2,1)$-labeling of $C_6 \boxtimes C_{13}$ depicted in Figure 1.

From Lemma 2 it follows that $\lambda(C_6 \boxtimes C_m) \le 12$ if $m \equiv 0 \pmod{13}$. Analogously, we have found 12-$L(2,1)$-labelings of $C_6 \boxtimes C_{40}$, $C_6 \boxtimes C_{41}$, $C_6 \boxtimes C_{29}$, $C_6 \boxtimes C_{30}$, $C_6 \boxtimes C_{44}$, $C_6 \boxtimes C_{45}$, $C_6 \boxtimes C_{33}$, $C_6 \boxtimes C_{34}$, $C_6 \boxtimes C_{48}$, $C_6 \boxtimes C_{36}$, $C_6 \boxtimes C_{50}$, and $C_6 \boxtimes C_{38}$. Any of these labelings restricted to the first 13 copies of C_6 induces a 12-$L(2,1)$-labeling of $C_6 \boxtimes C_{13}$. From Lemma 2 it follows that $\lambda(C_6 \boxtimes C_{13k+1}) \le 12$ for $k \ge 3$, $\lambda(C_6 \boxtimes C_{13k+2}) \le 12$ for $k \ge 3$, $\lambda(C_6 \boxtimes C_{13k+3}) \le 12$ for $k \ge 2$, $\lambda(C_6 \boxtimes C_{13k+4}) \le 12$ for $k \ge 2$, $\lambda(C_6 \boxtimes C_{13k+5}) \le 12$ for $k \ge 3$, $\lambda(C_6 \boxtimes C_{13k+6}) \le 12$ for $k \ge 3$, $\lambda(C_6 \boxtimes C_{13k+7}) \le 12$ for $k \ge 2$, $\lambda(C_6 \boxtimes C_{13k+8}) \le 12$ for $k \ge 2$, $\lambda(C_6 \boxtimes C_{13k+9}) \le 12$ for $k \ge 3$, $\lambda(C_6 \boxtimes C_{13k+10}) \le 12$ for $k \ge 2$, $\lambda(C_6 \boxtimes C_{13k+11}) \le 12$ for $k \ge 3$, and $\lambda(C_6 \boxtimes C_{13k+12}) \le 12$ for $k \ge 2$.

Since we have also found 12-$L(2,1)$-labelings of $C_6 \boxtimes C_{27}$, $C_6 \boxtimes C_{28}$, $C_6 \boxtimes C_{31}$, $C_6 \boxtimes C_{32}$, $C_6 \boxtimes C_{35}$, and $C_6 \boxtimes C_{37}$, we conclude that $\lambda(C_6 \boxtimes C_m) \le 12$ for $m \ge 27$ and the proof is complete. □

Theorem 14. *If* $m \ge 7$, *then*

$$\lambda(C_7 \boxtimes C_m) = \begin{cases} 13, & m = 7, 8, 9, 14, 15, 16, 17, \\ & \quad 18, 19, 27, 28, 29 \\ 12, & \textit{otherwise.} \end{cases} \tag{8}$$

Proof. The results for $m \le 26$ follow from Table 2. We have also established the results for $m = 27, 28, 29$ by solving the SAT instances transformed from the corresponding $L(2,1)$-labeling problems. Since $\lambda(C_7 \boxtimes P_7) = 12$ as we can see in Table 2, it follows by Lemma 1 that $\lambda(C_7 \boxtimes C_m) \ge 12$ for $m \ge 7$. In order to see that $\lambda(C_7 \boxtimes C_m) \le 12$ for other m of interest, see as an example a 12-$L(2,1)$-labeling of $C_7 \boxtimes C_{21}$ depicted in Figure 3. This labeling restricted to the first ten copies of C_7 induces a 12-$L(2,1)$-labeling of $C_7 \boxtimes C_{10}$. From Lemma 2 it follows that $\lambda(C_7 \boxtimes C_{10k+1}) \le 12$, $k \ge 2$, and $\lambda(C_7 \boxtimes C_{10k}) \le 12$, $k \ge 1$.

Analogously, we have found 12-$L(2,1)$-labelings of $C_7 \boxtimes C_{22}$, $C_7 \boxtimes C_{23}$, $C_7 \boxtimes C_{34}$, $C_7 \boxtimes C_{35}$, $C_7 \boxtimes C_{36}$, $C_7 \boxtimes C_{47}$, $C_7 \boxtimes C_{48}$, and $C_7 \boxtimes C_{49}$. Any of these labelings restricted to the first ten copies of C_7 is a 12-$L(2,1)$-labeling of $C_7 \boxtimes C_{10}$. From Lemma 2 it follows that $\lambda(C_7 \boxtimes C_{10k+2}) \le 12$ for $k \ge 2$, $\lambda(C_7 \boxtimes C_{10k+3}) \le 12$ for $k \ge 2$, $\lambda(C_7 \boxtimes C_{10k+3}) \le 12$ for $k \ge 2$, $\lambda(C_7 \boxtimes C_{10k+4}) \le 12$ for $k \ge 3$, $\lambda(C_7 \boxtimes C_{10k+5}) \le 12$ for $k \ge 3$, $\lambda(C_7 \boxtimes C_{10k+6}) \le 12$ for $k \ge 3$, $\lambda(C_7 \boxtimes C_{10k+7}) \le 12$ for $k \ge 4$, $\lambda(C_7 \boxtimes C_{10k+8}) \le 12$ for $k \ge 4$, and $\lambda(C_7 \boxtimes C_{10k+9}) \le 12$ for $k \ge 4$.

0	2	4	0	2	4	6	8	12	10	1	4	6	8	12	10	2	4	6	8	12
5	7	9	11	7	9	11	1	3	5	7	9	11	3	0	5	7	9	11	1	3
11	1	3	5	1	3	5	7	9	11	0	2	5	7	9	11	3	0	5	7	9
4	6	8	10	12	8	10	12	2	4	6	8	10	12	4	1	6	8	10	12	2
0	12	2	4	0	2	4	6	8	10	12	1	3	6	8	10	12	2	4	6	8
5	7	9	11	7	9	11	1	3	5	7	9	11	0	2	5	7	9	11	1	3
11	1	3	5	1	3	5	7	9	11	0	2	5	7	9	11	0	3	5	7	9
6	8	10	12	8	10	0	2	4	6	8	12	10	1	4	6	8	12	10	2	4

Figure 4: 12-$L(2,1)$-labeling of $C_8 \boxtimes C_{21}$.

0	7	1	6	0	7	1	6	0	12	1	6	0	5	12	4	11	5	12	6	0	12	1	6	0	5	12	6
2	12	3	11	2	12	3	9	2	10	3	11	2	7	1	6	2	7	1	8	2	7	3	11	2	7	1	8
4	10	5	9	4	10	5	11	4	8	5	9	4	10	3	8	0	10	3	11	4	10	5	8	4	9	3	11
6	1	7	0	6	1	7	0	6	1	12	0	6	12	5	11	4	12	5	0	6	1	12	0	6	12	5	0
8	3	11	2	8	3	9	2	10	3	7	2	8	1	7	2	6	1	7	2	8	3	7	2	10	1	7	2
10	5	9	4	10	5	11	4	8	5	9	4	10	3	9	0	8	3	9	4	10	5	9	4	8	3	9	4
12	1	7	0	12	1	7	0	12	1	11	0	12	5	11	4	12	5	11	0	12	1	11	0	12	5	11	0
8	3	11	2	8	3	9	2	10	3	7	2	8	1	7	2	10	1	7	2	8	3	7	2	10	1	7	2
10	5	9	4	10	5	11	4	8	5	9	4	10	3	9	0	8	3	9	4	10	5	9	4	8	3	9	4

Figure 5: 12-$L(2,1)$-labeling of $C_9 \boxtimes C_{28}$.

Since we have also found 12-$L(2,1)$-labelings of $C_7 \boxtimes C_{27}$, $C_7 \boxtimes C_{37}$, $C_7 \boxtimes C_{38}$, and $C_7 \boxtimes C_{39}$, it follows that $\lambda(C_7 \boxtimes C_m) \leq 12$ for all graphs of interest and the proof is complete. □

Proposition 15. $\lambda(C_8 \boxtimes C_m) = 11$ *only if* $m \equiv 0 \pmod 6$.

Proof. The graph $D^2_{8,11}$ with 8157632 vertices and the largest outdegree 8 has been computed. Since breadth first search algorithm has found only cycles of length, six, twelve, and twenty-four, Theorem 9 yields the proof. □

Theorem 16. *If* $m \geq 8$, *then*

$$\lambda(C_8 \boxtimes C_m) = \begin{cases} 11, & m \equiv 0 \pmod 6 \\ 13, & m = 10 \\ 12, & \textit{otherwise.} \end{cases} \tag{9}$$

Proof. Since $\lambda(C_8 \boxtimes P_7) = 11$ (see Table 2), it follows from Lemma 1 that $\lambda(C_8 \boxtimes C_m) \geq 11$. Proposition 15 says that $\lambda(C_8 \boxtimes C_m) = 11$ only if $m \equiv 0 \pmod 6$, while the results for $m \leq 26$ follow from Table 2. Figure 4 shows a 12-$L(2,1)$-labeling of $C_8 \boxtimes C_{21}$. This labeling restricted to the first nine copies of C_8 induces a 12-$L(2,1)$-labeling of $C_8 \boxtimes C_9$. From Lemma 2 it follows that $\lambda(C_8 \boxtimes C_{9k+3}) \leq 12$, $k \geq 2$, and $\lambda(C_8 \boxtimes C_{9k}) \leq 12$, $k \geq 1$.

Moreover, we have found 12-$L(2,1)$-labelings of $C_8 \boxtimes C_{46}$, $C_8 \boxtimes C_{47}$, $C_8 \boxtimes C_{40}$, $C_8 \boxtimes C_{50}$, $C_8 \boxtimes C_{24}$, $C_8 \boxtimes C_{43}$, and $C_8 \boxtimes C_{44}$. Any of these labelings restricted to the first nine copies of C_8 is a 12-$L(2,1)$-labeling of $C_8 \boxtimes C_9$. From Lemma 2 it follows that $\lambda(C_8 \boxtimes C_{9k+1}) \leq 12$ for $k \geq 5$, $\lambda(C_8 \boxtimes C_{9k+2}) \leq 12$ for $k \geq 5$, $\lambda(C_8 \boxtimes C_{9k+4}) \leq 12$ for $k \geq 4$, $\lambda(C_8 \boxtimes C_{9k+5}) \leq 12$ for $k \geq 5$, $\lambda(C_8 \boxtimes C_{9k+6}) \leq 12$ for $k \geq 2$, $\lambda(C_8 \boxtimes C_{9k+7}) \leq 12$ for $k \geq 4$, and $\lambda(C_8 \boxtimes C_{9k+8}) \leq 12$ for $k \geq 4$.

Since we have also found 12-$L(2,1)$-labelings of $C_8 \boxtimes C_{28}$, $C_8 \boxtimes C_{29}$, $C_8 \boxtimes C_{31}$, $C_8 \boxtimes C_{32}$, $C_8 \boxtimes C_{34}$, $C_8 \boxtimes C_{35}$, $C_8 \boxtimes C_{37}$, $C_8 \boxtimes C_{38}$, and $C_8 \boxtimes C_{41}$, we establish the desired upper bound for all graphs of interest and the proof is complete. □

Theorem 17. *If* $m \geq 9$, *then*

$$\lambda(C_9 \boxtimes C_m) = \begin{cases} 13, & m = 9, 11, 15, 14, 18, 19, 22, 23 \\ 13 \textit{ or } 14, & m = 10 \\ 12 \textit{ or } 13, & m = 27, 31, 35 \\ 12, & \textit{otherwise.} \end{cases} \tag{10}$$

Proof. Since $\lambda(C_9 \boxtimes P_7) = 12$ (see Table 2), it follows from Lemma 1 that $\lambda(C_9 \boxtimes C_m) \geq 12$. The results for $m \leq 26$ follow from Table 2. We have also established by solving the SAT instances transformed from the corresponding $L(2,1)$-labeling problems that $\lambda(C_9 \boxtimes C_{10})$ is either 13 or 14, while for $m = 27, 31, 35$ the value of $\lambda(C_9 \boxtimes C_m)$ is either 12 or 13. Figure 5 shows a 12-$L(2,1)$-labeling of $C_9 \boxtimes C_{28}$. This labeling restricted to first 12 copies of C_9 induces a 12-$L(2,1)$-labeling of $C_9 \boxtimes C_{12}$. From Lemma 2 it follows that $\lambda(C_9 \boxtimes C_{12k+4}) \leq 12$, $k \geq 2$, and $\lambda(C_9 \boxtimes C_{12k}) \leq 12$, $k \geq 1$.

We have also found 12-$L(2,1)$-labelings of $C_9 \boxtimes C_{49}$, $C_9 \boxtimes C_{50}$, $C_9 \boxtimes C_{63}$, $C_9 \boxtimes C_{29}$, $C_9 \boxtimes C_{66}$, $C_9 \boxtimes C_{55}$, $C_9 \boxtimes C_{32}$, $C_9 \boxtimes C_{33}$, $C_9 \boxtimes C_{58}$, and $C_9 \boxtimes C_{59}$. Any of these labelings restricted to first 12 copies of C_9 induces a 12-$L(2,1)$-labeling of $C_9 \boxtimes C_{12}$. From Lemma 2 it follows that $\lambda(C_9 \boxtimes C_{12k+1}) \leq 12$ for $k \geq 4$, $\lambda(C_9 \boxtimes C_{12k+2}) \leq 12$ for $k \geq 4$, $\lambda(C_9 \boxtimes C_{12k+3}) \leq 12$ for $k \geq 5$, $\lambda(C_9 \boxtimes C_{12k+5}) \leq 12$ for $k \geq 2$, $\lambda(C_9 \boxtimes C_{12k+6}) \leq 12$ for $k \geq 5$, $\lambda(C_9 \boxtimes C_{12k+7}) \leq 12$ for $k \geq 4$, $\lambda(C_9 \boxtimes C_{12k+8}) \leq 12$ for $k \geq 2$, $\lambda(C_9 \boxtimes C_{12k+9}) \leq 12$ for $k \geq 2$, $\lambda(C_9 \boxtimes C_{12k+10}) \leq 12$ for $k \geq 4$, and $\lambda(C_9 \boxtimes C_{12k+11}) \leq 12$ for $k \geq 4$.

Since we have also found 12-$L(2,1)$-labelings of $C_9 \boxtimes C_{30}$, $C_9 \boxtimes C_{34}$, $C_9 \boxtimes C_{37}$, $C_9 \boxtimes C_{38}$, $C_9 \boxtimes C_{39}$, $C_9 \boxtimes C_{42}$, $C_9 \boxtimes C_{43}$, $C_9 \boxtimes C_{46}$, $C_9 \boxtimes C_{47}$, $C_9 \boxtimes C_{51}$, and $C_9 \boxtimes C_{54}$, we establish that $\lambda(C_9 \boxtimes C_m) \leq 12$ for all m of interest and the proof is complete. □

Theorem 18. *If $m \geq 10$, then*

$$\lambda(C_{10} \boxtimes C_m) = \begin{cases} 13, & m = 10, 11, 13, 15, 16, 17, 22, 26 \\ 12 \text{ or } 13, & 27 \leq m \leq 395 \\ 12, & \text{otherwise.} \end{cases} \tag{11}$$

Proof. We can see in Table 2 that $\lambda(C_{10} \boxtimes P_5) = 12$ hence, it follows by Lemma 1 that $\lambda(C_{10} \boxtimes C_m) \geq 12$. The results for $m \leq 26$ are given in Table 2.

The following pattern represents a 12-$L(2,1)$-labeling of $C_{10} \boxtimes C_{37}$, while the leftmost 12 columns of the pattern represents a 12-$L(2,1)$-labeling of $C_{10} \boxtimes C_{12}$:

8	3	1	12	8	5	7	2	0	11	9	6	8	3	1	12	10	7	4	2	0	11	9	6	8	3	1	12	10	5	7	4	2	0	11	9	6
0	11	9	6	3	1	12	10	5	7	4	2	0	11	9	6	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10	5	7	4	2
7	4	2	0	11	9	6	8	3	1	12	10	5	7	2	0	11	9	6	8	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10
1	12	10	5	7	4	2	0	11	9	6	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10	5	7	4	2	0	11	9	6	3
9	6	8	3	1	12	10	5	7	2	0	11	9	6	8	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10	5	7	2	0	11
4	2	0	11	9	6	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10	5	7	4	2	0	11	9	6	3	1	12	10	5	7
12	10	5	7	4	0	11	9	4	8	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10	5	7	2	0	11	9	6	8	3	1
6	8	3	1	12	8	5	7	2	0	11	9	6	8	3	1	12	10	5	7	4	2	0	11	6	8	3	1	12	10	5	7	4	2	0	11	9
2	0	11	9	6	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12	10	7	4	2	0	11	9	6	8	3	1	12	10	5	7	4
10	5	7	4	0	11	9	4	8	3	1	12	10	5	7	4	2	0	11	6	8	3	1	12	10	5	7	4	2	0	11	9	6	8	3	1	12

By Lemma 2, we have $\lambda(C_{10} \boxtimes C_{12\alpha+37\beta}) \leq 12$ for integers α and β. Finally, thanks to Lemma 3, we get $\lambda(C_{10} \boxtimes C_m) \leq 12$ for $m \geq (37-1) \cdot (12-1) = 396$.

We have found 13-$L(2,1)$-labelings of $C_{10} \boxtimes C_m$ for $27 \leq m \leq 46$ and we can construct 13-$L(2,1)$-labelings of $C_{10} \boxtimes C_m$ for $m \geq 36$ as follows. We have found 13-$L(2,1)$-labelings of $C_{10} \boxtimes C_{25}$, $C_{10} \boxtimes C_{50}$, $C_{10} \boxtimes C_{27}$, $C_{10} \boxtimes C_{28}$, $C_{10} \boxtimes C_{29}$, $C_{10} \boxtimes C_{54}$, and $C_{10} \boxtimes C_{39}$. Any of these labelings restricted to the first eight copies of C_{10} is a 13-$L(2,1)$-labeling of $C_{10} \boxtimes C_8$. From Lemma 2 it follows that $\lambda(C_{10} \boxtimes C_{8k}) \leq 13$ for $k \geq 1$, $\lambda(C_{10} \boxtimes C_{8k+1}) \leq 13$ for $k \geq 3$, $\lambda(C_{10} \boxtimes C_{8k+2}) \leq 13$ for $k \geq 6$, $\lambda(C_{10} \boxtimes C_{8k+3}) \leq 13$ for $k \geq 3$, $\lambda(C_{10} \boxtimes C_{8k+4}) \leq 13$ for $k \geq 3$, $\lambda(C_{10} \boxtimes C_{8k+5}) \leq 13$ for $k \geq 3$, $\lambda(C_{10} \boxtimes C_{8k+6}) \leq 13$ for $k \geq 6$, and $\lambda(C_{10} \boxtimes C_{8k+7}) \leq 13$ for $k \geq 4$. These observations complete the proof. □

Proposition 19. $\lambda(C_{11} \boxtimes C_m) = 10$ *only if* $m \equiv 0 \pmod{11}$.

Proof. The graph $D^2_{11,10}$ with 380 vertices and the largest out-degree 2 has been computed. Since breadth first search algorithm has found only cycles of length length 11, Theorem 9 yields the proof. □

Theorem 20. *If $m \geq 11$, then*

$$\lambda(C_{11} \boxtimes C_m) = \begin{cases} 10, & m \equiv 0 \pmod{11} \\ 12, & m \in \{12, 13, 14, 15, 16, 18, 19, \\ & \quad 20, 21, 22, 23, 25, 26\} \\ 11 & m \geq 270 \\ 11 \text{ or } 12, & \text{otherwise.} \end{cases} \tag{12}$$

Proof. For $m \in \{12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 25, 26\}$ the λ numbers are obtained by using the SAT reduction as depicted in Table 1. Since $\lambda(C_{11} \boxtimes P_3) = 10$, from Lemma 1 it follows that $\lambda(C_{11} \boxtimes C_m) \geq 10$, while from Proposition 19 it follows that $\lambda(C_{11} \boxtimes C_m) \geq 11$ if $m \not\equiv 0 \pmod{11}$.

The result for $m \equiv 0 \pmod{11}$ can be obtained by the fact that $\lambda(C_{11} \boxtimes C_{11}) = 10$ and by Lemma 2.

Figure 6 represents an 11-$L(2,1)$-labeling of $C_{11} \boxtimes C_{28}$, where the leftmost 11 columns of the figure represent an 11-$L(2,1)$-labeling of $C_{11} \boxtimes C_{11}$. By Lemma 2, we have $\lambda(C_{11} \boxtimes C_{11\alpha+28\beta}) \leq 11$ for integers α and β. Finally, thanks to Lemma 3, we get $\lambda(C_{11} \boxtimes C_m) \leq 11$ for $m \geq (28-1) \cdot (11-1) = 270$.

In order to find the general upper bound, we present the constructions showing that $\lambda(C_{11} \boxtimes C_m) \leq 12$ for $m \geq 26$. In particular, Figure 7 shows a 12-$L(2,1)$-labeling of $C_{11} \boxtimes C_{37}$. This labeling restricted to the first 12 copies of C_{11} induces a 12-$L(2,1)$-labeling of $C_{11} \boxtimes C_{12}$. From Lemma 2 it follows that $\lambda(C_{11} \boxtimes C_{12k+1}) \leq 12$, $k \geq 3$, and $\lambda(C_{11} \boxtimes C_{12k}) \leq 12$, $k \geq 1$.

Analogously, we have found 12-$L(2,1)$-labelings of $C_{11} \boxtimes C_{38}$, $C_{11} \boxtimes C_{39}$, $C_{11} \boxtimes C_{40}$, $C_{11} \boxtimes C_{41}$, $C_{11} \boxtimes C_{42}$, $C_{11} \boxtimes C_{43}$, $C_{11} \boxtimes C_{44}$, $C_{11} \boxtimes C_{45}$, $C_{11} \boxtimes C_{46}$, and $C_{11} \boxtimes C_{35}$. Any of these labelings restricted to the first 12 copies of C_{11} is a 12-$L(2,1)$-labeling of $C_{11} \boxtimes C_{12}$.

From Lemma 2 it follows that $\lambda(C_{11} \boxtimes C_{12k+2}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+3}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+4}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+5}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+6}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+7}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+8}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+9}) \leq 12$ for $k \geq 3$, $\lambda(C_{11} \boxtimes C_{12k+10}) \leq 12$ for $k \geq 3$, and $\lambda(C_{11} \boxtimes C_{12k+11}) \leq 12$ for $k \geq 2$.

Since we have also found 12-$L(2,1)$-labelings of $C_{11} \boxtimes C_{27}$, $C_{11} \boxtimes C_{28}$, $C_{11} \boxtimes C_{29}$, $C_{11} \boxtimes C_{30}$, $C_{11} \boxtimes C_{31}$, $C_{11} \boxtimes C_{32}$, $C_{11} \boxtimes C_{33}$, and $C_{11} \boxtimes C_{34}$, we establish that $\lambda(C_{11} \boxtimes C_m) \leq 12$ for all $m \geq 27$ and the proof is complete. □

4.3. λ-Numbers of $P_n \boxtimes C_m$

Proposition 21. *If $m \geq 3$, then*

$$\lambda(P_3 \boxtimes C_m) = \begin{cases} 14, & m = 5 \\ 11, & m \in \{3, 4, 6, 7, 8, 9, 10, 13, 14, 18, 19\} \\ 10, & \text{otherwise.} \end{cases} \tag{13}$$

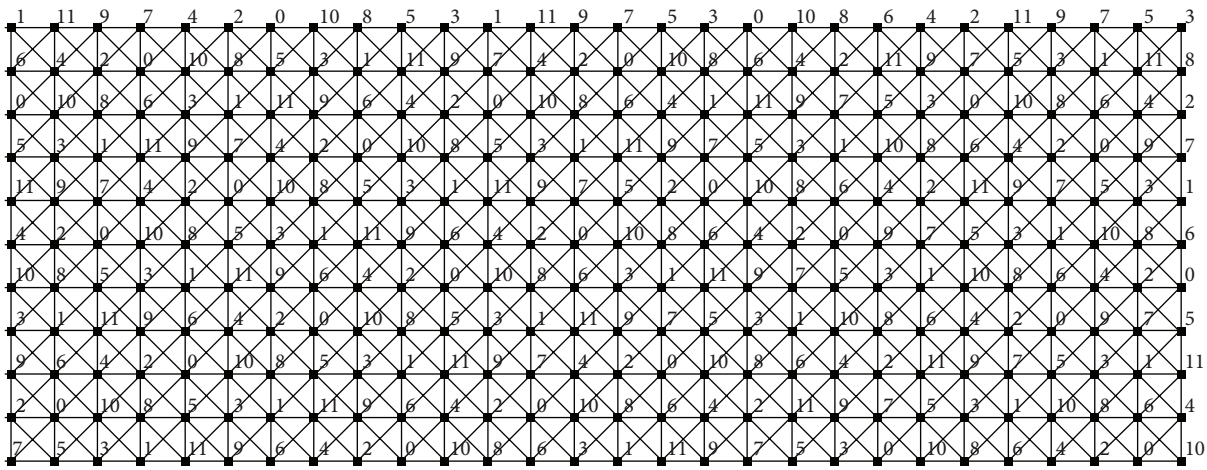

Figure 6: 11-$L(2,1)$-labeling of $C_{11} \boxtimes C_{28}$.

0	2	4	6	8	12	1	3	12	10	7	9	0	2	4	6	8	10	12	1	3	5	7	9	11	0	2	4	6	8	12	1	3	12	10	7	9
5	7	9	0	2	4	9	6	8	1	3	12	5	7	9	0	2	4	6	8	10	12	1	3	5	7	9	11	0	2	4	9	6	8	1	3	12
1	11	3	5	7	11	0	2	4	11	6	8	1	11	3	5	7	9	11	0	2	4	6	8	10	12	1	3	5	7	11	0	2	4	11	6	8
4	6	8	10	1	3	5	10	7	0	2	10	4	6	8	10	1	3	5	7	9	11	0	2	4	6	8	10	12	1	3	5	10	7	0	2	10
0	2	12	4	6	9	12	1	3	9	5	7	0	2	12	4	6	8	10	12	1	3	5	7	9	11	0	2	4	6	9	12	1	3	9	5	7
11	5	7	0	11	2	4	8	6	11	1	3	11	5	7	0	11	2	4	6	8	10	12	1	3	5	7	9	11	0	2	4	8	6	11	1	3
9	1	10	3	5	7	10	0	2	4	8	6	9	1	10	3	5	7	9	11	0	2	4	6	8	10	12	1	3	5	7	10	0	2	4	8	6
12	4	6	8	1	12	3	5	7	12	0	2	12	4	6	8	1	12	3	5	7	9	11	0	2	4	6	8	10	12	1	3	5	7	12	0	2
7	0	2	11	4	6	8	11	1	3	10	5	7	0	2	11	4	6	8	10	12	1	3	5	7	9	11	0	2	4	6	8	11	1	3	10	5
3	10	5	7	9	0	2	4	9	6	8	1	3	10	5	7	9	0	2	4	6	8	10	12	1	3	5	7	9	11	0	2	4	9	6	8	1
6	8	12	1	3	5	10	7	0	2	4	11	6	8	12	1	3	5	7	9	11	0	2	4	6	8	10	12	1	3	5	10	7	0	2	4	11

Figure 7: 12-$L(2,1)$-labeling of $C_{11} \boxtimes C_{37}$.

Proof. For $m \leq 12$ the results are obtained by solving the SAT instances transformed from the corresponding $L(2,1)$-labeling problems (see Table 1).

The graph $D'^{2}_{3,10}$ with 9080 vertices and the largest outdegree 16 has been created in order to find 10-$L(2,1)$-labelings in $P_3 \boxtimes C_m$. Matrix multiplication has been applied in order to find closed directed walks in the graph. The algorithm has found no closed directed walk of length from the set $\{3,4,6,7,8,9,10,13,14,18,19\} \cup \{5\}$. It follows that $\lambda(P_3 \boxtimes C_n) \geq 11$ for any $m \in \{3,4,6,7,8,9,10,13,14,18,19\}$. The upper bounds for $m \in \{13,14,18,19\}$ follow from the labelings depicted in Figure 8.

We have found 10-$L(2,1)$-labelings of $P_3 \boxtimes C_{23}$, $P_3 \boxtimes C_{35}$, $P_3 \boxtimes C_{47}$, $P_3 \boxtimes C_{26}$, $P_3 \boxtimes C_{38}$, $P_3 \boxtimes C_{39}$, $P_3 \boxtimes C_{40}$, $P_3 \boxtimes C_{41}$, $P_3 \boxtimes C_{31}$, and $P_3 \boxtimes C_{43}$. Any of these labelings restricted to the first 11 copies of P_3 is a 10-$L(2,1)$-labeling of $P_3 \boxtimes C_{11}$. From Lemma 2 it follows that $\lambda(P_3 \boxtimes C_{11k}) \leq 10$ for $k \geq 1$, $\lambda(P_3 \boxtimes C_{11k+1}) \leq 10$ for $k \geq 2$, $\lambda(P_3 \boxtimes C_{11k+2}) \leq 10$ for $k \geq 3$, $\lambda(P_3 \boxtimes C_{11k+3}) \leq 10$ for $k \geq 4$, $\lambda(P_3 \boxtimes C_{11k+4}) \leq 10$ for $k \geq 2$, $\lambda(P_3 \boxtimes C_{11k+5}) \leq 10$ for $k \geq 3$, $\lambda(P_3 \boxtimes C_{11k+6}) \leq 10$ for $k \geq 3$, $\lambda(P_3 \boxtimes C_{11k+7}) \leq 10$ for $k \geq 3$, $\lambda(P_3 \boxtimes C_{11k+8}) \leq 10$ for $k \geq 3$, $\lambda(P_3 \boxtimes C_{11k+9}) \leq 10$ for $k \geq 2$, and $\lambda(P_3 \boxtimes C_{11k+10}) \leq 10$ for $k \geq 3$.

Since we have also found 10-$L(2,1)$-labelings of $P_3 \boxtimes C_{13}$, $P_3 \boxtimes C_{14}$, $P_3 \boxtimes C_{15}$, $P_3 \boxtimes C_{16}$, $P_3 \boxtimes C_{17}$, $P_3 \boxtimes C_{18}$, $P_3 \boxtimes C_{19}$, $P_3 \boxtimes C_{20}$, $P_3 \boxtimes C_{21}$, $P_3 \boxtimes C_{24}$, $P_3 \boxtimes C_{25}$, $P_3 \boxtimes C_{27}$, $P_3 \boxtimes C_{28}$, $P_3 \boxtimes C_{29}$, $P_3 \boxtimes C_{30}$, $P_3 \boxtimes C_{32}$, and $P_3 \boxtimes C_{36}$, we establish that $\lambda(P_3 \boxtimes C_m) \leq 10$ for all $m \geq 13$ and the proof is complete. □

Proposition 22. *If $m \geq 3$, then*

$$\lambda(P_4 \boxtimes C_m) = \begin{cases} 14, & m = 5 \\ 10, & m \equiv 0 \pmod{11} \\ 11, & \textit{otherwise.} \end{cases} \tag{14}$$

Proof. For $m = 5$ the result is obtained by solving the SAT instance transformed from the corresponding $L(2,1)$-labeling problem. For $m \equiv 0 \pmod{11}$ the result follows from Lemma 2 and from the fact that $\lambda(C_{11} \boxtimes C_{11}) = 10$.

In order to find 10-$L(2,1)$-labelings in $P_4 \boxtimes C_m$, the graph $D'^{2}_{4,10}$ with 16792 vertices and the largest outdegree 3 has been created. Since breadth first search algorithm has found only cycles of length 11, the upper bound follows.

For all $m \not\equiv 0 \pmod{11}$ and $m \geq 13$ we can construct 11-$L(2,1)$-labelings of $P_4 \boxtimes C_m$ as described below. We have found 11-$L(2,1)$-labelings of $P_4 \boxtimes C_{13}$, $P_4 \boxtimes C_{14}$, $P_4 \boxtimes C_{15}$, $P_4 \boxtimes C_{16}$, and $P_4 \boxtimes C_{17}$. Any of these labelings restricted to the first six copies of P_4 is an 11-$L(2,1)$-labeling of $P_4 \boxtimes C_{11}$. From Lemma 2 it follows that $\lambda(P_4 \boxtimes C_{6k+1}) \leq 11$ for $k \geq 2$, $\lambda(P_4 \boxtimes C_{6k+2}) \leq 11$ for $k \geq 2$, $\lambda(P_4 \boxtimes C_{6k+3}) \leq 11$ for $k \geq 2$, $\lambda(P_4 \boxtimes C_{6k+4}) \leq 11$ for $k \geq 2$, and $\lambda(P_4 \boxtimes C_{6k+5}) \leq 11$ for $k \geq 2$. These conclusions complete the proof. □

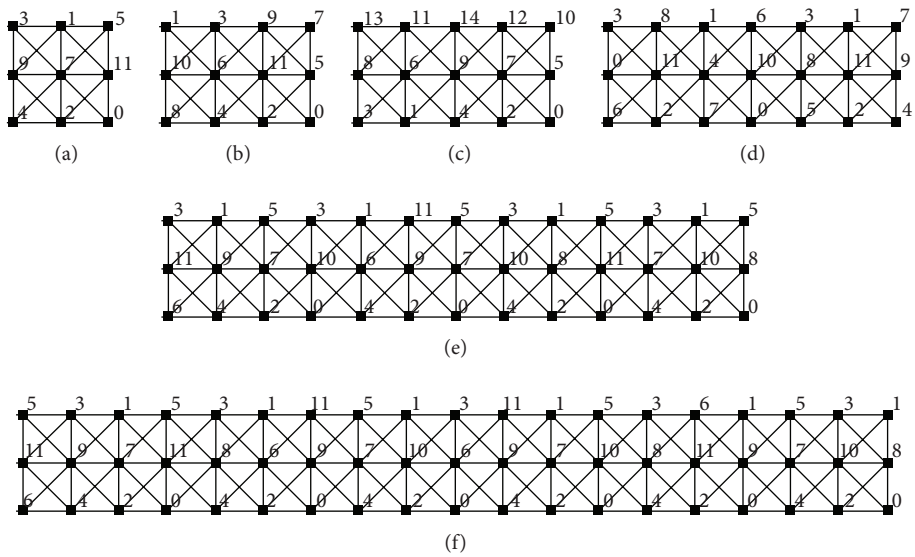

Figure 8: (a) 11-$L(2,1)$-labeling of $P_3 \boxtimes C_3$, (b) 11-$L(2,1)$-labeling of $P_3 \boxtimes C_4$, (c) 14-$L(2,1)$-labeling of $P_3 \boxtimes C_5$, (d) 11-$L(2,1)$-labeling of $P_3 \boxtimes C_7$, (e) 11-$L(2,1)$-labeling of $P_3 \boxtimes C_{13}$, and (f) 11-$L(2,1)$-labeling of $P_3 \boxtimes C_{19}$.

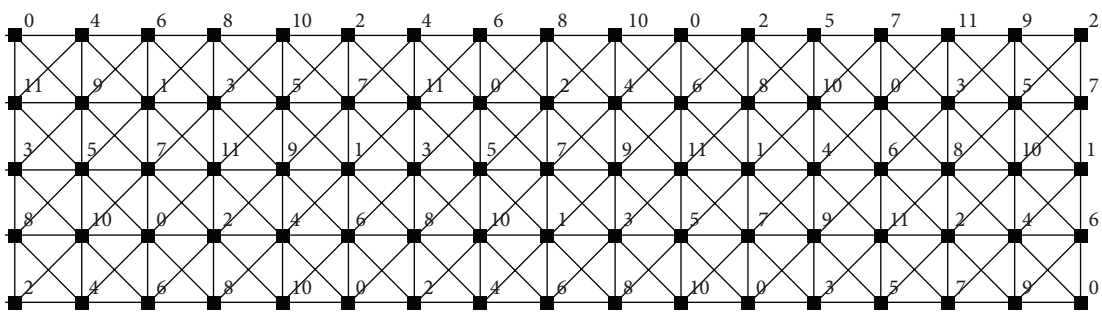

Figure 9: 11-$L(2,1)$-labeling of $P_5 \boxtimes C_{17}$.

Corollary 23. *Let $n, m \geq 4$.*

(i) *If $m \not\equiv 0 \pmod{11}$, then $\lambda(P_n \boxtimes C_m) \geq 11$,*

(ii) *If $n \not\equiv 0 \pmod{11}$ or $m \not\equiv 0 \pmod{11}$, then $\lambda(C_n \boxtimes C_m) \geq 11$.*

From Theorem 6 now we have the following

Corollary 24. *If $m \geq 24$ and $n \geq 26$, then*

$$\lambda(C_n \boxtimes C_m) = \begin{cases} 10, & n \equiv 0 \pmod{11} \\ & m \equiv 0 \pmod{11} \\ 11 \text{ or } 12, & \text{otherwise.} \end{cases} \tag{15}$$

Proposition 25. *If $m \geq 3$, then*

$$\lambda(P_5 \boxtimes C_m) = \begin{cases} 14, & m = 5 \\ 12, & m = 3, 9, 10 \\ 10, & m \equiv 0 \pmod{11} \\ 11, & m = 4, 6, 7, 8, 12 \text{ or } m \geq 80 \\ 11 \text{ or } 12, & \text{otherwise.} \end{cases} \tag{16}$$

Proof. For $m \leq 12$ the results follow from Table 1. For $m \equiv 0 \pmod{11}$ the result follows from Lemma 2 and from the fact that $\lambda(C_{11} \boxtimes C_{11}) = 10$.

Figure 9 represents an 11-$L(2,1)$-labeling of $P_5 \boxtimes C_{17}$, where the leftmost six columns of the figure represent an 11-$L(2,1)$-labeling of $P_5 \boxtimes C_6$.

By Lemma 2, we have $\lambda(P_5 \boxtimes C_{6\alpha+17\beta}) \leq 11$ for integers α and β. Finally, thanks to Lemma 3, we get $\lambda(P_5 \boxtimes C_m) \leq 11$ for $m \geq (17-1) \cdot (6-1) = 80$.

In order to complete the proof note that from Theorem 16 and Lemma 1 it follows that $\lambda(P_5 \boxtimes C_m) \leq 12$ for $m \geq 11$. □

Proposition 26. *If $m \geq 3$, then*

$$\lambda(P_6 \boxtimes C_m) = \begin{cases} 14, & m = 5 \\ 12, & m = 3, 9, 10 \\ 10, & m \equiv 0 \pmod{11} \\ 11, & m = 4, 6, 7, 8, 12 \text{ or } m \geq 154 \\ 11 \text{ or } 12, & \text{otherwise.} \end{cases} \tag{17}$$

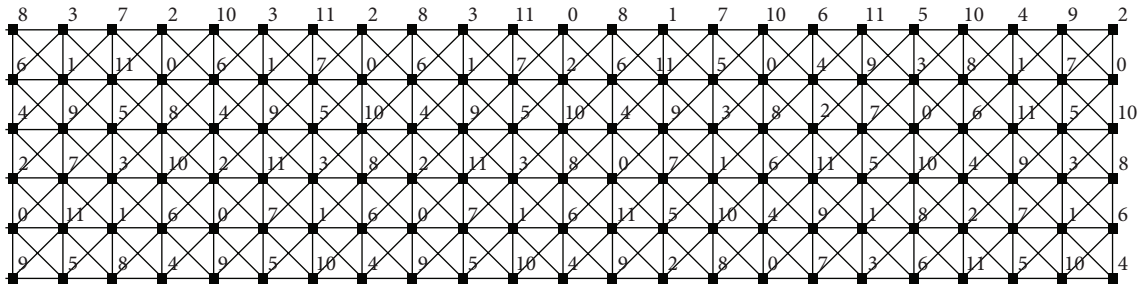

Figure 10: 11-$L(2,1)$-labeling of $P_6 \boxtimes C_{23}$.

Proof. For $m \leq 12$ the results follow from Table 1. For $m \equiv 0 \pmod{11}$ the result follows Lemma 2 and from the fact that $\lambda(C_{11} \boxtimes C_{11}) = 10$.

Figure 10 represents a 11-$L(2,1)$-labeling of $P_6 \boxtimes C_{23}$, where the leftmost eight columns of the figure represent an 11-$L(2,1)$-labeling of $P_6 \boxtimes C_8$.

By Lemma 2, we have $\lambda(P_6 \boxtimes C_{8\alpha+23\beta}) \leq 11$ for integers α and β. From Lemma 3 it follows that $\lambda(P_6 \boxtimes C_m) \leq 11$ for $m \geq (23-1)\cdot(8-1) = 154$.

We complete the proof by noting that from Theorem 16 and Lemma 1 it follows $\lambda(P_6 \boxtimes C_m) \leq 12$ for $m \geq 11$. □

Values in Table 1, the results from Section 4.2, Theorems 6 and 20, and Corollary 23 provide lower and upper bounds for the λ-number of $P_n \boxtimes C_m$. The results are summarized in the following.

Theorem 27. *If $n \geq 7$, then*

$$\lambda(P_n \boxtimes C_m) = \begin{cases} 10, & m \equiv 0 \pmod{11} \\ 14, & m = 5 \\ 12, & m = 3, 7, 9, 10 \\ 11, & m = 4, 6, 8, 12 \text{ or } m \geq 270 \\ 11 \text{ or } 12, & \text{otherwise.} \end{cases} \tag{18}$$

5. Conclusion

In this paper, the $L(2,1)$-labeling problem of the strong product of paths and cycles is studied. The problem derives from the more general Frequency Assignment Problem (FAP) which requires assigning frequencies to transmitters in a wireless network. It is well known that some interesting wireless networks are closely connected to the strong product of graphs. For example, an octagonal grid is the strong product of two paths and an octagonal torus is the strong product of two cycles.

By using various computational approaches, we succeed in solving the problem (except for the final number of cases) for the strong product of a path and a cycle, as well as for the the strong product of two cycles, where one of the cycles is of length at most eleven. Moreover, the obtained results enable us to improve the bounds on the λ-number for the strong product of two cycles, where both cycles are sufficiently long. Finding the exact λ-numbers for these graphs is therefore an interesting and challenging avenue of further research.

Conflict of Interests

The authors wish to confirm that there is no known conflicts of interests associated with this paper and there has been no significant financial support for this work that could have influenced its outcome. They confirm that the paper has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. They further confirm that the order of authors listed in the paper has been approved by all of us. They confirm that they have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing they confirm that they have followed the regulations of their institutions concerning intellectual property.

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