An \( L(2,1) \)-labeling of a graph \( G = (V,E) \) is a function \( f \) from the vertex set \( V(G) \) to the set of nonnegative integers such that the labels on adjacent vertices differ by at least two and the labels on vertices at distance two differ by at least one. The span of \( f \) is the difference between the largest and the smallest numbers in \( f(V) \). The \( \lambda \)-number of \( G \), denoted by \( \lambda(G) \), is the minimum span over all \( L(2,1) \)-labelings of \( G \). We consider the \( \lambda \)-number of \( P_n \bowtie C_m \) and for \( n \leq 11 \) the \( \lambda \)-number of \( C_n \bowtie C_m \). We determine \( \lambda \)-numbers of graphs of interest with the exception of a finite number of graphs and we improve the bounds on the \( \lambda \)-number of \( C_n \bowtie C_m \), \( m \geq 24 \) and \( n \geq 26 \).

1. Introduction

The Frequency Assignment Problem (FAP) requires assigning frequencies to transmitters in a wireless network. In a broadcasting network, each transmitter is assigned a frequency channel for its transmissions. Two transmissions can interfere if their channels are too close. That means that even if two transmitters use different channels, there still may be interference if the two transmitters are located close to each other [1, 2].

The spectrum of frequencies gets more and more scarce because of increasing demands, both civil and military. Thus the task is to minimize the span of frequencies while avoiding interference. One of the graph-theoretical models of FAP which is well elaborated is the concept of distance-constrained labeling of graphs [1]. Many variants of this concept have been proposed; however, the \( L(2,1) \)-labeling problem where adjacent vertices must be assigned colors of distance at least two apart and vertices of distance two must be assigned different colors has attracted the most of interest [3, 4].

An \( L(2,1) \)-labeling of a graph \( G \) is a function \( f \) from the vertex set \( V(G) \) to the set of nonnegative integers \( C \) (called labels or colors) such that for any two vertices \( u \) and \( v \)
\[
|f(u) - f(v)| \geq 2 \quad \text{if } \delta(u, v) = 1,
\]
\[
|f(u) - f(v)| \geq 1 \quad \text{if } \delta(u, v) = 2.
\] (1)

A \( k \)-\( L(2,1) \)-labeling is a \( L(2,1) \)-labeling of \( G \) such that \( C = \{0, \ldots, k\} \). An optimal \( L(2,1) \)-labeling of \( G \) is a \( k \)-\( L(2,1) \)-labeling with \( k \) smallest possible. The largest label used by an optimal \( L(2,1) \)-labeling is called the \( \lambda \)-number of \( G \) and denoted by \( \lambda(G) \).

There is a number of studies on the algorithms for \( L(2,1) \)-labeling problem [1, 5, 6]. It is known to be \( \text{NP} \)-hard for general graphs [4]. Even for some relatively simple families of graphs such as planar graphs, bipartite graphs, chordal graphs [5], and graphs of treewidth two [7], the problem is also \( \text{NP} \)-hard.

Product graphs are considered in order to gain global information from the factor graphs [8]. Many interesting wireless networks are based on product graphs with simple factors, such as paths and cycles. In particular, any square
grid (resp., torus) is the Cartesian product of two paths (resp., cycles) and any octagonal grid (resp., torus) is the strong product of two paths (resp., cycles) [9]. For the Cartesian product of these factors the \( \lambda \) numbers have been completely determined [10–12], while for the strong and the direct product only partial results have been found [13–15].

The paper is organized as follows. In Section 2, we give definitions and concepts needed in this paper. We also report on the known results for the \( \lambda \) numbers of the graphs of interest. In Section 3, two main computer search methods applied in the paper are described: the dynamic algorithm and the SAT reduction. Finally, in Section 4, we present the results on the \( \lambda \)-number of \( P_n \otimes C_m \) and the \( \lambda \)-number of \( C_n \otimes L_m \).

2. Preliminaries and Previous Results

For a graph \( G = (V, E) \), \( V(G) \) and \( E(G) \) are the sets of vertices and edges of \( G \), respectively. A directed graph \( D \) consists of vertices \( V(D) \) together with a set of arcs \( A(D) \subseteq V(D) \times V(D) \). We write \( G \) also to stand for the vertex set of the graph \( G \). In this paper, only directed and undirected graphs without multiple edges or loops are considered.

The strong product of graphs \( G \) and \( H \) is the graph \( G \otimes H \) with vertex set \( G \times H \) and \( (x_1, x_2)(y_1, y_2) \in E(G \otimes H) \) whenever \( x_1y_1 \in E(G) \) and \( x_2 = y_2 \), or \( x_2y_2 \in E(H) \) and \( x_1 = y_1 \), or \( x_1y_1 \in E(G) \) and \( x_2y_2 \in E(H) \). The strong product is commutative and associative, having the trivial graph as a unit (cf. [8]). The subgraph of \( G \otimes H \) induced by \( u \times V(H) \) is isomorphic to \( H \). It is called an \( H \)-fiber and denoted by \( H^n \).

The path \( P_n \) is the graph whose vertices are \( 0, 1, \ldots, n - 1 \) and for which two vertices are adjacent precisely if their difference is \( \pm 1 \). For an integer \( n \geq 3 \), the cycle of length \( n \) is the graph \( C_n \), whose vertices are \( 0, 1, \ldots, n - 1 \) and whose edges are the pairs \( i, i + 1 \), where the arithmetic is done modulo \( n \). Note that the strong product \( C_6 \otimes C_{13} \) depicted in Figure 1, can be regarded as a graph composed of six copies of \( C_{13} \) (denoted by \( C_{13}^0 \), \( C_{13}^1 \), \( C_{13}^2 \), \( C_{13}^3 \), \( C_{13}^4 \), \( C_{13}^5 \)) or a graph composed of 13 copies of \( C_6 \) (denoted by \( C_6^0 \), \( C_6^1 \), \( C_6^2 \), \( C_6^3 \), \( C_6^4 \), \( C_6^5 \), \( C_6^6 \), \( C_6^7 \), \( C_6^8 \), \( C_6^9 \), \( C_6^{10} \), \( C_6^{11} \), \( C_6^{12} \)).

A walk in a directed graph \( D \) is a sequence of (not necessarily distinct) vertices \( v_1, v_2, \ldots, v_t \) such that \( v_i v_{i+1} \in A(D) \) for \( 1, 2, \ldots, n - 1 \). If \( v_i = v_n \), we say it is a closed walk.

If \( P \) is a path (resp., walk), then its length is its number of edges (resp., arcs).

The following simple lemma is well known.

Lemma 1. If \( H \) is a subgraph of \( G \), then \( \lambda(H) \leq \lambda(G) \).

Let \( f \) denote a \( k-L(2, 1) \)-labeling of \( C_n \otimes G \). We denote by \( f_{i,p} \) the restriction of \( f \) to \( G^{i+1}, G^{i+1}, \ldots, G^{i+p-1}, i = 0, 1, \ldots, n-1 \) and \( p = 1, \ldots, n - p \). Note that \( G^i \) is isomorphic to \( G \); that is, \( G^i \) is the subgraph of \( G \otimes C_n \) induced by \( V(G) \times i \). We will also write \( f \) for \( f_{i,1} \).

The following lemma provides an upper bound for the \( \lambda \)-number of the strong product of a graph with a cycle.

Lemma 2. Let \( t \geq 1 \) and \( f \) be a \( k-L(2, 1) \)-labeling of \( C_n \otimes G \). If \( f_{0,p} \) is a \( k-L(2, 1) \)-labeling of \( C_p \otimes G \), then \( \lambda(C_n \otimes (t-1)p \otimes G) \leq k \).

Proof. Let \( f' \) be a function from \( V(C_n \otimes (t-1)p \otimes G) \) onto the set \( \{0, 1, \ldots, k\} \) and \( f'' \) the restriction of \( f' \) to \( V(G) \). The function \( f'' \) is defined as follows:

\[
f''_i = \begin{cases} f_i' & i < n \\ f_{(i-n) \mod p}' & i \geq n. \end{cases}
\]

It is not difficult to see that \( f'' \) is a \( k-L(2, 1) \)-labeling of \( C_n \otimes (t-1)p \otimes G \).

Given two integers \( r \) and \( s \), let \( S(r, s) \) denote the set of all nonnegative integer combinations of \( r \) and \( s \):

\[
S(r, s) = \{ \alpha r + \beta s : \alpha, \beta \in \mathbb{Z}^+ \}.
\]

We will need the result of Sylvester [16].

Lemma 3. If \( r, s > 1 \) are relatively prime integers, then \( t \in S(r, s) \) for all \( t \geq (s-1)(r-1) \).
Some partial results on the $\lambda$-number for the strong products of two cycles are given in [15].

**Theorem 4.** Let $m \geq 3$. Then

$$\lambda (C_3 \boxtimes C_m) = \begin{cases} 
16, & m = 3, 6 \\
14, & m = 5, 7, 10, 11, 15 \\
13, & m = 9, 14, 18, 19, 22, 23, 27, 31, 35 \\
12, & \text{otherwise.} 
\end{cases}$$

**Theorem 5.** Let $m \geq 3$. Then

$$\lambda (C_4 \boxtimes C_m) = \begin{cases} 
19, & m = 5 \\
15, & m = 4, 8 \\
14, & m = 11 \\
13, & m = 7, 10, 14, 17, 20, 23 \\
11, & m \equiv 0 \pmod{6} \\
12, & \text{otherwise.} 
\end{cases}$$

**Theorem 6.** If $m \geq 24$ and $n \geq 36$, then $\lambda(C_n \boxtimes C_m) \leq 12$.

For the strong product of more than two cycles the following result presented in [14] is known.

**Theorem 7.** If $k \geq 1$ and $m_0, \ldots, m_{k-1}$ are each multiple of $3^k + 2$, then $\lambda(C_{m_0} \boxtimes \cdots \boxtimes C_{m_{k-1}}) = 3^k + 1$.

### 3. Computer Search

#### 3.1. Dynamic Algorithm.

The idea is introduced in [10] in a more general framework and later used several times, for example, [13, 15]. In order to make the paper self-contained we first describe its basic definitions and results.

We define a digraph $D_{nk}$ as follows. Its vertices are the $k$-$L(2,1)$-labelings of $C_n \boxtimes P_2$. Let $u = u_1u_2$ be a vertex of $D_{nk}$. Then $u_1$ and $u_2$ represent the $k$-$L(2,1)$-labeling of $C_n \boxtimes P_2$ restricted to the first and second copies of $C_n$, respectively.

Let $u$ and $v$ be two vertices of $D_{nk}$. Then $\overline{uv}$ denotes the labeling of $C_n \boxtimes P_2$ obtained by applying $u_1, u_2$, and $v_2$ to the consecutive copies of $C_n$. (Note that $\overline{uv}$ is not always a $k$-$L(2,1)$-labeling of $C_n \boxtimes P_2$.) We make an arc from $u$ to $v$ in $D_{nk}$ if and only if the following two conditions are fulfilled:

(i) $u_2$ equals $v_1$; (ii) $\overline{uv}$ is a $k$-$L(2,1)$-labeling of $C_n \boxtimes P_3$.

Analogously, we define a digraph $D'_{nk}$ with the vertex set composed by $k$-$L(2,1)$-labelings of $P_n \boxtimes P_2$. In other words, if $u = u_1u_2$ is a vertex of $D'_{nk}$, then $u_1$ and $u_2$ represent the $k$-$L(2,1)$-labeling of $P_n \boxtimes P_2$ restricted to the first and second copies of $P_n$, respectively. The set of arcs of $D'_{nk}$ is formed analogously as the set of arcs of $D_{nk}$.

Figure 2(a) shows two vertices of $D_{6,12}$ denoted by $u$ and $v$. We can see that the labeling of the second copy of $C_6$ in $u$ equals the labeling of the first copy of $C_6$ in $v$. Moreover, the labeling of $u$ and the labeling of the second copy of $C_6$ in $v$ induce a $12$-$L(2,1)$-labeling of $C_6 \boxtimes P_3$. It follows that $A(D_{6,12})$ admits an arc from $u$ to $v$.

The next theorem follows from the results presented in [10].

**Theorem 8.** $C \boxtimes C_6$ (resp., $P_1 \boxtimes C_6$) admits a $k$-$L(2,1)$-labeling if and only if $D_{6,k}$ (resp. $D'_{6,k}$) contains a closed directed walk of length $\ell$.

The dynamic algorithm first generates all $k$-$L(2,1)$-labelings of $C_n \boxtimes P_2$ which are the vertices of $D_{nk}$. Since a main building block $C_i \boxtimes P_2$ is usually relatively small, a simple method, for example, backtracking, can be applied for this step. In the next step, the set of edges of $D_{nk}$ has to be generated. The procedure for this step is described in [15].

The described algorithm however has the time complexity $O(m^2)$, where $m$ denotes the number of vertices in $D_{nk}$. Note that $m$ can be very large even for $k$ and $\ell$ of a moderate size. Some examples for $k$ and $\ell$ of interest are $|D_{5,14}| = 114984000$, $|D_{6,11}| = 1925760$, and $|D_{8,11.1}^2| = 107253264$. The complexity of the algorithm does not allow a computation of $A(D_{6,k})$ in a reasonable time for these cases. We have therefore improved this method as described in the sequel.

Let $V_{ij,3}^3$ denote the set of all $k$-$L(2,1)$-labelings of $C_i \boxtimes P_5$. If $u$ is an element of $V_{ij,3}^3$, then $u_1, u_2, u_3$ denote the restriction of $u$ to the first, second, and third copies of $C_i$ in $C_i \boxtimes P_5$.
Procedure CREATEGRAPH(\(i, k, N\));
begin
\(V^3_{i,k}\) := set of all \(k\)-\(L(2, 1)\)-labelings of \(C_i \boxtimes P_3\);
for all \(v = v_1v_2v_3 \in V^3_{i,k}\) do Insert \(v_1v_2\) in \(V^3_{i,k}\);
Sort vertices \(u_i, u_3\) of \(V^3_{i,k}\) with respect to \(u_1\), vertices with the same value of \(u_i\) sort with respect to \(u_2\);
for all \(v = v_1v_2v_3 \in V^3_{i,k}\) do
Find \(u = u_iu_3\) in \(V^3_{i,k}\) such that \(u_1 = v_1\) and \(u_2 = v_2\);
Find \(w = w_iw_3\) in \(V^3_{i,k}\) such that \(u_1 = v_1\) and \(u_2 = v_3\); Insert \(w\) in \(N_{u_i}\);
end.

Algorithm 1

respectively. Note that \(V^3_{i,k}\) contains symmetric labelings of \(C_i \boxtimes P_3\), that is, if \(u \in V^3_{i,k}\), then \(v \in V^3_{i,k}\) exists, such that \(u_1 = v_3, u_3 = v_1,\) and \(u_2 = v_2\).

We now define the digraph \(D^2_{i,k}\) as follows. Let \(V(D^2_{i,k})\) denote the set of \(k\)-\(L(2, 1)\)-labelings of \(C_i \boxtimes P_3\) obtained from \(V^3_{i,k}\) in the following way: \(x = x_1x_2x_3\) belongs to \(V(D^2_{i,k})\) if and only if there exist \(u, v \in V^3_{i,k}\) such that \(u_1 = x_1, u_2 = x_2,\) and \(v_3 = x_3, v_1 = x_2\).

Figure 2(b) shows two vertices of \(V^3_{6,12}\) denoted by \(u\) and \(v\). We can see that the labelings of the first and the second copies of \(C_i\) in \(u\) equal the third and the second copies of \(C_i\) in \(v\), respectively. It follows that \(V(D^2_{6,12})\) possesses the vertex \(x\) comprising these two labelings.

Let \(x, y \in V(D^2_{i,k})\). We make an arc from \(x = x_1x_2x_3\) to \(y = y_1y_2y_3\) if and only if \(x_2 = y_1\) and \(x_1x_2y_2\) belongs to \(V^3_{i,k}\).

Note that analogous as above we can improve the method for \(P_i \boxtimes C_\ell\). The graph obtained with this procedure (a subgraph of \(D^2_{i,k}\)) will be denoted by \(D^2_{i,k}\) in the sequel.

For a vertex \(v\) of a directed graph \(D\), the number of inward (resp., outward) directed arcs from \(v\) in \(D\) is called an indegree (resp., outdegree) and denoted by \(\text{indeg}_{D(v)}\) (resp., \(\text{outdeg}_{D(v)}\)).

We obtain the main result of this section.

**Theorem 9.** \(C_i \boxtimes C_\ell\) (resp., \(P_i \boxtimes C_\ell\)) admits a \(k\)-\(L(2, 1)\)-labeling if and only if \(D^2_{i,k}\) (resp., \(D^2_{i,k}\)) contains a closed directed walk of length \(\ell\).

**Proof.** It is easy to see that \(V(D^2_{i,k}) \subseteq V(D_{i,k})\) and \(A(D^2_{i,k}) \subseteq A(D_{i,k})\). From the definition of \(V^3_{i,k}\) for \(x \in V(D^2_{i,k})\) it follows that \(\text{indeg}_{D_{i,k}}(x) > 0\) and \(\text{outdeg}_{D_{i,k}}(x) > 0\).

Suppose now for \(y \in V(D_{i,k})\) that \(\text{indeg}_{D_{i,k}}(y) > 0\) and \(\text{outdeg}_{D_{i,k}}(y) > 0\). Since \(V^3_{i,k}\) contains all \(k\)-\(L(2, 1)\)-labelings of \(C_i \boxtimes P_3\) and \(\text{outdeg}_{D_{i,k}}(y) > 0\), there has to be a vertex \(v \in V^3_{i,k}\) such that \(v_1 = y_1\) and \(v_2 = y_2\). Moreover, since \(\text{indeg}_{D_{i,k}}(y) > 0\), there has to be a vertex \(u \in V^3_{i,k}\) such that \(u_2 = y_1\) and \(u_3 = y_2\). It follows that \(y\) belongs to \(V(D_{i,k})\).

We have therefore proven that \(x \in V(D^2_{i,k})\) if and only if \(\text{indeg}_{D_{i,k}}(x) > 0\) and \(\text{outdeg}_{D_{i,k}}(x) > 0\).

Analogously as above we can show that \(xy\) is an arc in \(D^2_{i,k}\) if and only if \(xy\) is an arc in \(D_{i,k}\). In other words, we can show that \(D^2_{i,k}\) is isomorphic to subgraph of \(D_{i,k}\) induced by \(V^3_{i,k}\). It follows that \(D^2_{i,k}\) contains a closed directed walk of length \(\ell\) if and only if \(D_{i,k}\) contains a closed directed walk of the same size and the proof for \(D_{i,k}\) is settled.

Since the proof for \(D^2_{i,k}\) is analogous, the proof of the theorem is complete.

The algorithm for generating the graph \(D^2_{i,k}\) is depicted in Algorithm 1 (Procedure CREATE GRAPH).

Note that the number of vertices of \(D^2_{i,k}\) can be much smaller than in \(D_{i,k}\). Some examples for \(i\) and \(k\) of interest are: \(|D^2_{5,14}| = 386345700, |D^2_{6,11}| = 12336,\) and \(|D^2_{7,11}| = 8157632\). However, this reduction is not the only positive effect of the new approach. It is also of a great importance that the running time of CREATE GRAPH is \(O(n \log n)\), where \(n\) the number of vertices of \(V^3_{i,k}\). In order to see this, note that the running time of an efficient sorting algorithm is also within this time bound. Moreover, this is also the running time of the duration of loop, since a single search in an ordered list with \(n\) elements requires \(O(\log n)\) time.

The final step of the approach is the search for closed directed walks in \(D^2_{i,k}\). We can find these walks by applying a matrix multiplication of the adjacancy matrix of \(D^2_{i,k}\) or a breadth (depth) first search in \(D^2_{i,k}\). Since graphs \(D^2_{i,k}\) are relatively sparse for \(i\) and \(k\) of interest, the later approach has been applied in order to compute the results of this paper.

### 3.2. SAT Reduction for \(k\)-\(L(2, 1)\)-Labeling

The approach is proposed in [17] for the distance-constrained labeling problem. Here we present this method adapted for \(L(2, 1)\)-labeling.

Let \(G = (V, E)\) be a graph and \(k\) a positive integer. For every \(v \in V\) and every \(i \in \{0, 1, 2, \ldots, k\}\) introduce an atom \(x_{v,i}\). Intuitively, this atom shows that the vertex \(v\) is assigned the color \(i\). Consider the following propositional formulas:

1. (1) for all \(v \in V, x_{v,0} \lor \neg x_{v,k}\);
2. (2) for all \(v \in V, 0 \leq i < j \leq k, x_{v,i} \lor \neg x_{v,j}\);
3. (3) for all \(v, u \in V, 0 \leq i, j \leq k\) with \(d(v, u) = 1\) and \(|i - j| < 2\) or \(d(v, u) = 2\) and \(|i - j| < 1, \neg x_{v,i} \lor \neg x_{v,j}\).

Clauses (1) and (2) ensure that each vertex is labeled with exactly one label. Clause (3) guarantees that an obtained
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Table 1: Values of $\lambda(P_n \bowtie C_m)$.

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Table 2: Values of $\lambda(C_n \bowtie C_m)$.

| n \ m | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 5     | 24 | $14^a$ | 17 | 19 | 14 | 16 | 18 | 14 | 16 | 17 | 14 | 15 | 16 | 14 | 15 | 16 | 14 | 15 | 16 | 14 | 15 | 16 |
| 6     | 16 | $14^a$ | 11 | 13 | 13 | 13 | 11 | 13 | 12 | 13 | 12 | 13 | 12 | 13 | 12 | 13 | 12 | 13 | 12 | 13 | 12 | 13 |
| 7     | 13 | 13 | 13 | 12 | 12 | 12 | 13 | 13 | 13 | 13 | 13 | 13 | 12 | 12 | 12 | 13 | 13 | 13 | 13 | 13 | 13 | 13 |
| 8     | 13 | 13 | 13 | 12 | 12 | 11 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |
| 9     | 13 | 13 | $13/14$ | 13 | 12 | 13 | 12 | 12 | 12 | 13 | 13 | 12 | 12 | 13 | 13 | 12 | 12 | 13 | 13 | 12 | 12 | 13 |
| 10    | 13 | 13 | 12 | 12 | 12 | 12 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 |
| 11    | 10 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 10 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |
| 12    | 11 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |

4. Results

4.1. SAT Reduction. We solve the SAT instances transformed from $L(2,1)$-labeling problems described in Section 4.2 by using the software MiniSat [18]. As a result, we have obtained the $\lambda$-numbers of $P_n \bowtie C_m$ presented in Table 1 and the $\lambda$-numbers of $C_n \bowtie C_m$ presented in Table 2.

The values in Table 2 marked with $a$ denote the results already obtained in [15], while the entry with $13/14$ means that the corresponding value is either 13 or 14.

4.2. $\lambda$-Labeling of $C_n \bowtie C_m$

Proposition 10. $\lambda(C_5 \bowtie C_m) = 14$ only if $m \equiv 0 \pmod{3}$.

Proof. Note that $\lambda(C_5 \bowtie P_2) = 14$. We can see that $\lambda(C_5 \bowtie P_3) \leq 14$ from the fact that every pair of vertices $u, v \in V(C_5 \bowtie P_3)$ is at distance at most two. Let $f$ denote a $14$-$L(2,1)$-labeling of $C_5 \bowtie C_m$ and $f_k$ its restriction to $C_5\times C^{k}_m$. Let also $L_k$ denote the set of labels used in $f_k$. Since $\lambda(C_5 \bowtie P_3) = 14$, we have $|L_1| = 5$ and $|L_2| + |L_{k+1}| = 10$. Therefore, the restriction of $f$ to $C^{k}_5 \times C^{k}_m$ has to comprise the same set of labels as the restriction of $f$ to $C_5$ or, more formally, $L_k = L_{k+1}$. It is straightforward to see that this equality can be satisfied in $C_5 \bowtie C_m$ only if $m \equiv 0 \pmod{3}$.

Theorem 11. Let $m \geq 5$. Then

$$\lambda(C_5 \bowtie C_m) = \begin{cases} 14, & m \equiv 0 \pmod{3}, \\ 17, & m = 7, 14, \\ 19, & m = 8, \\ 18, & m = 11, \\ 16, & m = 10, 13, 17, 20, 23, 26, 29, \\ 15, & \text{otherwise}. \end{cases}$$

Proof. Note that the values for $m \leq 26$ are given in Table 2. We can also show by using the SAT reduction that $\lambda(C_5 \bowtie C_{29}) = 16$. Since $\lambda(C_5 \bowtie C_j) = 14$ only if $j = 0 \pmod{3}$, we construct below a $14$-$L(2,1)$-labeling for $C_5 \bowtie C_{3j}$, $j \geq 1$, a $15$-$L(2,1)$-labeling for $C_5 \bowtie C_{3j+1}$, $j \geq 5$, and a $15$-$L(2,1)$-labeling for $C_5 \bowtie C_{3j+2}$, $j \geq 10$.

Let $[a, b]$ for $b \geq a$ denote the set $\{a, a + 1, \ldots, b\}$.

Let $f$ denote a function from $V(C_5 \bowtie C_m)$ to $[0, 14]$ and $f_k$ its restriction to $C_5^k$, $k \leq m - 1$. Let also $L_k$ denote the set of labels used in $f_k$. If we set for $i \geq 0$ and $s \in \{0, 1, 2\}$: $L_{3i+1} = [5s, 5s + 4]$, then $f$ is a $14$-$L(2,1)$-labeling of $C_5 \bowtie C_{3j}$ for $j \geq 1$.

Let $f'$ denote a function from $V(C_5 \bowtie C_m)$ to $[0, 15]$ and $f'_k$ its restriction to $C_5^k$, $k \leq m - 1$. Let also $L_k$ denote the set of labels used in $f'_k$. If we set for $0 \leq i \leq 3$: $L'_{3i} = [15 - i, 15] \cup [0, 3 - i], L'_{3i+1} = [4 - i, 8 - i], L'_{3i+2} = [9 - i, 13 - i]$ and for $i \geq 4$ and $s \in \{0, 1, 2\}$: $L'_{3i+1} = [5s, 5s + 4]$, then $f'$ is a $15$-$L(2,1)$-labeling of $C_5 \bowtie C_{3j+1}$ for $j \geq 5$. As an example, observe the following pattern representing a $15$-$L(2,1)$-labeling of $C_5 \bowtie C_{22}$:
The following results partially depend on comprehensive constructions which provide labels of interest. These constructions are mostly not included in this paper and can be obtained by the authors.

**Proposition 12.** $\lambda(C_6 \otimes C_m) = 11$ only if $m \equiv 0 \pmod{4}$.

**Proof.** The graph $D_{6,4}$ with 12336 vertices and the largest outdegree six has been computed. Since breadth first search algorithm has found only cycles of length four, Theorem 9 yields the proof.

**Theorem 13.** If $m \geq 6$, then

$$\lambda(C_6 \otimes C_m) = \begin{cases} 11, & m \equiv 0 \pmod{4} \\ 16, & m = 6 \\ 14, & m = 7 \\ 13, & m = 9, 10, 11, 14 \\ 12, & \text{otherwise}. \end{cases}$$

**Proof.** The results for $m = 6, 7, 9, 10, 11, 14$ follow from Table 2. We can also see in Table 1 that $\lambda(C_6 \otimes P_5) = 11$; thus, from Lemma 1 it follows that $\lambda(C_6 \otimes C_m) \geq 11$. Moreover, Proposition 12 says that $\lambda(C_6 \otimes C_m) = 11$ only if $m \equiv 0 \pmod{4}$. In order to see that $\lambda(C_6 \otimes C_m)$ is ten for other $m$ of interest, see as an example a 12-$L(2, 1)$-labeling of $C_6 \otimes C_{13}$ depicted in Figure 1.

Let $f''$ denote a function from $V(C_5 \otimes C_m)$ to $[0, 15]$ and $f''_k$ its restriction to $C_5^k, k \leq m - 1$. Let also $L''_k$ denote the set of labels used in $f''_k$. If we set for $0 \leq k \leq 31$: $L_k := L''_k$ and for $i \geq 11$ and $s \in \{0, 1, 2\}: L_{3j+1} := [5s, 5s+4]$, then $f''_6$ is a $15$-$L(2, 1)$-labeling of $C_5 \otimes C_{3j+2}$ for $j \geq 10$. This assertion concludes the proof.

Since we have found 12-$L(2, 1)$-labelings of $C_6 \otimes C_{27}$, $C_6 \otimes C_{28}$, $C_6 \otimes C_{31}$, $C_6 \otimes C_{32}$, $C_6 \otimes C_{35}$, and $C_6 \otimes C_{37}$, we conclude that $\lambda(C_6 \otimes C_m)$ is twelve for $m \geq 27$ and the proof is complete.

**Theorem 14.** If $m \geq 7$, then

$$\lambda(C_6 \otimes C_m) = \begin{cases} 13, & m = 7, 8, 9, 14, 15, 16, 17, \\ 18, 19, 27, 28, 29, & \text{otherwise}. \end{cases}$$

**Proof.** The results for $m \leq 26$ follow from Table 2. We have also established the results for $m = 27, 28, 29$ by solving the SAT instances transformed from the corresponding $L(2, 1)$-labeling problems. Since $\lambda(C_7 \otimes P_5) = 12$ as we can see in Table 2, it follows by Lemma 1 that $\lambda(C_7 \otimes C_m) \geq 12$ for $m \geq 7$.

In order to see that $\lambda(C_7 \otimes C_m)$ is twelve for other $m$ of interest, see as an example a 12-$L(2, 1)$-labeling of $C_7 \otimes C_{10}$ depicted in Figure 3.

| 15 | 4 | 9 | 14 | 3 | 8 | 13 | 2 | 7 | 12 | 1 | 6 | 11 | 0 | 5 | 10 | 0 | 5 | 10 | 0 | 5 | 10 |
| 0 | 5 | 10 | 15 | 4 | 9 | 14 | 3 | 8 | 13 | 2 | 7 | 12 | 1 | 6 | 11 | 0 | 5 | 10 | 0 | 5 | 10 |
| 1 | 6 | 11 | 0 | 5 | 10 | 15 | 4 | 9 | 14 | 3 | 8 | 13 | 2 | 7 | 12 | 1 | 6 | 11 | 0 | 5 | 10 |
| 2 | 7 | 12 | 1 | 6 | 11 | 0 | 5 | 10 | 15 | 4 | 9 | 14 | 3 | 8 | 13 | 3 | 8 | 13 | 3 | 8 | 13 |
| 3 | 8 | 13 | 2 | 7 | 12 | 1 | 6 | 11 | 0 | 5 | 10 | 15 | 4 | 9 | 14 | 4 | 9 | 14 | 4 | 9 | 14 |

**Figure 3:** 12-$L(2, 1)$-labeling of $C_7 \otimes C_{21}$. 

| $C_7 \otimes C_{13k+1}$ | $C_7 \otimes C_{13k+2}$ | $C_7 \otimes C_{13k+3}$ | $C_7 \otimes C_{13k+4}$ | $C_7 \otimes C_{13k+5}$ | $C_7 \otimes C_{13k+6}$ | $C_7 \otimes C_{13k+7}$ | $C_7 \otimes C_{13k+8}$ | $C_7 \otimes C_{13k+9}$ | $C_7 \otimes C_{13k+10}$ | $C_7 \otimes C_{13k+11}$ | $C_7 \otimes C_{13k+12}$ |
| 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |

| $C_7 \otimes C_{10k+1}$ | $C_7 \otimes C_{10k+2}$ | $C_7 \otimes C_{10k+3}$ | $C_7 \otimes C_{10k+4}$ | $C_7 \otimes C_{10k+5}$ | $C_7 \otimes C_{10k+6}$ | $C_7 \otimes C_{10k+7}$ | $C_7 \otimes C_{10k+8}$ | $C_7 \otimes C_{10k+9}$ | $C_7 \otimes C_{10k+10}$ | $C_7 \otimes C_{10k+11}$ | $C_7 \otimes C_{10k+12}$ |
| 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |

**Figure 3:** This labeling restricted to the first ten copies of $C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

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**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.

**Figure 3:** $L(2, 1)$-labeling of $C_7 \otimes C_6$.$C_7 \otimes C_6$.
Since we have also found $12\cdot L(2,1)$-labelings of $C_{9} \bowtie C_{27}$, $C_{9} \bowtie C_{23}$, $C_{9} \bowtie C_{28}$, and $C_{9} \bowtie C_{38}$, it follows that $\lambda(C_{9} \bowtie C_{m}) \leq 12$ for all graphs of interest and the proof is complete.

**Proposition 15.** $\lambda(C_{9} \bowtie C_{m}) = 11$ only if $m \equiv 0 \pmod{6}$.

**Proof.** The graph $D_{8}^{11}$ with 8157632 vertices and the largest outdegree 8 has been computed. Since breadth first search algorithm has found only cycles of length, six, twelve, and twenty-four, Theorem 9 yields the proof.

**Theorem 16.** If $m \geq 8$, then

$$\lambda(C_{9} \bowtie C_{m}) = \begin{cases} 11, & m \equiv 0 \pmod{6} \\ 13, & m = 10 \\ 12, & \text{otherwise.} \end{cases} \quad (9)$$

**Proof.** Since $\lambda(C_{9} \bowtie P_{7}) = 11$ (see Table 2), it follows from Lemma 1 that $\lambda(C_{9} \bowtie C_{m}) \geq 11$. Proposition 15 says that $\lambda(C_{9} \bowtie C_{m}) = 11$ only if $m \equiv 0 \pmod{6}$, while the results for $m \leq 26$ follow from Table 2. Figure 4 shows a $12\cdot L(2,1)$-labeling of $C_{9} \bowtie C_{21}$. This labeling restricted to the first nine copies of $C_{9}$ induces a $12\cdot L(2,1)$-labeling of $C_{9} \bowtie C_{9}$. From Lemma 2 it follows that $\lambda(C_{9} \bowtie C_{9k+3}) \leq 12$, $k \geq 2$, and $\lambda(C_{9} \bowtie C_{9k}) \leq 12$, $k \geq 1$.

Moreover, we have found $12\cdot L(2,1)$-labelings of $C_{9} \bowtie C_{46}$, $C_{9} \bowtie C_{47}$, $C_{9} \bowtie C_{48}$, $C_{9} \bowtie C_{49}$, $C_{9} \bowtie C_{50}$, $C_{9} \bowtie C_{51}$, $C_{9} \bowtie C_{52}$, $C_{9} \bowtie C_{53}$, and $C_{9} \bowtie C_{54}$. Any of these labelings restricted to the first nine copies of $C_{9}$ is a $12\cdot L(2,1)$-labeling of $C_{9} \bowtie C_{9}$. From Lemma 2 it follows that $\lambda(C_{9} \bowtie C_{9k+1}) \leq 12$ for $k \geq 5$, $\lambda(C_{9} \bowtie C_{9k+2}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{9k+3}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{9k+4}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{9k+5}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{9k+6}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{9k+7}) \leq 12$ for $k \geq 4$, and $\lambda(C_{9} \bowtie C_{9k+8}) \leq 12$ for $k \geq 4$.

Since we have also found $12\cdot L(2,1)$-labelings of $C_{9} \bowtie C_{28}$, $C_{9} \bowtie C_{29}$, $C_{9} \bowtie C_{31}$, $C_{9} \bowtie C_{32}$, $C_{9} \bowtie C_{34}$, $C_{9} \bowtie C_{35}$, $C_{9} \bowtie C_{37}$, $C_{9} \bowtie C_{38}$, and $C_{9} \bowtie C_{41}$, we establish the desired upper bound for all graphs of interest and the proof is complete.

**Theorem 17.** If $m \geq 9$, then

$$\lambda(C_{9} \bowtie C_{m}) = \begin{cases} 13, & m = 9, 11, 15, 14, 18, 19, 22, 23 \\ 13 \text{ or } 14, & m = 10 \\ 12 \text{ or } 13, & m = 27, 31, 35 \\ 12, & \text{otherwise.} \end{cases} \quad (10)$$

**Proof.** Since $\lambda(C_{9} \bowtie P_{7}) = 12$ (see Table 2), it follows from Lemma 1 that $\lambda(C_{9} \bowtie C_{m}) \geq 12$. The results for $m \leq 26$ follow from Table 2. We have also established by solving the SAT instances transformed from the corresponding $L(2,1)$-labeling problems that $\lambda(C_{9} \bowtie C_{29})$ is either 13 or 14, while for $m = 27, 31, 35$ the value of $\lambda(C_{9} \bowtie C_{29})$ is either 12 or 13. Figure 5 shows a $12\cdot L(2,1)$-labeling of $C_{9} \bowtie C_{28}$. This labeling restricted to first 12 copies of $C_{9}$ induces a $12\cdot L(2,1)$-labeling of $C_{9} \bowtie C_{12}$. From Lemma 2 it follows that $\lambda(C_{9} \bowtie C_{12k+1}) \leq 12$, $k \geq 2$, and $\lambda(C_{9} \bowtie C_{12k}) \leq 12$, $k \geq 1$.

We have also found $12\cdot L(2,1)$-labelings of $C_{9} \bowtie C_{29}$, $C_{9} \bowtie C_{30}$, $C_{9} \bowtie C_{32}$, $C_{9} \bowtie C_{33}$, $C_{9} \bowtie C_{35}$, $C_{9} \bowtie C_{38}$, and $C_{9} \bowtie C_{39}$. Any of these labelings restricted to first 12 copies of $C_{9}$ induces a $12\cdot L(2,1)$-labeling of $C_{9} \bowtie C_{12}$. From Lemma 2 it follows that $\lambda(C_{9} \bowtie C_{12k+2}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+3}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+4}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+5}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+6}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+7}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+8}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+9}) \leq 12$ for $k \geq 4$, $\lambda(C_{9} \bowtie C_{12k+10}) \leq 12$ for $k \geq 4$, and $\lambda(C_{9} \bowtie C_{12k+11}) \leq 12$ for $k \geq 4$.

Since we have also found $12\cdot L(2,1)$-labelings of $C_{9} \bowtie C_{30}$, $C_{9} \bowtie C_{34}$, $C_{9} \bowtie C_{37}$, $C_{9} \bowtie C_{38}$, $C_{9} \bowtie C_{39}$, $C_{9} \bowtie C_{42}$, $C_{9} \bowtie C_{43}$, $C_{9} \bowtie C_{46}$, $C_{9} \bowtie C_{47}$, $C_{9} \bowtie C_{51}$, and $C_{9} \bowtie C_{44}$, we establish that $\lambda(C_{9} \bowtie C_{m}) \leq 12$ for all $m$ of interest and the proof is complete.
Theorem 18. If \( m \geq 10 \), then

\[
\lambda(C_{10} \boxdot C_m) = \begin{cases} 
13, & m = 10, 11, 13, 15, 16, 17, 22, 26 \\
12 \text{ or } 13, & 27 \leq m \leq 395 \\
12, & \text{otherwise.} 
\end{cases}
\]

(11)

By Lemma 2, we have \( \lambda(C_{10} \boxdot C_{12k+37}) \geq 12 \) for integers \( \alpha \) and \( \beta \). Finally, thanks to Lemma 3, we get \( \lambda(C_{10} \boxdot C_m) \leq 12 \) for \( m \geq (37 - 1) \cdot (12 - 1) = 396 \).

We have found 13-\( L(2, 1) \)-labelings of \( C_{10} \boxdot C_m \) for \( 27 \leq m \leq 46 \) and we can construct 13-\( L(2, 1) \)-labelings of \( C_{10} \boxdot C_m \) for \( m \geq 36 \) as follows. We have found 13-\( L(2, 1) \)-labelings of \( C_{10} \boxdot C_{25}, C_{10} \boxdot C_{50}, C_{10} \boxdot C_{27}, C_{10} \boxdot C_{28}, C_{10} \boxdot C_{29}, C_{10} \boxdot C_{54}, \) and \( C_{10} \boxdot C_{39} \). Any of these labelings restricted to the first eight copies of \( C_{10} \) is a 13-\( L(2, 1) \)-labeling of \( C_{8} \). From Lemma 2 it follows that \( \lambda(C_{10} \boxdot C_{8k}) \leq 13 \) for \( k \geq 1 \), \( \lambda(C_{10} \boxdot C_{8k+1}) \leq 13 \) for \( k \geq 1 \), \( \lambda(C_{10} \boxdot C_{8k+2}) \leq 13 \) for \( k \geq 1 \), \( \lambda(C_{10} \boxdot C_{8k+3}) \leq 13 \) for \( k \geq 1 \), \( \lambda(C_{10} \boxdot C_{8k+4}) \leq 13 \) for \( k \geq 1 \), \( \lambda(C_{10} \boxdot C_{8k+5}) \leq 13 \) for \( k \geq 1 \), and \( \lambda(C_{10} \boxdot C_{8k+7}) \leq 13 \) for \( k \geq 4 \). These observations complete the proof.

Proposition 19. \( \lambda(C_{11} \boxdot C_m) = 10 \) only if \( m \equiv 0 \) (mod 11).

Proof. The graph \( D_{11,10}^{12} \) with 380 vertices and the largest out-degree 2 has been computed. Since breadth first search algorithm has found only cycles of length 11, Theorem 9 yields the proof.

Theorem 20. If \( m \geq 11 \), then

\[
\lambda(C_{11} \boxdot C_m) = \begin{cases} 
10, & m \equiv 0 \pmod{11} \\
12, & m \in \{12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 25, 26\} \\
11, & m \geq 270 \\
11 \text{ or } 12, & \text{otherwise.} 
\end{cases}
\]

(12)

Proof. For \( m \in \{12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 25, 26\} \) the \( \lambda \) numbers are obtained by using the SAT reduction as depicted in Table 1. Since \( \lambda(C_{11} \boxdot P_5) = 10 \), from Lemma 1 it follows that \( \lambda(C_{11} \boxdot C_m) \geq 10 \), while from Proposition 19 it follows that \( \lambda(C_{11} \boxdot C_m) \geq 11 \) if \( m \not\equiv 0 \pmod{11} \).

The result for \( m \equiv 0 \pmod{11} \) can be obtained by the fact that \( \lambda(C_{11} \boxdot C_{11}) = 10 \) and Lemma 2.

Figure 6 represents an 11-\( L(2, 1) \)-labeling of \( C_{11} \boxdot C_{28} \), where the leftmost 11 columns of the figure represent an 11-\( L(2, 1) \)-labeling of \( C_{11} \boxdot C_{11} \). By Lemma 2, we have \( \lambda(C_{11} \boxdot C_{11}) \geq 11 \) for integers \( \alpha \) and \( \beta \). Finally, thanks to Lemma 3, we get \( \lambda(C_{11} \boxdot C_m) \leq 11 \) for \( m \geq (28 - 1) \cdot (11 - 1) = 270 \).

In order to find the general upper bound, we present the constructions showing that \( \lambda(C_{11} \boxdot C_m) \leq 12 \) for \( m \geq 26 \). In particular, Figure 7 shows a 12-\( L(2, 1) \)-labeling of \( C_{11} \boxdot C_{37} \). This labeling restricted to the first 2 copies of \( C_{11} \) induces a 12-\( L(2, 1) \)-labeling of \( C_{11} \boxdot C_{12} \). From Lemma 2 it follows that \( \lambda(C_{11} \boxdot C_{12k+1}) \leq 12, k \geq 3, \) and \( \lambda(C_{11} \boxdot C_{12k}) \leq 12, k \geq 1 \).

Analogously, we have found 12-\( L(2, 1) \)-labelings of \( C_{11} \boxdot C_{38}, C_{11} \boxdot C_{39}, C_{11} \boxdot C_{40}, C_{11} \boxdot C_{41}, C_{11} \boxdot C_{42}, C_{11} \boxdot C_{43}, C_{11} \boxdot C_{44}, C_{11} \boxdot C_{45}, C_{11} \boxdot C_{46}, \) and \( C_{11} \boxdot C_{45} \). Any of these labelings restricted to the first 2 copies of \( C_{11} \) is a 12-\( L(2, 1) \)-labeling of \( C_{11} \boxdot C_{12} \).

From Lemma 2 it follows that \( \lambda(C_{11} \boxdot C_{12k+2}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+3}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+4}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+5}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+6}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+7}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+8}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+9}) \leq 12 \) for \( k \geq 3 \), \( \lambda(C_{11} \boxdot C_{12k+10}) \leq 12 \) for \( k \geq 3 \), and \( \lambda(C_{11} \boxdot C_{12k+11}) \leq 12 \) for \( k \geq 2 \).

Since we have also found 12-\( L(2, 1) \)-labelings of \( C_{11} \boxdot C_{27}, C_{11} \boxdot C_{28}, C_{11} \boxdot C_{29}, C_{11} \boxdot C_{30}, C_{11} \boxdot C_{31}, C_{11} \boxdot C_{32}, C_{11} \boxdot C_{33}, \) and \( C_{11} \boxdot C_{34}, \) we establish that \( \lambda(C_{11} \boxdot C_m) \leq 12 \) for all \( m \geq 27 \) and the proof is complete.

4.3. \( \lambda \)-Numbers of \( P_n \boxdot C_m \)

Proposition 21. If \( m \geq 3 \), then

\[
\lambda(P_3 \boxdot C_m) = \begin{cases} 
14, & m = 5 \\
11, & m \in \{3, 4, 6, 7, 8, 9, 10, 13, 14, 18, 19\} \\
10, & \text{otherwise.} 
\end{cases}
\]
The upper bounds for degree 16 has been created in order to find \( m = \{3,4,6,7,8,9,10,13,14,18,19\} \cup \{5\}. It follows that \( \lambda(P_3 \boxtimes C_m) \geq 11 \) for any \( m \in \{3,4,6,7,8,9,10,13,14,18,19\} \). The upper bounds for \( m \in \{13,14,18,19\} \) follow from the labelings depicted in Figure 8.

We have found 10\( -L(2,1) \)-labelings of \( P_4 \boxtimes C_{23}, P_4 \boxtimes C_{35}, \) \( P_4 \boxtimes C_{37}, P_4 \boxtimes C_{39}, P_4 \boxtimes C_{40}, P_4 \boxtimes C_{41}, P_4 \boxtimes C_{43}, \) and \( P_4 \boxtimes C_{45}. \) Any of these labelings restricted to the first 11 cycles of \( P_4 \) is a 10\( -L(2,1) \)-labeling of \( P_4 \boxtimes C_{11}. \) From Lemma 2 it follows that \( \lambda(P_3 \boxtimes C_{11k}) \leq 10 \) for \( k \geq 1, \lambda(P_3 \boxtimes C_{11k+2}) \leq 10 \) for \( k \geq 2, \lambda(P_3 \boxtimes C_{11k+4}) \leq 10 \) for \( k \geq 4, \) \( \lambda(P_3 \boxtimes C_{11k+6}) \leq 10 \) for \( k \geq 3, \lambda(P_3 \boxtimes C_{11k+8}) \leq 10 \) for \( k \geq 4, \) \( \lambda(P_3 \boxtimes C_{11k+10}) \leq 10 \) for \( k \geq 4 \).

Since we have also found 10\( -L(2,1) \)-labelings of \( P_4 \boxtimes C_{13}, P_4 \boxtimes C_{15}, P_4 \boxtimes C_{16}, P_4 \boxtimes C_{17}, P_4 \boxtimes C_{18}, P_4 \boxtimes C_{19}, P_4 \boxtimes C_{20}, P_4 \boxtimes C_{21}, P_4 \boxtimes C_{24}, P_4 \boxtimes C_{25}, P_4 \boxtimes C_{27}, P_4 \boxtimes C_{28}, P_4 \boxtimes C_{29}, P_4 \boxtimes C_{30}, P_4 \boxtimes C_{32}, \) and \( P_4 \boxtimes C_{36}, \) we establish that \( \lambda(P_3 \boxtimes C_m) \leq 10 \) for all \( m \geq 13 \) and the proof is complete.

**Proposition 22.** If \( m \geq 3, \) then

\[
\lambda(P_3 \boxtimes C_m) = \begin{cases} 
14, & m = 5 \\
10, & m \equiv 0 \pmod{11} \\
11, & \text{otherwise.}
\end{cases}
\]

**Proof.** For \( m = 5 \) the result is obtained by solving the SAT instance transformed from the corresponding \( L(2,1) \)-labeling problem. For \( m \equiv 0 \pmod{11} \) the result follows from Lemma 2 and from the fact that \( \lambda(C_{11} \boxtimes C_{11}) = 10 \).

In order to find 10\( -L(2,1) \)-labelings in \( P_4 \boxtimes C_{m}, \) the graph \( D_{170}^2 \) with 16792 vertices and the largest outdegree has been created. Since breadth first search algorithm has found only cycles of length 11, the upper bound follows.

For all \( m \not\equiv 0 \pmod{11} \) and \( m \geq 13 \) we can construct 11\( -L(2,1) \)-labelings of \( P_4 \boxtimes C_{m} \) as described below. We have found 11\( -L(2,1) \)-labelings of \( P_4 \boxtimes C_{13}, P_4 \boxtimes C_{14}, P_4 \boxtimes C_{15}, P_4 \boxtimes C_{16}, \) \( P_4 \boxtimes C_{17}, \) and \( P_4 \boxtimes C_{18}. \) Any of these labelings restricted to the first six copies of \( P_4 \) is an 11\( -L(2,1) \)-labeling of \( P_4 \boxtimes C_{11}. \) From Lemma 2 it follows that \( \lambda(P_3 \boxtimes C_{6k+1}) \leq 11 \) for \( k \geq 2, \lambda(P_3 \boxtimes C_{6k+3}) \leq 11 \) for \( k \geq 2, \lambda(P_3 \boxtimes C_{6k+4}) \leq 11 \) for \( k \geq 2, \) and \( \lambda(P_3 \boxtimes C_{6k+5}) \leq 11 \) for \( k \geq 2. \) These conclusions complete the proof.

---

**Figure 6:** 11\( -L(2,1) \)-labeling of \( C_{11} \boxtimes C_{28}. \)

**Figure 7:** 12\( -L(2,1) \)-labeling of \( C_{11} \boxtimes C_{37}. \)
Corollary 23. Let $n, m \geq 4$.

(i) If $m \not\equiv 0 \pmod{11}$, then $\lambda(P_n \boxtimes C_m) \geq 11$.

(ii) If $n \not\equiv 0 \pmod{11}$ or $m \not\equiv 0 \pmod{11}$, then $\lambda(C_n \boxtimes C_m) \geq 11$.

From Theorem 6 now we have the following

Corollary 24. If $m \geq 24$ and $n \geq 26$, then

$$
\lambda(C_n \boxtimes C_m) = \begin{cases} 
10, & n \equiv 0 \pmod{11} \\
11 \text{ or } 12, & m \equiv 0 \pmod{11} \\
11 \text{ or } 12, & \text{otherwise.}
\end{cases}
$$

(15)

Proposition 25. If $m \geq 3$, then

$$
\lambda(P_5 \boxtimes C_m) = \begin{cases} 
14, & m = 5 \\
12, & m = 3, 9, 10 \\
10, & m \equiv 0 \pmod{11} \\
11, & m = 4, 6, 7, 8, 12 \text{ or } m \geq 80 \\
11 \text{ or } 12, & \text{otherwise.}
\end{cases}
$$

(16)

Proposition 26. If $m \geq 3$, then

$$
\lambda(P_6 \boxtimes C_m) = \begin{cases} 
14, & m = 5 \\
12, & m = 3, 9, 10 \\
10, & m \equiv 0 \pmod{11} \\
11, & m = 4, 6, 7, 8, 12 \text{ or } m \geq 154 \\
11 \text{ or } 12, & \text{otherwise.}
\end{cases}
$$

(17)
Proof. For $m \leq 12$ the results follow from Table 1. For $m \equiv 0 \pmod{11}$ the result follows Lemma 2 and from the fact that $\lambda(C_{11} \boxtimes C_{11}) = 10$.

Figure 10 represents a $11-L(2,1)$-labeling of $P_6 \boxtimes C_{23}$, where the leftmost eight columns of the figure represent an $11-L(2,1)$-labeling of $P_6 \boxtimes C_8$.

By Lemma 2, we have $\lambda(P_6 \boxtimes C_{\alpha+23}) \leq 11$ for integers $\alpha$ and $\beta$. From Lemma 3 it follows that $\lambda(P_6 \boxtimes C_m) \leq 11$ for $m \geq (23 - 1) \cdot (8 - 1) = 154$.

We complete the proof by noting that from Theorem 16 and Lemma 1 it follows $\lambda(P_6 \boxtimes C_m) \leq 12$ for $m \geq 11$.

Values in Table 1, the results from Section 4.2, Theorems 6 and 20, and Corollary 23 provide lower and upper bounds for the $\lambda$-number of $P_n \boxtimes C_m$. The results are summarized in the following.

**Theorem 27.** If $n \geq 7$, then

$$\lambda(P_n \boxtimes C_m) = \begin{cases} 10, & m \equiv 0 \pmod{11} \\ 14, & m = 5 \\ 12, & m = 3, 7, 9, 10 \\ 11, & m = 4, 6, 8, 12 \text{ or } m \geq 270 \\ 11 \text{ or } 12, & \text{ otherwise}. \end{cases}$$

(18)

**5. Conclusion**

In this paper, the $L(2,1)$-labeling problem of the strong product of paths and cycles is studied. The problem derives from the more general Frequency Assignment Problem (FAP) which requires assigning frequencies to transmitters in a wireless network. It is well known that some interesting wireless networks are closely connected to the strong product of graphs. For example, an octagonal grid is the strong product of two paths and an octagonal torus is the strong product of two cycles.

By using various computational approaches, we succeed in solving the problem (except for the final number of cases) for the strong product of a path and a cycle, as well as for the the strong product of two cycles, where one of the cycles is of length at most eleven. Moreover, the obtained results enable us to improve the bounds on the $\lambda$-number for the strong product of two cycles, where both cycles are sufficiently long.

Finding the exact $\lambda$-numbers for these graphs is therefore an interesting and challenging avenue of further research.

**Conflict of Interests**

The authors wish to confirm that there is no known conflicts of interests associated with this paper and there has been no significant financial support for this work that could have influenced its outcome. They confirm that the paper has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. They further confirm that the order of authors listed in the paper has been approved by all of us. They confirm that they have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing they confirm that they have followed the regulations of their institutions concerning intellectual property.

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