Research Article

On the Exponential Inequality for Weighted Sums of a Class of Linearly Negative Quadrant Dependent Random Variables

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The exponential inequality for weighted sums of a class of linearly negative quadrant dependent random variables is established, which extends and improves the corresponding ones obtained by Ko et al. (2007) and Jabbari et al. (2009). In addition, we also give the relevant precise asymptotics.

1. Introduction

Lehmann [1] introduced a natural definition of negative dependence: two random variables X and Y are said to be negative quadrant dependent (NQD, say) if $P(X > x, Y > y) \leq P(X > x)P(Y > y)$ for all $x, y \in \mathbf{R}$. Based on the concept of NQD, another notion of negative dependence was formulated by Newman [2] as follows: a sequence $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be linearly negative quadrant dependent (LNQD, say) if, for any disjoint subsets A_1 and A_2 of $\{1, 2, \dots, n\}$ and positive r_j 's, $\sum_{i \in A_1} r_i X_i$, and $\sum_{j \in A_2} r_j X_j$ are NQD. Recall that a finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA, say) if, for every pair of disjoint subsets B_1 and B_2 of $\{1, 2, \dots, n\}$,

$$\text{Cov} \left\{ f_1(X_i, i \in B_1), f_2(X_j, j \in B_2) \right\} \leq 0, \quad (1)$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is NA if every finite subfamily is NA. The concept of negative association was introduced by Joag-Dev and Proschan [3]. It is obvious to observe that NA sequences are LNQD and LNQD sequences are not necessarily NA, as it can be seen from the examples in Newman [2] or Joag-Dev and Proschan [3]. Hence, it is

of interest to investigate the exponential inequality and its relevant result for LNQD sequences.

It is well-known that the exponential inequalities for partial sum $\sum_{i=1}^n (X_i - EX_i)$ play a very important role in various proofs of limit theorems. One can refer to Yang and Wang [4], T.-S. Kim and H.-C. Kim [5], Sung [6], Jabbari et al. [7], Xing et al. [8], Sung [9], and so on for further comprehension. As for the limit results about LNQD sequence, one can refer to Newman [2], Zhang [10], H. Kim and T. Kim [11], Wang and Zhang [12], and references therein.

Recently, Ko et al. [13] gave a Bernstein-Hoeffding type inequality for uniformly bounded LNQD random variables, by which they obtained the almost sure convergence rate $O(1)n^{-1/2}(p_n \log n)^{1/2}$ of $\sum_{i=1}^n X_i/n$, where $p_n \rightarrow \infty$ as $n \rightarrow \infty$. Motivated by the paper above, we establish the exponential inequality for weighted sums of uniformly bounded LNQD random variables. The result obtained extends and improves the corresponding ones given by Ko et al. [13] and Jabbari et al. [7]. Furthermore, we give the precise asymptotics with respect to the rate $n^{-1/2}(\log n)^{1/2}$.

Throughout this paper, we always suppose that C and C_1 denote positive constants independent of n but whose value may vary over cases, $[x]$ denotes the integral part of x , $S_n = \sum_{i=1}^n X_i$, $\sigma_n^2 = ES_n^2$, and $u(n) = \sup_{i \geq 1} \sum_{j: |j-i| \geq n} |\text{Cov}(X_i, X_j)|$ and denote $\log n = \ln(n \vee e)$. This paper is organized

as follows. Section 2 contains our main results. Section 3 contains the corresponding proofs.

2. Main Results

In this section, our main results will be given. For formulation of the theorems obtained, some assumptions are needed, which are listed below.

(A1) Let $\{X_i, i \geq 1\}$ be a sequence of stationary LNQD random variables with $|X_i| \leq M$ and let $\{a_{ni} : 1 \leq i \leq n, n \geq 1\}$ be a triangular array of numbers satisfying $|a_{ni}| \leq M_1$, where M and M_1 are generic positive constants.

(A2) Let $\{p_n, n \geq 1\}$ be a positive integer sequence satisfying $p_n \leq n/2$ and $p_n \rightarrow \infty$ as $n \rightarrow \infty$.

Theorem 1. Suppose that the assumption (A1) holds. Then for any $0 < \varepsilon < 1$, one has

$$P\left(\left|\sum_{i=1}^n a_{ni}(X_i - EX_i)\right| > n\varepsilon\right) \leq C_1 \exp\left\{-\frac{n\varepsilon^2}{16C}\right\}, \quad (2)$$

where C_1 and C are positive constants.

Taking

$$\varepsilon = \varepsilon_n := 4\sqrt{\frac{(\alpha C \log n)}{n}}, \quad \text{for some } \alpha > 1, \quad (3)$$

in Theorem 1, then by Borel-Cantelli lemma, we have

Corollary 2. Assume that the assumptions (A1) and (A2) hold.

Let $p_n \leq \sqrt{Cn/(\alpha M^2 \log n)}$, for some $\alpha > 1$. Suppose that ε is as in (3). Then one has

$$\sum_{i=1}^n a_{ni}(X_i - EX_i) = O(1)(n \log n)^{1/2} \quad \text{a.s.} \quad (4)$$

In particular, one has

$$\sum_{i=1}^n (X_i - EX_i) = O(1)(n \log n)^{1/2} \quad \text{a.s.}, \quad (5)$$

when $a_{ni} = 1$ for $1 \leq i \leq n$.

Remark 3. (1) Theorem 1 generalizes Theorem 2.1 in Ko et al. [13] to weighted case. On the other hand, by (5), we can obtain that the strong convergence rate of $\sum_{i=1}^n (X_i - EX_i)/n$ is $O(1)n^{-1/2} \log^{1/2} n$, which is obviously faster than the corresponding one $n^{-1/2}(p_n \log n)^{1/2}$ that Ko et al. [13] obtained, where $p_n \leq n/2$ and $p_n \rightarrow \infty$ as $n \rightarrow \infty$.

(2) Since LNQD sequences are strictly weaker than NA sequences, as mentioned in Section 1, Theorem 1 extends Theorem 2.1 in Jabbari et al. [7] from strictly stationary negatively associated setting to weighted LNQD case. In addition, by the analysis mentioned above, we know that the strong convergence rate $O(1)n^{-1/2} \log^{1/2} n$ of $\sum_{i=1}^n (X_i - EX_i)/n$ is

much faster than the relevant one $O(1)n^{-1/3}(\log n)^{2/3}$ Jabbari et al. [7] obtained only for the special case of geometrically decreasing covariances.

(3) For the sequence of extended negatively dependent (END, say) or wide orthant dependent (WOD, say) random variables, similar results can also be obtained.

Regarding the convergence rate $n^{-1/2} \log^{1/2} n$ obtained in (5), we can give the following precise asymptotics.

Theorem 4. Let $\{X_i, i \geq 1\}$ be a sequence of identically distributed LNQD random variables with $EX_1 = 0$ and $E|X_1|^{2+\delta} < \infty$ for some $0 < \delta \leq 1$, $\inf_{n \geq 1} \sigma_n^2/n > 0$, $u(n) \leq Cn^{-\theta}$ for some $\theta > 1$, and

$$\sigma^2 := EX_1^2 + 2 \sum_{i=2}^{\infty} \text{Cov}(X_1, X_i) > 0. \quad (6)$$

Then for $\beta > -1$, one has

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\beta+2} \sum_{n=2}^{\infty} \frac{(\log n)^{\beta}}{n} P(|S_n| \geq \varepsilon \sigma \sqrt{n \log n}) = \frac{E|N|^{2\beta+2}}{\beta+1}, \quad (7)$$

where N denotes the standard normal random variable.

3. Proofs

Firstly, the following lemma is needed, which will be used in what follows.

Lemma 5 (see [10]). Let $\{X_i, i \geq 1\}$ be an LNQD sequence with zero mean and finite u th moment. Then, for $u > 2$, there exists a positive constant C which only depends on u such that

$$E\left|\sum_{i=1}^n X_i\right|^u \leq Cn^{u/2} \max_{1 \leq i \leq n} E|X_i|^u, \quad (8)$$

for any $n \geq 1$.

Next, we give some notations used later. Define $X_{ni} = X_i$ for $1 \leq i \leq n$ and $X_{ni} = 0$ for $i > n$. Set $r_n = \lfloor n/(2p_n) \rfloor + 1$ and then define

$$Y_{nj} = \sum_{i=2(j-1)p_n+1}^{2(j-1)p_n+p_n} a_{ni}(X_{ni} - EX_{ni}), \quad (9)$$

$$Z_{nj} = \sum_{i=2(j-1)p_n+p_n+1}^{2jp_n} a_{ni}(X_{ni} - EX_{ni}),$$

for $j = 1, 2, \dots, r_n$ and

$$S_{1n} = \sum_{j=1}^{r_n} Y_{nj}, \quad S_{2n} = \sum_{j=1}^{r_n} Z_{nj}. \quad (10)$$

Clearly, $\sum_{i=1}^n a_{ni}(X_i - EX_i) = S_{1n} + S_{2n}$ and $n < 2r_n p_n \leq 2n$.

Now, we can obtain the following lemma.

Lemma 6. Let $\{X_i, i \geq 1\}$ be a sequence of stationary LNQD random variables with $|X_i| \leq M$ and let $\{a_{ni} : 1 \leq i \leq n, n \geq 1\}$ be a triangular array of numbers satisfying $|a_{ni}| \leq M_1$, where M and M_1 are generic positive constants. If $0 < \lambda p_n M M_1 \leq 1/2$ for some $\lambda > 0$, then on account of Definitions (9) and (10), one has

$$\begin{aligned} E(\exp(\lambda S_{1n})) &\leq \exp(C\lambda^2 n), \\ E(\exp(\lambda S_{2n})) &\leq \exp(C\lambda^2 n). \end{aligned} \quad (11)$$

Proof. Without loss of generality, assume that $a_{ni} \geq 0$. Applying $EY_{nj} = 0$ and $\lambda p_n M M_1 \leq 1/2$, we have by $|\lambda Y_{nj}| \leq 2\lambda p_n M M_1$ that

$$\begin{aligned} E(\exp(\lambda Y_{nj})) &= \sum_{k=0}^{\infty} \frac{E(\lambda Y_{nj})^k}{k!} = 1 + \sum_{k=2}^{\infty} \frac{E(\lambda Y_{nj})^k}{k!} \\ &\leq 1 + E(\lambda Y_{nj})^2 \sum_{k=2}^{\infty} \frac{1}{k!} \\ &\leq 1 + \lambda^2 EY_{nj}^2 \leq \exp(\lambda^2 EY_{nj}^2). \end{aligned} \quad (12)$$

Therefore, in terms of the result above and the concept of LNQD,

$$\begin{aligned} \prod_{j=1}^{r_n} E(\exp(\lambda Y_{nj})) &\leq \exp\left(\lambda^2 \sum_{j=1}^{r_n} EY_{nj}^2\right) \\ &\leq \exp\left(M_1^2 \lambda^2 \sum_{j=1}^{r_n} \sum_{i=2(j-1)p_n+1}^{2(j-1)p_n+p_n} \text{Var}(X_{ni})\right) \\ &\leq \exp\left(M_1^2 \lambda^2 \sum_{j=1}^{r_n} \sum_{i=2(j-1)p_n+1}^{2(j-1)p_n+p_n} EX_1^2\right) \\ &\leq \exp(C\lambda^2 n). \end{aligned} \quad (13)$$

Also, since $a_{ni} \geq 0$ and $\lambda > 0$, $\{\lambda Y_{nj}, 1 \leq j \leq r_n\}$ is LNQD. Therefore, from Lemma 3.1 in Ko et al. [13], we have

$$E(\exp(\lambda S_{1n})) \leq \prod_{j=1}^{r_n} E(\exp(\lambda Y_{nj})). \quad (14)$$

Combining (13) and (14), we can get the desired result. Similarly, we can get the same result for S_{2n} . The proof is completed. $\square$

By Lemma 6, we can easily get the following result.

Lemma 7. Let $\{X_i, i \geq 1\}$ be a sequence of stationary LNQD random variables with $|X_i| \leq M$ and let $\{a_{ni} : 1 \leq i \leq n, n \geq 1\}$ be a triangular array of numbers satisfying $|a_{ni}| \leq M_1$, where

M and M_1 are generic positive constants. If $0 < \lambda p_n M M_1 \leq 1/2$ for some $\lambda > 0$, then, for any $0 < \varepsilon < 1$ and some $C_1 > 0$,

$$\begin{aligned} P\left(|S_{1n}| > \frac{n\varepsilon}{2}\right) &\leq C_1 \exp\left\{-\frac{n\varepsilon^2}{16C}\right\}, \\ P\left(|S_{2n}| > \frac{n\varepsilon}{2}\right) &\leq C_1 \exp\left\{-\frac{n\varepsilon^2}{16C}\right\}. \end{aligned} \quad (15)$$

Proof. Applying Markov inequality and Lemma 6, we obtain

$$\begin{aligned} P\left(|S_{1n}| > \frac{n\varepsilon}{2}\right) &= P\left(S_{1n} > \frac{n\varepsilon}{2}\right) + P\left(-S_{1n} > \frac{n\varepsilon}{2}\right) \\ &\leq P\left(e^{\lambda S_{1n}} > e^{\lambda n\varepsilon/2}\right) + P\left(e^{-\lambda S_{1n}} > e^{\lambda n\varepsilon/2}\right) \\ &\leq C_1 \exp\left(C\lambda^2 n - \frac{\lambda n\varepsilon}{2}\right). \end{aligned} \quad (16)$$

Optimizing the exponent in the term of this upper-bound, we find that $\lambda = \varepsilon/(4C)$, so that this exponent becomes equal to $-n\varepsilon^2/(16C)$ as desired. The proof is completed. $\square$

Lemma 8 (see [10]). Under the conditions of Theorem 4, one has

$$\frac{\sigma_n^2}{n} \longrightarrow \sigma^2, \quad \frac{S_n}{\sigma_n} \longrightarrow N \quad \text{in distribution}, \quad (17)$$

where N is a standard normal random variable.

Lemma 9 (see [12]). Under the conditions of Theorem 4, we have for $x \in \mathbb{R}$

$$\sup_x \left| P\left(\frac{S_n}{\sigma_n} < x\right) - \Phi(x) \right| \leq Cn^{-\delta/(2+3\delta)}, \quad (18)$$

where $\Phi(x)$ denotes the standard normal distribution function.

Based on the above lemmas, the proofs of Theorems 1 and 4 can be given as follows.

Proof of Theorem 1. Let $\lambda > 0$ which satisfies $\lambda p_n M M_1 \leq 1/2$; then it follows from Lemma 7 that

$$\begin{aligned} P(|S_n| > n\varepsilon) &\leq P\left(|S_{1n}| > \frac{n\varepsilon}{2}\right) + P\left(|S_{2n}| > \frac{n\varepsilon}{2}\right) \\ &\leq C_1 \exp\left\{-\frac{n\varepsilon^2}{16C}\right\}, \end{aligned} \quad (19)$$

which completes the proof. $\square$

Proof of Theorem 4. Without loss of generality, set $\sigma = 1$ in what follows. Since

$$\begin{aligned} & \epsilon^{2\beta+2} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \\ &= \epsilon^{2\beta+2} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} \left\{ P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \right. \\ & \quad \left. - P\left(|N| \geq \epsilon \sqrt{\log n}\right) \right\} \\ & \quad + \epsilon^{2\beta+2} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} P\left(|N| \geq \epsilon \sqrt{\log n}\right), \end{aligned} \quad (20)$$

it is sufficient to prove that

$$\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} P\left(|N| \geq \epsilon \sqrt{\log n}\right) = \frac{E|N|^{2(\beta+1)}}{\beta+1}, \quad (21)$$

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \right. \\ & \quad \left. - P\left(|N| \geq \epsilon \sqrt{\log n}\right) \right| = 0, \end{aligned} \quad (22)$$

for any $\beta > -1$.

To prove (21), we need only to show that

$$\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} P\left(N \geq \epsilon \sqrt{\log n}\right) = \frac{E|N|^{2(\beta+1)}}{2(\beta+1)}, \quad (23)$$

by noting $P(|N| \geq x) = 2P(N \geq x)$ for any $x > 0$. It is easy to observe that

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} P\left(N \geq \epsilon \sqrt{\log n}\right) \\ &= \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \int_2^\infty \frac{(\log x)^\beta}{x} P\left(N \geq \epsilon \sqrt{\log x}\right) dx \\ &= \lim_{\epsilon \searrow 0} \int_\epsilon^\infty 2y^{2\beta+1} P(N \geq y) dy \\ &= \frac{E|N|^{2(\beta+1)}}{2(\beta+1)}, \end{aligned} \quad (24)$$

for any $\beta > -1$, which implies (21).

Next, we will prove (22). Let $J(\epsilon) = \exp(M/\epsilon^2)$, where $M > 4$ and $0 < \epsilon < 1/4$. Obviously,

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \right. \\ & \quad \left. - P\left(|N| \geq \epsilon \sqrt{\log n}\right) \right| \\ &= \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \right. \\ & \quad \left. - P\left(|N| \geq \epsilon \sqrt{\log n}\right) \right| \\ & \quad + \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \right. \\ & \quad \left. - P\left(|N| \geq \epsilon \sqrt{\log n}\right) \right| \\ &\leq \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \right. \\ & \quad \left. - P\left(|N| \geq \epsilon \sqrt{\log n}\right) \right| \\ & \quad + \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \\ & \quad + \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} P\left(|N| \geq \epsilon \sqrt{\log n}\right) \\ &:= I_1 + I_2 + I_3; \end{aligned} \quad (25)$$

thus it suffices to prove that

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_1 = 0, \quad \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_2 = 0, \\ & \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_3 = 0, \end{aligned} \quad (26)$$

respectively. We consider firstly I_1 . Set $\Delta_n = \sup_x |P(|S_n| \geq x \sqrt{n}) - P(|N| \geq x)|$. Noticing Lemma 8, we have $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$. It follows that

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_1 \\ &\leq \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \Delta_n \\ &= \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} (\log J(\epsilon))^{\beta+1} \frac{1}{(\log J(\epsilon))^{\beta+1}} \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \Delta_n \\ &\leq \lim_{\epsilon \searrow 0} M^{\beta+1} \frac{1}{(\log J(\epsilon))^{\beta+1}} \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \Delta_n \rightarrow 0, \end{aligned} \quad (27)$$

which implies that $\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_1 = 0$. Turn to I_2 . By Lemma 9, it follows that

$$\begin{aligned}
 & P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \\
 & \leq \left| P\left(\left|\frac{S_n}{\sigma_n}\right| \geq \frac{\epsilon \sqrt{n \log n}}{\sigma_n}\right) - P\left(|N| \geq \frac{\epsilon \sqrt{n \log n}}{\sigma_n}\right) \right| \\
 & \quad + P\left(|N| \geq \frac{\epsilon \sqrt{n \log n}}{\sigma_n}\right) \\
 & \leq C n^{-\delta/(2+3\delta)} + 2P\left(N \geq \frac{\epsilon \sqrt{n \log n}}{\sigma_n}\right) \\
 & \leq C n^{-\delta/(2+3\delta)} + C n^{-\epsilon^2/2} \leq C n^{-\epsilon^2/2},
 \end{aligned} \tag{28}$$

for sufficiently small ϵ . Thus we obtain

$$\begin{aligned}
 & \lim_{M \rightarrow \infty} \epsilon^{2(\beta+1)} I_2 \\
 & = \lim_{M \rightarrow \infty} \epsilon^{2(\beta+1)} \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} P\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) \\
 & \leq C \lim_{M \rightarrow \infty} \epsilon^{2(\beta+1)} \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} \exp\left(-\frac{\epsilon^2}{2} \log n\right) \\
 & \leq C \lim_{M \rightarrow \infty} \epsilon^{2(\beta+1)} \int_{J(\epsilon)}^{\infty} \frac{(\log x)^\beta}{x} \exp\left(-\frac{\epsilon^2}{2} \log x\right) dx \\
 & \leq C \lim_{M \rightarrow \infty} \epsilon^{2(\beta+1)} \int_{M/2\epsilon^2}^{\infty} y^\beta \exp\left(-\frac{\epsilon^2}{2} y\right) dy \\
 & \leq C \lim_{M \rightarrow \infty} \int_{M/4}^{\infty} t^\beta e^{-t} dt \rightarrow 0.
 \end{aligned} \tag{29}$$

Hence, $\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_2 \rightarrow 0$ when $M \rightarrow \infty$. On the other hand, noting that $M > 4$ and $0 < \epsilon < 1/4$ imply $J(\epsilon) - 1 \geq \sqrt{J(\epsilon)}$, we can obtain

$$\begin{aligned}
 & \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_3 \\
 & \leq \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \int_{J(\epsilon)}^{\infty} \frac{(\log x)^\beta}{x} P\left(|N| \geq \epsilon \sqrt{\log x}\right) dx
 \end{aligned}$$

$$\begin{aligned}
 & \leq \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \int_{J(\epsilon)-1}^{\infty} \frac{(\log x)^\beta}{x} P\left(|N| \geq \epsilon \sqrt{\log x}\right) dx \\
 & \leq \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \int_{\sqrt{J(\epsilon)}}^{\infty} \frac{(\log x)^\beta}{x} P\left(|N| \geq \epsilon \sqrt{\log x}\right) dx \\
 & \leq \text{Clim}_{\epsilon \searrow 0} \int_{\epsilon \sqrt{M/(2\epsilon^2)}}^{\infty} y^{2\beta+1} P(|N| \geq y) dy \\
 & \leq C \int_{\sqrt{M}/2}^{\infty} y^{2\beta+1} P(|N| \geq y) dy \rightarrow 0 \quad \text{as } M \rightarrow \infty,
 \end{aligned} \tag{30}$$

uniformly for $0 < \epsilon < 1/4$. Thus we have $\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_3 \rightarrow 0$, when $M \rightarrow \infty$. Combining the earlier results together yields (26). The proof is complete. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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