Research Article

On Thompson's Conjecture for Alternating Groups A_{p+3}

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Let G be a group. Denote by $\pi(G)$ the set of prime divisors of $|G|$. Let $GK(G)$ be the graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We set $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. Denote by $N(G)$ the set of nonidentity orders of conjugacy classes of elements in G . Alavi and Daneshkhah proved that the groups, A_n where $n = p, p+1, p+2$ with $s(G) \geq 2$, are characterized by $N(G)$. As a development of these topics, we will prove that if G is a finite group with trivial center and $N(G) = N(A_{p+3})$ with $p+2$ composite, then G is isomorphic to A_{p+3} .

1. Introduction

In this paper, all groups considered are finite and simple groups are nonabelian simple groups. Let G be a finite group and let $Z(G)$ be its center. For any $1 \neq x \in G$, suppose that x^G denotes the conjugacy classes in G containing x and $C_G(x)$ denotes the centralizer of x in G . We will use $N(G)$ to denote the set $\{n : G \text{ has a conjugacy class of size } n\}$. Thompson in 1987 gave the following conjecture with respect to $N(G)$.

Thompson's Conjecture (see [1], Question 12.38). If L is a finite simple nonabelian group, G is a finite group with trivial center, and $N(G) = N(L)$, then $G \cong L$.

Let $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $GK(G)$ be the graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We set $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. A classification of all finite simple groups with disconnected prime graph was obtained in [2, 3]. Based on these results, Thompson's conjecture was proved valid for many finite simple groups with $s(G) \geq 2$ (see [4, 5]). In Ahanjideh and Xu's works, the groups $L_n(q)$, $D_n(q)$, B_n , C_n , ${}^2D_n(q)$, and $E_7(q)$ are true for Thompson's conjecture (see [6–10]). Alavi and Daneshkhah proved that the groups A_n with $n = p, p+1, p+2$ and $s(G) \geq 2$ are characterized by $N(G)$ (see [11]). So is there a group with connected prime graph for which Thompson's

conjecture would be true? Recently, the groups A_{10} , A_{16} , A_{22} , and A_{26} were proved valid for this conjecture (see [12–15]). As a development of these topics, we will prove that Thompson's conjecture is true for the alternating groups A_{p+3} of degree $p+3$ with $p+2$ composite.

We will introduce some notations used in the proof of the main theorem. Let G be a group and p a prime. Then denote by G_p the Sylow p -subgroup of G . Let $\text{Aut}(G)$ and $\text{Out}(G)$ denote the automorphism and outer automorphism groups of G , respectively. Let $\omega(G)$ denote the set of element orders of G . The other notations are standard (see [16], for instance).

2. Preliminary Results

In this section we will give some preliminary results.

Lemma 1. Let $x, y \in G$, $(|x|, |y|) = 1$, and $xy = yx$. Then

- (1) $C_G(xy) = C_G(x) \cap C_G(y)$;
- (2) $|x^G|$ divides $|(xy)^G|$;
- (3) if $|x^G| = |(xy)^G|$, then $C_G(x) \leq C_G(y)$.

Proof. See [12, Lemma 1.2] and [7, Lemma 2.3]. □

Lemma 2. If P and H are finite groups with trivial centers and $N(P) = N(H)$, then $\pi(P) = \pi(H)$.

Proof. See [12, Lemma 3]. $\square$

Lemma 3. Suppose that G is a finite group with trivial center and p is a prime from $\pi(G)$ such that p^2 does not divide $|x^G|$ for all x in G . Then a Sylow p -subgroup of G is elementary abelian.

Proof. See [12, Lemma 4]. $\square$

Lemma 4. Let K be a normal subgroup of G , and $\bar{G} = G/K$.

- (1) If $\bar{x}$ is the image of an element x of G in $\bar{G}$, then $|\bar{x}^{\bar{G}}|$ divides $|x^G|$.
- (2) If $(|x|, |K|) = 1$, then $C_{\bar{G}}(\bar{x}) = C_G(x)K/K$.
- (3) If $y \in K$, then $|y^K|$ divides $|y^G|$.

Proof. See [12, Lemma 5]. $\square$

Let $\exp(n, r)$ denote the nonnegative integer a such that $r^a \mid n$ but $r^{a+1} \nmid n$.

Lemma 5. Let A_{p+3} be the alternating group of degree $p+3$, where p is a prime. Then the following hold.

- (1) $\exp(|A_{p+3}|, 2) = \sum_{i=1}^{\infty} [(p+3)/2^i] - 1$; in particular, $\exp(|A_{p+3}|, 2) \leq p+2$.
- (2) $\exp(|A_{p+3}|, r) = \sum_{i=1}^{\infty} [(p+3)/r^i]$ for each $r \in \pi(A_{p+3}) \setminus \{2\}$. Furthermore, $\exp(|A_{p+3}|, r) < (p-1)/2$, where $3 \leq r \in \pi(A_{p+3})$. In particular, if $r > [(p+3)/2]$, then $\exp(|A_{p+3}|, r) = 1$.

Proof. (1) By the definition of Gaussian integer function, we have that

$$\begin{aligned} \exp(|A_{p+3}|, 2) &= \sum_{i=1}^{\infty} \left[\frac{p+3}{2^i} \right] - 1 \\ &= \left(\left[\frac{p+3}{2} \right] + \left[\frac{p+3}{2^2} \right] + \left[\frac{p+3}{2^3} \right] + \cdots \right) - 1 \\ &\leq \left(\frac{p+3}{2} + \frac{p+4}{2^2} + \frac{p+3}{2^3} + \cdots \right) - 1 \\ &= (p+3) \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) - 1 \\ &= p+2. \end{aligned} \tag{1}$$

(2) Similarly, as (1), we have that

$$\begin{aligned} \exp(|A_{p+3}|, r) &\leq (p+3) \left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots \right) \\ &= \frac{p+3}{r-1} \leq \frac{p+3}{2} \end{aligned} \tag{2}$$

for an odd prime $r \in \pi(A_{p+3})$.

If $r > [(p+3)/2]$, $\exp(|A_{p+3}|, r) = 1$.

The proof is completed. $\square$

Let S_n be the symmetric group of degree n . Assume that the cycle has c_1 1-cycles, c_2 2-cycles, and so on, up to c_k k -cycles, where $1c_1 + 2c_2 + \cdots + kc_k = n$. Then the number of conjugacy classes in S_n is

$$z = n! \left(\prod_{i=1}^k i^{c_i} \prod_{i=1}^k c_i! \right)^{-1}. \tag{3}$$

Let A_n be the alternating group of degree n .

Lemma 6. Let $x \in A_n$. Then, for the size of the conjugacy class x^G of x in A_n , one has the following.

- (1) If, for all even i , $c_i = 0$ and, for all odd i , $i \in \{0, 1\}$, then $|x^G| = z/2$.
- (2) In all other cases, $|x^G| = z$.

In particular, $|x^G| \geq z/2$.

Proof. See [17]. $\square$

Lemma 7. If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $(n+1)/2 < p_i < n$. Here

- (i) $s(n) = 6$ for $n \geq 48$;
- (ii) $s(n) = 5$ for $42 \leq n \leq 47$;
- (iii) $s(n) = 4$ for $38 \leq n \leq 41$;
- (iv) $s(n) = 3$ for $18 \leq n \leq 37$;
- (v) $s(n) = 2$ for $14 \leq n \leq 17$;
- (vi) $s(n) = 1$ for $6 \leq n \leq 13$.

In particular, for every natural number $n > 6$, there exists a prime p such that $(n+1)/2 < p < n-1$, and, for every natural number $n > 3$, there exists an odd prime number p such that $n-p < p < n$.

Proof. See Lemma 1 of [18]. $\square$

Lemma 8. Let a, b , and n be positive integers such that $(a, b) = 1$. Then there exists a prime p with the following properties:

- (i) p divides $a^n - b^n$,
- (ii) p does not divide $a^k - b^k$ for all $k < n$,

with the following exceptions: $a = 2$, $b = 1$; $n = 6$ and $a + b = 2^k$; $n = 2$.

Proof. See [19]. $\square$

Lemma 9. With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation

$$p^m - 2q^n = \pm 1; \quad p, q \text{ prime}; \quad m, n > 1, \tag{4}$$

has exponents $m = n = 2$; that is, it comes from a unit $p - q \cdot 2^{1/2}$ of the quadratic field $Q(2^{1/2})$ for which the coefficients p and q are primes.

Proof. See [20, 21]. $\square$

Let L be a nonabelian simple group and let O denote the order of the outer automorphism group of L .

Lemma 10. *Let L be a nonabelian simple group. Then the orders and their outer automorphism of L are as listed in Tables 1, 2, and 3.*

Proof. See [22]. $\square$

3. Main Theorem and Its Proof

In this section, we give the main theorem and its proof.

Theorem 11. *Let G be a finite group with trivial center and $N(G) = N(A_{p+3})$ with $p+2$ composite. Then G is isomorphic to A_{p+3} .*

Proof. We know that A_{p+3} are characterized by $N(G)$ if $p = 7, 13, 19$ (see [12–14]). Then in the following we only consider when $p \geq 23$. $\square$

We divide the proof into the following lemmas.

Lemma 12. *Let $L = A_{p+3}$. Then the following hold.*

- (1) *If $2 \neq r \leq [(p+3)/2]$, then we can write $p+3 = kr + m$ with $0 \leq m < r$ and conjugacy class sizes of r -elements of L are*

$$\frac{(p+3)!}{(p+3-ir)! \cdot r^i \cdot i!} \quad (5)$$

for possible i with $1 \leq i \leq k = [(p+3)/r]$.

In particular, if r is an odd prime divisor of $|G|$, then conjugacy class sizes of r -element of L are

$$\frac{(p+3)!}{2 \cdot k! \cdot r^2}, \quad (6)$$

where $p+3 = 2r + k$ and $0 \leq k < r$.

- (2) *If $r = 2$, then one can write $p+3 = 2k + m$ with $0 \leq m \leq 1$ and conjugacy class sizes of 2-elements of L are*

$$\frac{(p+3)!}{(p+3-2i)! \cdot 2^{2i} \cdot (2i)!} \quad (7)$$

for possible i with $1 \leq i \leq k = [(p+3)/2]$.

- (3) *If $r > [(p+3)/2]$, then one can write $p+3 = r + m$ with $0 \leq m < r$ and conjugacy class sizes of r -elements of L are*

$$\frac{(p+3)!}{(p+3-r)! \cdot r}. \quad (8)$$

In particular, if $r = p$, then the conjugacy class size of p -elements of L is

$$\frac{(p+3)!}{6p}. \quad (9)$$

- (4) *The following numbers from $N(G)$ are maximality with respect to divisibility.*

(a) *One of the following holds:*

$$(p+3)!/2r^2 \text{ if } 2 \cdot r = p+3;$$

$$(p+3)!/4r^2 \text{ if } 2 \cdot r + 2 = p+3;$$

$$(p+3)!/2(k-1)r^2 \text{ if } 2 \cdot r + k = p+3 \text{ and } k = 2n \text{ with } n \geq 2;$$

(b) $(p+3)!/6p$.

Proof. From (3) and Lemma 6, we get the desired results. $\square$

Lemma 13. *Let G be a finite group with trivial center and $N(G) = N(L)$. Then $|L| \mid |G|$ and $\pi(G) = \pi(L)$.*

Proof. Note that $|L| = \prod_{n \in N(L)} n$. Since $|x^G| \mid |C_G(x)| = |G|$, every member from $N(G)$ divides the order of G and $|L| \mid |G|$. So by Lemma 12, we have that $\pi(G) = \pi(L)$. $\square$

Lemma 14. *Suppose that G is a finite group with trivial center and $N(G) = N(L)$. Then the following hold.*

- (1) *There exist different primes r_1, r_2, p from $\pi(L)$ such that $r_1, r_2, p > [(p+3)/2]$. In particular, the Sylow r -subgroup S of G is a cyclic group of order r where $r \in \{r_1, r_2, p\}$. There does not exist an element of order $r_1 \cdot r_2, r_1 \cdot p$, or $r_2 \cdot p$.*
- (2) *For all $n \in N(G)$, if n is divisible at most by r^a , then the Sylow r -subgroup S of G is of order r^a .*

Proof. (1) By Lemma 7, there exist different prime numbers r_1, r_2, p from $\pi(G)$ such that $r_1, r_2, p > [(p+3)/2]$.

From Lemmas 12 and 13, we have that the primes r_1, r_2, p are prime divisors of $|G|$ and r_1^2, r_2^2, p^2 do not divide $|x^G|$ for all $x \in G$. Then by Lemma 3, S is elementary abelian. Therefore if $|x| = r$, then $|x^G|$ is an r' -number.

Let $|S| \geq p^2$. Consider an element y of G with

$$|y^G| = \frac{(p+3)!}{4 \cdot r^2} \quad \text{if } 2r + 2 = p+3, \quad (10)$$

$$|y^G| = \frac{(p+3)!}{2 \cdot r^2} \quad \text{if } 2r = p+3,$$

or

$$|y^G| = \frac{(p+3)!}{2(k-1)r^2} \quad \text{if } 2r + k = p+3, \quad (11)$$

$$k = 2n \quad \text{with } n \geq 2$$

by Lemma 12.

Assume that $p \nmid |y|$. Let x be an element of $C_G(y)$ having order p . Then $C_G(xy) = C_G(x) \cap C_G(y)$, $|x^G| \mid |(xy)^G|$, and

TABLE 1: The simple classical groups.

L	Lie; rank $\mathbf{L}$	d	O	$ L $
$L_n(q)$	$A_{n-1}(q)$	$(n, q-1)$	$2df$, if $n \geq 3$; df , if $n = 2$	$\frac{1}{d} q^{n(n-1)/2} \prod_{i=2}^n (q^i - 1)$
$U_n(q)$	${}^2A_{n-1}(q)$	$(n, q+1)$	$2df$, if $n \geq 3$; df , if $n = 2$	$\frac{1}{d} q^{n(n-1)/2} \prod_{i=2}^n (q^i - (-1)^i)$
$PSP_{2m}(q)$	$C_m(q)$	$(2, q-1)$	df , $m \geq 3$; $2f$, if $m = 2$	$\frac{1}{d} q^{m^2} \prod_{i=1}^m (q^{2i} - 1)$
$\Omega_{2m+1}(q)$	$B_m(q)$	2	$2f$	$\frac{1}{2} q^{m^2} \prod_{i=1}^m (q^{2i} - 1)$
q odd	m			
$P\Omega_{2m}^+(q)$	$D_m(q)$	$(4, q^m - 1)$	$2df$, if $m \neq 4$; $6df$, if $m = 4$	$\frac{1}{d} q^{m(m-1)(q^m-1)} \prod_{i=1}^{m-1} (q^{2i} - 1)$
$m \geq 3$	m			
$P\Omega_{2m}^-(q)$	${}^2D_m(q)$	$(4, q^m + 1)$	$2df$	$\frac{1}{d} q^{m(m-1)(q^m+1)} \prod_{i=1}^{m-1} (q^{2i} - 1)$
$m \geq 2$	$m-1$			

TABLE 2: The simple exceptional group.

L	$\mathbf{L}$	d	O	$ L $
$G_2(q)$	2	1	f , if $p \neq 3$; $2f$, if $p = 3$	$q^6 (q^2 - 1) (q^6 - 1)$
$F_4(q)$	4	1	$(2, p)f$	$q^{24} (q^2 - 1) (q^6 - 1) (q^8 - 1) (q^{12} - 1)$
$E_6(q)$	6	$(3, q-1)$	$2df$	$\frac{1}{d} q^{36} \prod_{i \in \{2,5,6,8,9,12\}} (q^i - 1)$
$E_7(q)$	7	$(2, q-1)$	df	$\frac{1}{d} q^{63} \prod_{i \in \{2,6,8,10,12,14,18\}} (q^i - 1)$
$E_8(q)$	8	1	f	$q^{120} \prod_{i \in \{2,8,12,14,18,20,24,30\}} (q^i - 1)$
${}^2B_2(q), q = 2^{2m+1}$	1	1	f	$q^2 (q^2 + 1) (q - 1)$
${}^2G_2(q), q = 3^{2m+1}$	1	1	f	$q^3 (q^3 + 1) (q - 1)$
${}^2F_4(q), q = 2^{2m+1}$	2	1	f	$q^{12} (q^6 + 1) (q^4 - 1) (q^3 + 1) (q - 1)$
${}^3D_4(q)$	2	1	$3f$	$q^{12} (q^8 + q^4 + 1) (q^6 - 1) (q^2 - 1)$
${}^2E_6(q)$	4	$(3, q+1)$	$2df$	$\frac{1}{d} q^{36} \prod_{i \in \{2,5,6,8,9,12\}} (q^i - (-1)^i)$

$|y^G| \mid |(xy)^G|$ by Lemma 1. Since S is abelian, $S \leq C_G(x)$. Hence, $p \nmid |x^G|$. It follows that $|x^G|$ equals $(p+3)!/6p$ or $(p+1)(p+2)(p+3)/3$ by Lemma 12.

If $|x^G|$ equals $(p+3)!/6p$, then $(p+3)!/6 \mid |(xy)^G|$. On the other hand, $|y^G| \mid |(xy)^G|$; then we have that

$$\begin{aligned} \frac{(p+3)!}{4 \cdot r^2} \mid |(xy)^G| & \quad \text{if } 2r+2 = p+3, \\ \frac{(p+3)!}{2 \cdot r^2} \mid |(xy)^G| & \quad \text{if } 2r = p+3, \end{aligned} \quad (12)$$

or

$$\begin{aligned} \frac{(p+3)!}{2(k-1)r^2} \mid |(xy)^G| & \quad \text{if } 2r+k = p+3, \\ k = 2n \quad \text{with } n \geq 2. & \end{aligned} \quad (13)$$

Obviously, there is no number from $N(G)$ such that $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$.

Therefore $|x^G|$ equals $(p+1)(p+2)(p+3)/3$. In the following, we will consider the following three cases.

Case 1. $|y^G| = (p+3)!/2 \cdot r^2$ if $2r = p+3$.

Obviously, $r \mid (p+1)(p+2)(p+3)/3$. Therefore $(p+3)!/r \mid |(xy)^G|$, a contradiction, since $|x^G| \mid |(xy)^G|$, $|y^G| \mid |(xy)^G|$, and the maximality of $|y^G| = (p+3)!/2 \cdot r^2$.

Case 2. $|y^G| = (p+3)!/4 \cdot r^2$ if $2r+2 = p+3$.

Obviously, $r \mid (p+1)(p+2)(p+3)/3$. Therefore $(p+3)!/r \mid |(xy)^G|$. Also we get a contradiction as in Case 1.

Case 3. $|y^G| = (p+3)!/2(k-1)r^2$ if $2r+k = p+3$ and $k = 2n$ with $n \geq 2$.

TABLE 3: The simple sporadic groups.

L	d	O	$ L $
M_{11}	1	1	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}	2	2	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	12	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}	1	1	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1	1	1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2	2	2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3	3	2	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	1	1	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	2	2	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Suz	6	2	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
McL	3	2	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Ru	2	1	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
$He(F_7)$	1	2	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ly	1	1	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
ON	3	2	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_1	2	1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	1	1	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	1	1	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	6	2	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Fi_{23}	1	1	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	3	2	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$HN(F_5)$	1	2	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
$Th(F_3)$	1	1	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
$BM(F_2)$	2	1	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$M(F_1)$	1	1	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

In this case, $r \nmid (p+1)(p+2)(p+3)/3$. It follows that $|y^G| = |(xy)^G|$. By Lemma 1, $C_G(y) \leq C_G(x)$ and so $|x^G| \mid |y^G|$, a contradiction.

Assume that $p \mid |y|$. Let $|y| = p \cdot t$. Since S is elementary abelian, the numbers p and t are coprime. Let

$$u = y^p, \quad v = y^t. \quad (14)$$

Then $y = uv$ and $C_G(uv) = C_G(u) \cap C_G(v)$. Therefore,

$$\begin{aligned} |v^G| \mid |y^G| &= \frac{(p+3)!}{2 \cdot r^2} \quad \text{if } 2r = p+3, \\ |v^G| \mid |y^G| &= \frac{(p+3)!}{2 \cdot r^2} \quad \text{if } 2r+2 = p+3, \end{aligned} \quad (15)$$

or

$$\begin{aligned} |v^G| \mid |y^G| &= \frac{(p+3)!}{2(k-1) \cdot r^2} \quad \text{if } 2r+k = p+3, \\ k &= 2n, \quad \text{with } n \geq 2. \end{aligned} \quad (16)$$

On the other hand, the element v of G is of order p . Since the Sylow p -subgroup of G is elementary abelian, then $p \nmid |v^G|$. It follows that

$$|v^G| = \frac{(p+3)!}{6p} \quad \text{or} \quad \frac{(p+1)(p+2)(p+3)}{3} \quad (17)$$

by Lemma 12.

If $|v^G| = (p+3)!/6p$, then $|v^G| \mid |y^G|$, a contradiction. Hence $|v^G| = (p+1)(p+2)(p+3)/3$. We consider the following three cases.

Case 1. $|y^G| = (p+3)!/2 \cdot r^2$ if $2r = p+3$.

Obviously, $r \mid (p+1)(p+2)(p+3)/3$. But $r \nmid (p+3)!/2r^2$, a contradiction, since $|x^G| \mid |(xy)^G|$, $|y^G| \mid |(xy)^G|$, and the maximality of $|y^G| = (p+3)!/2 \cdot r^2$.

Case 2. $|y^G| = (p+3)!/4 \cdot r^2$ if $2r+2 = p+3$.

Obviously, $r \mid (p+1)(p+2)(p+3)/3$. But $r \nmid (p+3)!/4r^2$, a contradiction, since $|x^G| \mid |(xy)^G|$, $|y^G| \mid |(xy)^G|$, and the maximality of $|y^G| = (p+3)!/4 \cdot r^2$.

Case 3. $|y^G| = (p+3)!/2(k-1)r^2$ if $2r+k = p+3$ and $k = 2n$ with $n \geq 2$.

In this case, $r \nmid (p+1)(p+2)(p+3)/3$. It follows that $|y^G| = |(xy)^G|$ since the maximality of $|v^G|$. By Lemma 1, $C_G(y) \leq C_G(x)$ and so $|x^G| \mid |y^G|$, a contradiction.

Therefore the Sylow p -subgroup of G is of order p .

Similarly we can prove the other two cases.

There does not exist an element of order $r_1 \cdot r_2$, $r_1 \cdot p$, or $r_2 \cdot p$.

(2) Without loss of generality, we assume that n is divisible at most by r^2 .

Assume that $|S| \geq r^3$. Consider an element x of G such that

$$|x^G| = \begin{cases} \frac{(p+3)!}{2r^2} & \text{if } 2r = p+3; \\ \frac{(p+3)!}{4r^2} & \text{if } 2r+2 = p+3; \\ \frac{(p+3)!}{2(k-1)r^2} & \text{if } 2r+k = p+3, \\ & \text{with } k = 2n, n \geq 2, \end{cases} \quad (18)$$

by Lemma 12.

Let $r \nmid |x|$. Then there is an element y of G of order r . By Lemma 1 we have that $C_G(xy) = C_G(x) \cap C_G(y)$, $|x^G| \mid |(xy)^G|$, and $|y^G| \mid |(xy)^G|$.

If y is an r -central element, then $S \leq C_G(y)$ and $|y^G|$ is an r' -number. We consider the following three cases.

Case 1. $2r = p+3$.

In this case $|y^G| = (p+3)!/2 \cdot r^2$. Then we have that $|x^G| = |y^G| = |(xy)^G|$ and so $r^2 \parallel |C_G(x)| = |C_G(y)|$ since y is an r -central element (in fact, if $r^3 \parallel |C_G(x)|$, then there is a conjugacy class size which is a multiple of the number $(p+3)!/r^3 \cdot 3!$ contradicting $2r = p+3$). Thus $r \mid |x^G| = |y^G|$ since $r^3 \mid |G|$, a contradiction.

Case 2. $2r+2 = p+3$.

We have that $|y^G| = (p+3)!/4 \cdot r^2$. We similarly can rule out this case as "Case 1: $2r = p+3$ ".

Case 3. $2r+k = p+3$ and $k = 2n$ with $n \geq 2$.

In this case, $|y^G| = (p+3)!/2(k-1)r^2$ or $(2r+1)(2r+k)/2(k-2)$ (we only choose the maximality from the r' -number in $N(G)$). If the former, then $|x^G| = |y^G| = |(xy)^G|$ and so $r^2 \parallel |C_G(x)| = |C_G(y)|$ since y is an r -central element (in fact, if $r^3 \parallel |C_G(x)|$, then there is a conjugacy class size which is a multiple of the number $(p+3)!/r^3 \cdot 3!$ contradicting $2r+k = p+3$). Thus $r \mid |x^G| = |y^G|$ since $r^3 \mid |G|$, a contradiction. Then $|y^G| = (2r+1)(2r+k)/2(k-2)$. But there is no member from $N(G)$ such that $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$.

If y is a noncentral r -element, then we choose an element z of order p such that $r^2 \parallel |z^G|$. By hypothesis, $r \mid |C_G(z)|$

and, obviously, $r \nmid |z|$. Then by Lemma 1.2 of [4], there is an r -element w such that $1 \neq w \in C_G(z)$, $C_G(wz) < C_G(w)$, and

$$r \mid \frac{|C_G(w)|}{|C_G(wz)|} = \frac{|(wz)^G|}{|w^G|}. \quad (19)$$

On the other hand, we also have $z \in C_G(w)$. Since $wz = zw$, then $|(wz)^G| = |z^G|$ and so $C_G(z) \leq C_G(w)$. Therefore $z \in C_G(z) \leq C_G(w)$. It follows that $|C_G(z)| = |C_G(wz)|$ since the maximality of $|z^G|$, and we get $r \mid |C_G(w)|/|C_G(wz)| = 1$, a contradiction.

Let $r \mid |x|$; then we write $|x| = rt$. If S is elementary abelian, then $(r, t) = 1$. Set

$$u = x^r, \quad v = x^t. \quad (20)$$

Then $x = uv$ and $C_G(x) = C_G(u) \cap C_G(v)$. Hence $|v^G| \mid |x^G|$ and $|u^G| \mid |x^G|$. Since v is an element of order r , then

$$|v^G| = \begin{cases} \frac{(p+3)!}{2r^2} & \text{if } 2r = p+3; \\ \frac{(p+3)!}{4r^2} & \text{if } 2r+2 = p+3; \\ \frac{(p+3)!}{2(k-1)r^2}, \text{ or } \frac{(2r+1)(2r+2) \cdots (2r+k)}{2(k-2)} & \text{if } 2r+k = p+3 \\ & \text{with } k = 2n, n \geq 2. \end{cases} \quad (21)$$

If $|v^G| = (p+3)!/2r^2$, $(p+3)!/4r^2$, or $(p+3)!/2(k-1)r^2$, then $|(uv)^G| = |v^G|$ and so $r^3 \mid |S| \mid |v^G|$. It follows that $r \nmid |C_G(v)| = |C_G(uv)| = |C_G(x)|$ contradicts the fact that x is an r -element. Therefore $|v^G| = (2r+1)(2r+2) \cdots (2r+k)/2(k-2)$. On the other hand, $|x^G| = (p+3)!/2(k-1)r^2$. Obviously, $r \mid (2r+1)(2r+2) \cdots (2r+k)/2(k-2)$ but $r \nmid (p+3)!/2(k-1)r^2$. It follows that $(p+3)!/2(k-1)r$ lies in $N(G)$, which contradicts the maximality of $(p+3)!/2(k-1)r^2$.

Therefore S is nonabelian. So we chose an element z of order p such that $r^2 \mid |z^G|$. By hypothesis, $r \mid |C_G(z)|$ and, obviously, $r \nmid |z|$. Then by Lemma 1.2 of [4], we also have that $|C_G(z)| = |C_G(wz)|$ since the maximality of $|z^G|$, and we get $r \mid |C_G(w)|/|C_G(wz)| = 1$, a contradiction.

The lemma is completed. $\square$

Lemma 15. Suppose that G is a finite group with trivial center and $N(G) = N(L)$. Let $\pi = \{2, 3\}$. Then $O_{\pi, \pi'}(G) = O_\pi(G)$. In particular, G is insoluble.

Proof. Let $K = O_\pi(G)$, $\bar{G} = G/K$ and denote by $\bar{x}$ and by $\bar{H}$ the images of an element x and a subgroup H of G in $\bar{G}$, respectively. Assume that the result is not true; then there is a prime $r \in \pi(L) \setminus \pi$ with $O_r(\bar{G}) \neq 1$.

Let $r \in \{r_1, r_2, p\}$ with $O_r(\bar{G}) \neq 1$. Then $\bar{G}$ contains a Hall $\{r, s\}$ -subgroup of order $r \cdot s$ with $s \in \{r_1, r_2, p\} - \{r\}$. However, Hall $\{r, s\}$ -subgroup must be cyclic contradicting Lemma 14.

Let P be a Sylow r -subgroup of $\bar{G}$ where $r \in \pi(L) \setminus \{r_1, r_2, p\}$. If $O_r(\bar{G}) \neq 1$, then $A = Z(O_r(\bar{G}))$ is a nontrivial

normal subgroup of $\bar{G}$. Let $\bar{x}$ be an element of order p in $\bar{G}$. So we have that $|\bar{x}^{\bar{G}}|$ is a divisor of

$$\frac{(p+3)!}{6p} \quad \text{or} \quad \frac{(p+1)(p+2)(p+3)}{3}. \quad (22)$$

By coprime action lemma, $A = C_A(\bar{x}) \times [A, \bar{x}]$. In the following, we consider two cases “ $r \leq [(p+3)/2]$ and $r \geq [(p+3)/2]$ ”.

- (i) $r > [(p+3)/2]$ and $r \neq r_1, r_2, p$. In this case, by Lemma 14, we have that the Sylow r -subgroup of G is of order r . Hence there is a Hall $\{r, p\}$ -subgroup H . Since H must be cyclic, then there is an element of order $r \cdot p$, a contradiction by the proof of Lemma 14.
- (ii) $r \leq [(p+3)/2]$ and $r \neq 2, 3$. If $2r = p+3, 2r+2 = p+3$, or $2r+k = p+3$ with $k = 2n$ and $n \geq 2$, then the index of $C_A(\bar{x})$ in A is at most r^2 . By Lemma 8, there exists a least divisor m of $\phi(p)$ such that p divides $r^m - 1$, and the subgroup $[A, \bar{x}] \langle \bar{x} \rangle$ must be abelian. It follows that $[A, \bar{x}] = 1$ and $A = C_A(\bar{x})$. If $\bar{y}$ is a nontrivial element of $Z(P) \cap A$, then the order of $C_{\bar{G}}(\bar{y})$ is a multiple of p . By Lemma 4, y lies in the center of a Sylow p -subgroup of G . This contradicts Lemma 12. Thus $O_r(\bar{G}) = 1$.

It follows that $O_r(\bar{G}) = 1$ for $r \in \{5, 7, \dots, p\}$.

Therefore $O_{\pi, \pi'}(G) = O_\pi(G)$. In particular, G is insoluble. $\square$

Lemma 16. *There is a normal series $1 \leq K \leq H \leq G$ such that $H/K \cong A_{p+3}$.*

Proof. By Lemmas 13 and 14, $|G| = (p+3)!/2$.

By Lemma 15, we have that $H/K \leq \bar{G} \leq \text{Aut}(H/K)$, where $M : H/K = S_1 \times S_2 \times \dots \times S_k$ is a direct product of nonabelian simple groups $S_1, S_2, \dots, S_k$. Since G cannot contain a Hall $\{r_1, r_2, p\}$ -subgroup, numbers r_1, r_2 , and p divide the order of exactly one of these groups that is listed as in Lemma 10, and so we assume that they divide S_1 . Since $S_1 \triangleleft \bar{G}$, we let G^* and M^* denote the factor groups $\bar{G}/S_1$ and M/S_1 , respectively. If $k > 1$, then a Sylow 2-subgroup of G^* is nontrivial and its center Z has a nontrivial intersection with M^* . Consider a nontrivial element y of $T = S_2 \times \dots \times S_k$ such that its image in $\bar{G}$ lies in Z . Since y centralizes S_1 , it lies in the center of a Sylow 2-subgroup of $\bar{G}$ and centralizes an element of order p , a contradiction. Thus $M = S_1$, and $\bar{G}$ is almost simple. Therefore

$$\frac{H}{K} \leq \bar{G} \leq \text{Aut}\left(\frac{H}{K}\right). \quad (23)$$

Obviously, $r_1, r_2, p \mid |H/K|$ (in fact, if $r_1, r_2, p \mid |G/H|$, then $r_1, r_2, p \mid |G/H| \mid |\text{Out}(H/K)|$, a contradiction, from Lemma 10; if $r_1, r_2, p \mid |K|$, then there is an element of order $r \cdot p$ with $r \in \{r_1, r_2\}$ contradicting Lemma 14). In the following, we always assume that $r \in \pi(G) = \pi(L)$. In the following, we consider S_1 which is listed as in Tables 1, 2, and 3.

(i) *Case 1.* $H/K \cong A_n$ with $n \geq 6$.

Then $n = p, p+1, \dots, p+k$ with $p+2, p+4, \dots$ composite and $p+k+1$ prime. If $k \geq 4$, then $(p+k)!/2 \mid (p+3)!$, a contradiction. Therefore H/K is isomorphic to A_p, A_{p+1}, A_{p+2} , or A_{p+3} .

Let x be an element of order p in H . Then $|x^H|$ is p' -number since $|H|_p = pH/K \cong A_p$.

Since $|A_p| \mid (p+3)!$, then $3 \mid |K|$. We have $|x^H| = (p-1)!/2$. On the other hand, $|x^G| = (p+3)!/6p$. It follows that $|x^K| \mid (p+1)(p+2)(p+3)/3$ and so there is an element of $r \cdot p$ or of order $r' \cdot p$ with $3 < r' < r < p$ and r and r' divide one of the prime divisors of the numbers $p+1, p+2$, or $p+3$, which contradicts Lemma 14. Similarly, we can rule out these cases “ $H/K \cong A_{p+1}$ and $H/K \cong A_{p+2}$ ”.

Therefore $H/K \cong A_{p+3}$.

(ii) *Case 2.* H/K is not isomorphic to a sporadic simple group according to Table 3.

(iii) *Case 3.* H/K is isomorphic to a simple group of Lie type. Let q be a prime power.

(1) $H/K \cong B_n(q)$ with $n \geq 2$.

In this situation, by Lemma 13, $\pi(G) = \{2, 3, 5, 7, \dots, p\}$ and so

$$\frac{1}{(2, q-1)} q^{n^2} \prod_{i=1}^n (q^{2i} - 1) \mid (p+3)!. \quad (24)$$

It follows that $p \mid q$ or $p \mid \prod_{i=1}^n (q^{2i} - 1)$. If $p \mid q$, then q is a power of p . Since $|G_p| = p$ by Lemma 14, this is impossible as $n \geq 2$. Therefore $p \mid \prod_{i=1}^n (q^{2i} - 1)$. It follows that $p \mid q^{2t} - 1$ for some $1 \leq t \leq n$ as p is prime. On the other hand, $q^{n^2} \mid r$ or $q^{n^2} \mid r^m$. If the former, then $q = r$ and $n = 1$, a contradiction. It follows that $q^{n^2} \mid r^m$. Hence $r \mid q$ and $m = kn^2$ for some integer $k \geq 1$. By Lemma 5, $kn^2 < p/2 < (q^{2n} - 1)/2$, but the equation has no solution in $\mathbb{N}$. Furthermore, since $C_n(q)$ has the same order as $B_n(q)$, we also can rule out.

(2) $H/K \cong D_n(q)$ with $n \geq 4$.

Therefore we have that $(1/(4, q^n - 1))q^{n(n-1)}(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1) \mid (p+3)!$. Since the Sylow p -subgroup of G is of order p , $p \nmid q$ as, otherwise, $q = p$ and thus $n = 1$, a contradiction. It follows that $p \mid q^n - 1$ or $p \mid q^{2t} - 1$ for some integer $1 \leq t \leq n-1$.

Let $p \mid q^n - 1$. Then $q^{n(n-1)} \mid r < p$ with r prime, and so $q^{n(n-1)} < p^n - 1$. Thus $n = 1$, a contradiction.

Let $p \mid q^{2t} - 1$ for some integer $1 \leq t \leq n-1$. Then $q^{n(n-1)} \mid r < p$ with r prime, and so $q^{n(n-1)} - 1 \leq q^{2t} - 1 < q^{2n} - 1$. It follows that $n = 1, 2, 3$, a contradiction.

(3) $H/K \cong A_n(q)$ with $n \geq 2$.

In this situation,

$$\frac{1}{(n+1, q+1)} q^{(1/2)n(n+1)} \prod_{i=1}^n (q^{i+1} - (-1)^{i+1}) \mid (p+3)!. \quad (25)$$

Since the Sylow p -subgroup of G is of order p and $n \geq 2$, we obtain that $p \mid q^{(1/2)n(n+1)}$ or $p \mid q^{t+1} - (-1)^{t+1}$ for some integer $1 \leq t \leq n$.

If the former, then we have that $q = p$ and $n = 1$, a contradiction.

Let $p \mid q^{t+1} - (-1)^{t+1}$ for some integer $1 \leq t \leq n$. Then $q^{(1/2)n(n+1)} \mid r^m$ for some m . If $m = 1$, then $q = r$ and $n = 1$, a contradiction. It follows that $q^{(1/2)n(n+1)} \mid r^m$ for some $m > 1$ and $r \mid q$. By Lemma 5, $r \leq [p/2] < p/2$ and $n(n+1)/2 \leq m \leq p/2$.

If t is odd, then $p \mid r^{t+1} - 1$. By Lemmas 8 and 9, $q = 2$, $t = 5$, and $r = 2$. Hence $p = 3$ or 7 . If $p = 3$, we can rule out this case since $p \geq 5$. If $p = 7$, then $n(n+1) \leq 3$ and so $n = 1, 2$, a contradiction, since $t = 5 < n$.

If t is even, then $p \mid r^{t+1} + 1$. So $p \mid r + 1$ or $p \mid r^t - r^{t-1} + \dots + 1$. If the former, then $p \mid r + 1 < [p/2] + 1$, a contradiction. If the latter, then $p < r^t + 1$. It follows that $p \leq r^t - 1$. Let $t = 2k$ with $1 < k < [p/2]$. Then $p \leq r^k + 1$ or $p \leq r^k - 1$ and so $p \mid r^k + 1$ or $p \mid r^k - 1$. If $p \mid r^k - 1$, then, by Lemma 8, $r = 2$ and $t = 5$; we also can rule out this case as above. So $p \mid r^k + 1$. It follows that $p \mid r^t - 1$. Similarly, as $p \mid r^k - 1$, we can rule out this case.

(4) $H/K \cong E_8(q)$.

Therefore we have that

$$q^{120} (q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1) \times (q^{14} - 1)(q^{12} - 1)(q^8 - 1)(q^2 - 1) \mid (p+3)!. \quad (26)$$

It follows that

$$p \mid q^{120} (q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1) \times (q^{12} - 1)(q^8 - 1)(q^2 - 1). \quad (27)$$

If $p \mid q^{120}$, then we can rule out this case since the Sylow p -subgroup of G is of order p . Hence $p \mid q^t - 1$, where $t \in \{2, 8, 12, 14, 18, 20, 24, 30\}$. On the other hand, $r^m \mid q^{120}$. If $m = 1$, then $q = r$ and $1 > 120$, a contradiction. If $m > 1$, then $q = r$ and $m \leq 120$. By Lemma 5, $120 \leq p$ and so $p \in \{5, 7, 11, 13, \dots, 101, 103, 107, 109, 113\}$. It is easy to rule out this case by considering the orders of G . Similarly, we can exclude that $H/K \cong E_6(q), E_7(q)$, and $F_4(q)$.

(5) $H/K \cong G_2(q)$.

Then we have $q^6(q^6 - 1)(q^2 - 1) \mid (p+3)!$. It follows that $q^6 \mid p$, $p \mid q^6 - 1$, or $p \mid q^2 - 1$.

If $q^6 \mid p$, we rule out this case.

If $p \mid q^6 - 1$, then there exists a prime r such that $q^6 \mid r^m$ for some integer m . Therefore $q = r$ and $6 \leq m < p$ by Lemma 5. It follows that $p = 5$ and so we have a contradiction by considering the order of G . Similarly, we also can rule out this case " $p \mid q^2 - 1$ ".

(6) $H/K \cong {}^2E_6(q)$.

It is easy to see that $(1/(3, q+1))q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1) \mid (p+3)!$. It follows that $p \mid q^t - 1$ with $t = 12, 8, 6, 2$, $p \mid q^k + 1$ with $k = 9, 5$, or $p \mid q^{36}$. If $p \mid q^{36}$, then we rule out this case since the Sylow p -subgroup of G is of order p . So $p \mid q^t - 1$ with $t = 12, 8, 6, 2$, $p \mid q^k + 1$ with $k = 9, 5$, and so there exists a prime r such that $q^{36} \mid r^m$ for some integer m . It means that $36 \leq m \leq p$. Therefore $p = 31, 29, 23, 19, 17, 13, 11, 7$. We also can rule out this case by order consideration.

(7) $H/K \cong {}^2B_2(q)$ with $q = 2^{2m+1}$.

It follows that $q^2(q^2 + 1)(q - 1) \mid p!$. Thus $q^2 \mid p$, $p \mid q^2 + 1$, or $p \mid q - 1$.

If $q^2 \mid p$, then we rule out this case.

If $p \mid q^2 + 1$, then there is a prime r such that $q^2 \mid r^m$ and so $2 \leq m \leq p$ by Lemma 5. Hence $p = 2$, a contradiction. Similarly we can rule out this case " $p \mid q + 1$ ".

Similarly $H/K \cong {}^2F_4(2^{2m+1})$.

(8) $H/K \cong {}^2G_2(q)$, $q = 3^{2n+1}$ with $n \geq 1$.

We see that $q^3(q^3 + 1)(q - 1) \mid p!$. Since the Sylow p -subgroup of G is of order p , then $p \nmid q^3$. It follows that $p \mid q^3 + 1$ or $p \mid q - 1$. If $p \mid q^3 + 1$, then there exists a prime r such that $q^3 \mid r^m$ for some integer m . If $m = 1$, then $1 > 3$, a contradiction. Hence $3 \leq m \leq p$ by Lemma 5, and so $p = 3$, a contradiction. If $p \mid q - 1$ and $r \mid q$, then there exists a Frobenius group of $r \cdot p$ with a kernel of order r and a complement of order p , respectively, and so there is an element of order $r \cdot p$, which contradicts Lemma 14.

(9) $H/K \cong {}^3D_4(q)$.

We have that $q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1) \mid p!$. In this case, since G has a Sylow p -subgroup of order p , then $p \mid q^8 + q^4 + 1$, $q \mid q^6 - 1$, or $p \mid q^2 - 1$. If $p \mid q^8 + q^4 + 1$, then there exists a prime r such that $r^m \mid q^{12}$ and so $m \leq 12$. By Lemma 5, $p \leq 12$. It follows that $p = 5, 7, 11$. Order consideration rules out these cases " $p = 5, 7, 11$ ". Similarly we can rule out this case " $p \mid q^2 - 1$ ".

(10) $H/K \cong A_n(q)$ with $n \geq 1$.

It is easy to get that

$$\frac{1}{(n+1, q-1)} q^{n(n+1)/2} \prod_{i=1}^n (q^{i+1} - 1) \mid p!. \quad (28)$$

It follows that $p \mid q^{n(n+1)/2}$ or $p \mid \prod_{i=1}^n (q^{i+1} - 1)$. If $p \mid q^{n(n+1)/2}$, then $n = 1$ since the Sylow p -subgroup of G is of order p , a contradiction. Hence $p \mid \prod_{i=1}^n (q^{i+1} - 1)$ and so $p \mid q^{t+1} - 1$ for some integer $1 \leq t \leq n$. It follows that there exists a prime r such that $q^{n(n+1)/2} \mid r^m$ and so $n(n+1)/2 \leq m \leq p/2$ by Lemma 5. Since the Sylow p -subgroup of G is of order p , then $n(n+1) \mid p$ and so $n = 1$, a contradiction.

This completes the proof of the lemma. $\square$

Lemma 17. Consider the following.

$$G \cong A_{p+3}.$$

Proof. By Lemma 16,

$$A_{p+3} \leq \bar{G} \leq \text{Aut}(A_{p+3}) \cong S_{p+3}. \quad (29)$$

If $\bar{G} \cong S_{p+3}$, then there exists an element $\bar{x}$ of $\bar{G}$ with

$$\bar{x}^{\bar{G}} = \frac{(p+3)!}{3p} \quad (30)$$

which contradicts Lemma 12.

So $\bar{G} \cong A_{p+3}$. Then we define the normal series $1 \leq K \leq G$ into the chief ones. We prove that $K = 1$. By Lemma 15, $\pi(K) \subseteq \{2, 3\}$.

If K is a 2-group, in this case, let $|\bar{x}| = p$. Then

$$\frac{(p+3)!}{6p} \mid |\bar{x}^{\bar{G}}|. \quad (31)$$

By Lemma 12,

$$|x^G| = |\bar{x}^{\bar{G}}| = \frac{(p+3)!}{6p}. \quad (32)$$

So x centralizes K . It follows that there is an element of $2 \cdot p$ which contradicts Lemma 12 (4).

If K is a 3-group. Then similarly as the case “ K is a 2-group”, we have that $|x^G| = |\bar{x}^{\bar{G}}|$ is maximal in $N(G)$ and $C_G(x)$ is abelian. So by Lemma 1.12 of [13], $K \leq Z(G) = 1$.

Therefore $K = 1$ and $G \cong A_{p+3}$.

This completes the proof of the lemma and also of the main theorem. $\square$

4. Some Applications and Problem

Y. Chen and G. Chen in [23] proved that the group A_{10} can be characterized by its order and two special conjugacy classes sizes. Then, obviously, we also have the following result.

Corollary 18. Let G be a finite group with trivial center. Assume that $N(G) = N(A_{p+3})$ and $|G| = |A_{p+3}|$. Then $G \cong A_{p+3}$.

One knows that the alternating groups A_n with $n = 10, 16, 22, 26$ are characterized by $N(G)$. Then by [4, 5, 15, 21], one has the following.

Corollary 19. Let G be a finite group with trivial center. Assume that $N(G) = N(A_n)$ with $n = p, p+1, p+2, p+3$. Then $G \cong A_n$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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