Research Article

Nonoscillatory Solutions for System of Neutral Dynamic Equations on Time Scales

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We will discuss nonoscillatory solutions to the \(n\)-dimensional functional system of neutral type dynamic equations on time scales. We will establish some sufficient conditions for nonoscillatory solutions with the property \(\lim_{t \to \infty} x_i(t) = 0\), \(i = 1, 2, \ldots, n\).

1. Introduction

The theory of dynamic equations on time scales was introduced by its founder Hilger in his PHD thesis [1] in 1988. The study of dynamic equations on time scales is an area of mathematics that has recently received a lot of attention. It has been created in order to unify continuous and discrete analysis. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales; we refer the reader to the papers [2–12]. In [13, 14], authors studied nonoscillatory solutions to the \(n\)-dimensional functional differential systems of neutral type and obtained some sufficient conditions for nonoscillatory solutions with the property \(\lim_{t \to \infty} x_i(t) = 0\), \(i = 1, 2, \ldots, n\). Using the idea and method of [13, 14], in this paper, we will study the nonoscillatory solutions for systems of neutral dynamic equations on time scales, which have the following form:

\[
\begin{align*}
\left[ x_1(t) - a(t) x_1(g(t)) \right]^2 &= p_1(t) x_2(t), \\
x_1^a(t) &= p_1(t) x_{i+1}(t), & i &= 2, 3, \ldots, n-1, \\
x_n^a(t) &= \Theta_n(t) f(x_1(h(t)) ), & t &\in \mathbb{T},
\end{align*}
\]

where natural number \(n \geq 3\), \(\Theta\) is a continuous, real-valued positive function defined on the time scale \(\mathbb{T}\) and through this paper we assume that

(a) \(g : \mathbb{T} \rightarrow \mathbb{T}\) is a continuous and increasing function with \(\lim_{t \to \infty} g(t) = \infty\) and \(\{ t \in \mathbb{T} : t \geq t_0 \} \subset g(\mathbb{T})\) for some \(t_0 \in \mathbb{T}\);

(b) \(p_i : \mathbb{T} \rightarrow \mathbb{R}^+ := [0, \infty), i = 1, 2, \ldots, n\), are continuous and increasing functions; \(p_n\) not identically equal to zero in any neighbourhood of infinity; \(\int_0^\infty p_k(t) \Delta t = \infty, k = 1, 2, \ldots, n-1\), hold for any \(t \in \mathbb{T}\);

(c) \(h : \mathbb{T} \rightarrow \mathbb{R}\) is a continuous and increasing function with \(\lim_{t \to \infty} h(t) = \infty\);

(d) \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a continuous function, and \(f(u)/u \geq K\) for \(u \neq 0\), where \(K\) is a positive constant.

2. Some Preliminary Results

A time scale \(\mathbb{T}\) is an arbitrary nonempty closed subset of the real numbers. Throughout this paper, as a matter of convenience, for any \(a, b \in \mathbb{T}, a < b\), we denote the sets \(\{ t \in \mathbb{T} : a \leq t \leq b\}\) by \([a, b]\), which is called a close interval in \(\mathbb{T}\). Open intervals and half-open intervals and so forth are defined accordingly.

A function \(z(t)\) is defined for \(x_i(t)\) as

\[
z(t) = x_1(t) - a(t) x_1(g(t)).
\]
A vector function \( x = (x_1, \ldots, x_n) \) is a solution to the system (1) if there is a \( t_1 \in T \) such that functions \( x_i(t), \) \( i = 1, 2, \ldots, n, \) are continuously differentiable and \( x \) satisfies (1) on \([t_1, \infty)_T\). Let \( W \) be the set of all solutions \( x = (x_1, \ldots, x_n) \) to the system (1) satisfying \( \sup\{\sum_{i=1}^n |x_i(t)| : t \in [T, \infty)_T\} > 0 \) for any \( T \in \mathbb{T} \). A solution \( x \in W \) is called nonoscillatory if there exists a \( T_1 \in \mathbb{T} \) such that every component is different from zero for \( t \in [T_1, \infty)_T \). Otherwise a solution \( x \in W \) is said to be oscillatory.

Since we restrict our attention to asymptotic properties of nonoscillatory solutions to the system (1), we suppose that the time scale under consideration is not bounded above; that is, it is a time scale interval of the form \([t_0, +\infty)_T\). On any time scale we define the forward operator and delta derivative as follows.

**Definition 1** (see [15]). Let \( \mathbb{T} \) be a time scale. For any \( t \in \mathbb{T} \), we define the forward jump operator \( \sigma : \mathbb{T} \rightarrow \mathbb{T} \) by

\[
\sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}.
\]

**Definition 2** (see [15]). Let \( \mathbb{T} \) be a time scale. Assume \( f : \mathbb{T} \rightarrow \mathbb{R} \) is a function and let \( t \in \mathbb{T} \). The (delta) derivative of \( f \) at \( t \) is defined by

\[
f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \in \mathbb{T} \setminus \{\sigma(t)\}.
\]

For some other preliminary concepts on time scales, one can refer to [15]. In the remainder of this section, we present some lemmas indispensable which will be used later.

**Lemma 3.** Let \( x \in W \) be a solution to (1) with \( x_1(t) \neq 0 \) on \([t_0, \infty)_T\). Then \( x \) is nonoscillatory and \( z(t), x_2(t), \ldots, x_n(t) \) are monotone on \([T, \infty)_T\) for some \( T \in [t_0, \infty)_T\).

**Proof.** Because \( x_1(t) \neq 0 \) on \([t_0, \infty)_T\), using the conditions (b) and (d), \( x_1^{\Delta}(t) \) does not change its sign and not identically equal to zero on \([T_1, \infty)_T\) for some \( T_1 \in [t_0, \infty)_T\). This implies that \( x_1(t) \) is monotone and \( x_1(t) \neq 0 \) on \([T_1, \infty)_T\) for some \( T_1 \in [T, \infty)_T\). Continue similarly as above and then conclude the desired results. The proof is complete.

**Lemma 4.** Assume that (a) holds and \( g(t) > t \) for \( t \in [t_0, \infty)_T\). Let \( y(t) \) be a nonoscillatory solution to the functional inequality

\[
y(t) \left[y(t) - a(t) y(g(t)) \right] > 0, \quad \text{for any } t \in [t_0, \infty)_T,
\]

where \( a : \mathbb{T} \rightarrow [0, \infty) \) is a continuous function. If \( 1 \leq a(t) \) for \( t \in [t_0, \infty)_T \), then \( y(t) \) is bounded on \([t_0, \infty)_T\). Moreover, if \( 0 < a(t) \leq \lambda < 1 \) for \( t \in [t_0, \infty)_T \), then \( \lim_{t \to \infty} y(t) = 0 \).

**Proof.** We firstly claim that there exists a negative integer \( n \) such that \( t = g^n(t) \) for some \( r \in [g^{-1}(t_0), t_0] \). In fact, if not the case, then there is \( T \in [t_0, \infty)_T \) such that \( g^{-n}(T) \notin [g^{-1}(t_0), t_0] \) for any \( n \in \mathbb{N} \). We need only to consider two cases: Case 1, \( g^{-n}(T) \in [t_0, T) \) for each \( n \in \mathbb{N} \). Then there exists \( b \in [t_0, T) \) satisfying \( \lim_{n \to \infty} g^{-n}(T) = b \), which implies \( g(b) = b \), a contradiction. Case 2, \( g^{-n}(T) < t_0 < g^{-n}(T) \). Then, by the monotonicity of \( g \), \( g^{-n}(T) = g(g^{-n+1}(T)) \leq g(g^{-1}(t_0)) = t_0 \), also a contradiction.

Without loss of generality, we may assume that \( y(t) \) is a positive solution of the functional inequality

\[
y(t) y(g(t)) > 0, \quad \text{for any } t \in [t_0, \infty)_T.
\]

**Lemma 5.** Assume that (a) holds and \( g(t) < t \) for \( t \in [t_0, \infty)_T \). Let \( y(t) \) be a nonoscillatory solution to the functional inequality

\[
y(t) \left[y(t) - a(t) y(g(t)) \right] < 0, \quad \text{for any } t \in [t_0, \infty)_T,
\]

where \( a : \mathbb{T} \rightarrow [0, \infty) \) is a continuous function. If \( 0 < a(t) \leq 1 \) for \( t \in [t_0, \infty)_T \), then \( y(t) \) is bounded on \([t_0, \infty)_T\). Moreover, if \( 0 < a(t) \leq \lambda < 1 \) for \( t \in [t_0, \infty)_T \), then \( \lim_{t \to \infty} y(t) = 0 \).

**Lemma 6.** Assume that \( q : \mathbb{T} \rightarrow \mathbb{R}^+ \) and \( \delta : \mathbb{T} \rightarrow \mathbb{T} \) are continuous functions and \( \delta(t) > \sigma(t) \) for all \( t \in \mathbb{T} \).

**Proof.** We only prove (i) for the proof of (ii) is similar. Suppose that the functional inequality (7) has an eventually positive solution \( x(t) \). Without loss of generality, we may assume that \( x(t) > 0 \) for any \( t \in \mathbb{T} \). Then, from (7),

\[
x^\Delta(t) \geq q(t)x(\delta(t)) \geq 0, \quad \text{for any } t \in \mathbb{T}.
\]

It follows that \( x(t) \) is nondecreasing on \( \mathbb{T} \).

Let \( X : \mathbb{R} \rightarrow \mathbb{R} \) be the linear extension of the function \( x(t) \), then \( X : \mathbb{T} \rightarrow \mathbb{R} \) is continuous, \( X : \mathbb{T} \rightarrow \mathbb{R} \) is delta differentiable, and \( X \Delta = X^\Delta \) on \( \mathbb{T} \). By [15, Theorem 1.87], for
any \( t \in T^x \), there is \( \xi_t \in [t, \sigma(t)] \) such that \([\ln x(t)]^A = \chi^A(\xi_t)/(X(\xi_t))\). Because \( \delta(t) > \sigma(t) \geq \xi_t \geq t \) for all \( t \in T^x \), \( x(\sigma(t)) \geq \chi(\sigma(t)) \geq x(t) > 0 \) for all \( t \in T^x \). Therefore,
\[
[\ln x(t)]^A \geq \chi^A(\xi_t)/(x(\xi_t)) \geq \chi^A(t)/(x(\delta(t))), \quad \text{for } t \in T^x. \tag{9}
\]
By (7), we have
\[
[\ln x(t)]^A \geq q(t), \quad \text{for } t \in T^x. \tag{10}
\]
Integrating the inequality above from \( \sigma(t) \) to \( \delta(t) \) on \( T \), we get, for \( t \in T^x \),
\[
x(\delta(t)) \geq x(\sigma(t)) \exp \left[ \int_{\sigma(t)}^{\delta(t)} q(s) \Delta s \right]. \tag{11}
\]
By the hypothesis, there exist \( t_1 \in T \) and a constant \( c \) such that
\[
\int_{\sigma(t)}^{\delta(t)} q(s) \Delta s \geq c > 1/e \text{ holds for } t \in [t_1, \infty). \tag{12}
\]
Here, to give the last inequality, we have used the inequality \( e^x > e^y \) for all \( y \geq 0 \). From (7) and (9), we have
\[
[\ln x(t)]^A \geq ec \cdot q(t), \quad \text{for } t \in [t_1, \infty). \tag{13}
\]
Integrating the inequality above from \( \sigma(t) \) to \( \delta(t) \) on \( T \), we get, for \( t \in [t_1, \infty) \),
\[
x(\delta(t)) \geq \exp \left[ \int_{\sigma(t)}^{\delta(t)} ec \cdot q(s) \Delta s \right] \geq e^{ec^2} \cdot x(\sigma(t)). \tag{14}
\]
Continuing this progress, we conclude that, for each natural number \( n \),
\[
x(\delta(t)) \geq (ec)^n x(\sigma(t)), \quad t \in [t_1, \infty). \tag{15}
\]
Since \( ec > 1 \), it follows that \( x(\delta(t)) = \infty \) for any \( t \in [t_1, \infty) \), a contradiction. The proof is complete.

Let \( x(t) \in W \) be a nonoscillatory solution to (1). It follows, from Lemma 3, that the function \( z(t) \) has to be eventually of constant sign. Hence, either
\[
x_1(t) \cdot z(t) > 0 \tag{16}
\]
or
\[
x_1(t) \cdot z(t) < 0 \tag{17}
\]
for sufficiently large \( t \in T \).

**Lemma 7.** Let \( x(t) \) be a nonoscillatory solution to (1) on \([t_0, \infty)\), and \( x_1(t)z(t) > 0 \) for \( t \in [t_0, \infty) \). Then there exist \( l \in \{1, 2, \ldots, n\} \) with \( i \cdot (-1)^{n-l+1} = 1 \) or \( l = n \) and some \( t_1 \in [t_0, \infty) \), such that for \( t \in [t_1, \infty) \),
\[
x_1(t) \cdot z(t) > 0, \quad i = 1, 2, \ldots, l;
\]
\[
(-1)^{i+l} x_1(t) \cdot z(t) > 0, \quad i = l + 1, \ldots, n. \tag{18}
\]

**Proof.** Putting \( A = \{k \in \{1, 2, \ldots, n\} : x_i(t)z(t) > 0 \text{ for } t \in [t_0, \infty) \text{ and } i = 1, 2, \ldots, k, \text{ then } A \neq \emptyset \} \) and so we can pick \( l = \max\{k : k \in A\} \). It is obvious that \( 1 \leq l \leq n \). Without loss of generality, we may assume that \( x_1(t) > 0 \) for \( t \in [t_0, \infty) \).

Note that \( z(t) > 0 \) for all \( t \in [t_0, \infty) \). If \( l = n \), it is obvious that (18) holds. Next, we assume \( 1 \leq l < n \), then \( x_1(t) > 0 \), \( x_1(t) < 0 \) for \( t \in [t_0, \infty) \). We will show recursively that (18) holds.

We firstly show that \( x_{l+1}(t) > 0 \) for \( t \in [t_0, \infty) \).

Otherwise, \( x_{l+1}(t) < 0 \) for \( t \in [t_0, \infty) \). Then, from (1) and (b), we have \( x_{l+1}(t) \leq x_{l+1}(t_0) < 0 \) for \( t \in [t_0, \infty) \). So, by (1) and (b),
\[
x_1(t_0) > -x_1(t) + x_1(t_0)
\]
\[
= - \int_{t_0}^t p(s) x_{l+1}(s) \Delta s \geq -x_{l+1}(t_0) \int_{t_0}^t p(s) \Delta s \tag{19}
\]
holds for all \( t \in [t_0, \infty) \), which contradicts the condition (b).

Hence, \( x_1(t_0) > 0 \) for \( t \in [t_0, \infty) \).

We now show that \( x_{l+2}(t) < 0 \) for \( t \in [t_0, \infty) \). Otherwise, \( x_{l+2}(t) > 0 \) for \( t \in [t_0, \infty) \). Then, from (1) and (b), we have \( x_{l+2}(t) \geq x_{l+2}(t_0) > 0 \) for \( t \in [t_0, \infty) \). So, by (1) and (b),
\[
x_{l+1}(t_0) < - \int_{t_0}^t p(s) x_{l+2}(s) \Delta s \leq -x_{l+2}(t_0) \int_{t_0}^t p(s) \Delta s \tag{20}
\]
holds for all \( t \in [t_0, \infty) \), which contradicts the condition (b). This implies that \( x_{l+2}(t) < 0 \) for \( t \in [t_0, \infty) \).

Continuing this progress, we can prove that (18) holds.

To complete the proof, it remains to show that \( i \cdot (-1)^{n-l+1} = 1 \). Suppose \( i \cdot (-1)^{n-l+1} = 1 \), then, from (1) and (d), we have 
\[
(-1)^{n-l+1} x_{n-l}(t) < 0 \text{ on } [t_2, \infty) \text{ for } t_2 \in [T_1, \infty). \tag{21}
\]
Note that \( (-1)^{n-l-1} x_{n-l}(t) > 0 \) and \( (-1)^{n-l} x_{n-l}(t) > 0 \) for all \( t \in [t_2, \infty) \). It is easy to show that \( x_n(t) \geq x_n(t_2) > 0 \) for all \( t \in [t_2, \infty) \).

But then,
\[
x_{n-l}(t_2) \geq \int_{t_2}^{t_0} p_n(s) x_n(s) \Delta s \tag{22}
\]
\[
= \int_{t_2}^{t_0} p_n(s) x_n(s) \Delta s \geq \int_{t_2}^{t_0} p_n(s) \Delta s \tag{23}
\]
holds for all \( t \in [t_2, \infty) \), which contradicts the condition (b).

This completes the proof.

**Lemma 8.** Let \( x(t) \) be a nonoscillatory solution to (1) on \([t_0, \infty)\), and \( x_1(t)z(t) < 0 \) for \( t \in [t_0, \infty) \). Then there exist \( l \in \{1, 2, \ldots, n\} \) with \( i \cdot (-1)^{n-l+1} = 1 \) or \( l = n \) and some \( t_1 \in [t_0, \infty) \), such that for \( t \in [t_1, \infty) \),
\[
x_1(t) \cdot z(t) < 0 \tag{24}
\]
or
\[
x_1(t) \cdot z(t) > 0, \quad x_1(t) \cdot z(t) > 0, \quad i = 2, 3, \ldots, l \tag{25}
\]
\[
(-1)^{i+l} x_1(t) \cdot z(t) > 0, \quad i = l + 1, \ldots, n. \tag{26}
\]


Proof. Without loss of generality, we may assume that \( x_1(t) > 0 \) for \( t \in [t_0, \infty) \). Note that \( z(t) < 0 \) for all \( t \in [t_0, \infty) \). We consider the following two cases.

(I) \( x_2(t) > 0 \) for \( t \in [t_0, \infty) \). We will show recursively that (22) holds. We firstly show that \( x_2(t) < 0 \) for \( t \in [t_0, \infty) \). Otherwise, \( x_2(t) > 0 \) for \( t \in [t_0, \infty) \). Then, from (1) and (b), it is easy to see that \( x_2(t) < x_2(t_1) < 0 \) for \( t \in [t_0, \infty) \). So, by (1) and (b) again,

\[
\begin{align*}
z(t) &< -z(t) + z(t_0) \\
&= -\int_{t_0}^{t} p_s(s) x_2(s) \Delta s \leq -x_2(t_0) \int_{t_0}^{t} p_s(s) \Delta s
\end{align*}
\]

holds for all \( t \in [t_0, \infty) \), which contradicts the condition (b). Hence, \( x_2(t) > 0 \) and \( x_3(t) < 0 \) for \( t \in [t_0, \infty) \). Continuing this progress similarly, we conclude that (22) holds.

(II) \( x_2(t) < 0 \) for \( t \in [t_0, \infty) \). Analogously as in the proof of Lemma 7, we can prove that (23) holds. This completes the proof.

For the sake of convenience, we denote by \( N^+ \), \( N^- \), and \( N_{ij}^- \) the set of all nonoscillatory solutions to the system (1) satisfying (18), (22), and (23) correspondingly. Denote by \( N \) the set of all nonoscillatory solutions to the system (1). From Lemmas 7 and 8, we have the classification of nonoscillatory solutions to the system (1) as follows:

(i) \( n \) is odd and \( i = 1 \),

\[
N = N^1 \cup N^+_{i=1} \cup \cdots \cup N^+_{i=n-1} \cup N^-_1 \cup N^-_2 \cup \cdots \cup N^-_{n-1};
\]

(ii) \( n \) is odd and \( i = -1 \),

\[
N = N^1 \cup N^+_{i=1} \cup \cdots \cup N^+_{i=n-1} \cup N^-_1 \cup N^-_2 \cup \cdots \cup N^-_{n-1} \cup N^-_n;
\]

(iii) \( n \) is even and \( i = 1 \),

\[
N = N^1 \cup N^+_{i=1} \cup \cdots \cup N^+_{i=n-1} \cup N^-_1 \cup N^-_2 \cup N^-_3 \cup \cdots \cup N^-_{n-1} \cup N^-_{n-1} \cup N^-_{n-2};
\]

(iv) \( n \) is even and \( i = -1 \),

\[
N = N^1 \cup N^+_{i=1} \cup \cdots \cup N^+_{i=n-1} \cup N^-_1 \cup N^-_2 \cup N^-_3 \cup \cdots \cup N^-_{n-1} \cup N^-_{n-2} \cup N^-_{n-3};
\]

Remark 9. Assume that \( g(t) < t \) and \( 0 < a(t) \leq \lambda < 1 \) for \( t \in [t_0, \infty) \), where \( \lambda \) is a constant. Then \( N^-_k = 0 \) for \( k \in \{2, 3, \ldots, n\} \).

In fact, if \( N^-_k \neq 0 \) for some \( k \in \{2, 3, \ldots, n\} \). Let \( x \in N^-_k \) and suppose, without loss of generality, that \( x_1(t) > 0 \) for \( t \in [t_0, \infty) \). Note that \( z(t) < 0 \) and \( x_2(t) > 0 \) for \( t \in [t_0, \infty) \). It follows that \( z(t) \leq z(t_1) < 0 \) for \( t \in [t_0, \infty) \). On the other hand, \( 0 > z(t) \geq -a(t)x_1(g(t)) \geq -\lambda x_1(g(t)) \) for \( t \in [t_0, \infty) \). By Lemma 5, \( \lim_{t \to \infty} x_i(t) = 0 \) which implies that \( \lim_{t \to \infty} z(t) = 0 \), a contradiction.

Lemma 10. Let \( x(t) \) be a nonoscillatory solution to (1) on \( [t_0, \infty) \). Suppose that \( \lim_{t \to \infty} |z(t)| = L_k \), and \( \lim_{t \to \infty} |x_i(t)| = L_k \) for \( k \in \{2, 3, \ldots, n\} \). Then

\[
L_k > 0, \quad k \in \{2, 3, \ldots, n\} \text{ implies } L_i = \infty,
\]

\[
i = 1, 2, \ldots, k - 1,
\]

\[
L_k < \infty, \quad k \in \{1, 2, \ldots, n-1\} \text{ implies } L_i = 0,
\]

\[
i = k + 1, \ldots, n.
\]

Proof. Suppose that \( L_k > 0 \), \( k \in \{2, 3, \ldots, n\} \), then there is a positive constant \( M_k \) such that \( |x_k(t)| \geq M_k \) for \( t \in [t_0, \infty) \). By (1) and (b), we have

\[
|x_{k-1}(t)| \geq \int_{t_0}^{t} p_{k-1}(s) x_k(s) \Delta s
\]

\[
\geq \int_{t_0}^{t} p_{k-1}(s) |x_k(s)| \Delta s \geq M_k \int_{t_0}^{t} p_{k-1}(s) \Delta s.
\]

It follows from (b) that \( L_{k-1} = \infty \). Obviously there is a positive constant \( M_{k-1} \) such that \( |x_{k-1}(t)| \geq M_{k-1} \) for \( t \in [t_0, \infty) \). Similarly as above, we can prove that \( L_{k-2} = \infty \). Continuing this progress, we conclude that (29) holds. Using (29), it is easy to see that (30) holds. The proof is complete.

3. Main Results and Proofs

We will now give the main results and their proofs. In the sequel, for the convenience of expressions, we define \( I_k \) and \( J_k \) by recursion formula as follows:

\[
I_0(s, t) \equiv 1,
\]

\[
I_k(s, t; h_1, h_2, \ldots, h_k) = \int_{s}^{t} h_1(x) I_{k-1}(x; t, h_2, h_3, \ldots, h_k) \Delta x,
\]

\[
J_0(s, t) \equiv 1,
\]

\[
J_k(s, t; h_1, h_2, \ldots, h_k)
\]

\[
= \int_{s}^{t} h_k(x) J_{k-1}(x; s, h_1, h_2, \ldots, h_{k-1}) \Delta x,
\]

where \( h_k : \overline{T} \to \mathbb{R}, k = 1, 2, \ldots, n \), are continuous functions and \( s, t \in \overline{T} \).

Remark II. From the definitions of \( I_k \) and \( J_k \), it is easy to show the following two properties:

(i) if \( h_i \geq 0 \) for each \( i = 1, 2, \ldots, k \), then

\[
I_k(s, t; h_1, h_2, \ldots, h_k) \geq I_k(u, v; h_1, h_2, \ldots, h_k) \geq 0
\]

for \( s \geq u \geq v \geq t, s, u, v, t \in \overline{T} \);

(ii) if \( h_i \geq 0 \) for each \( i = 1, 2, \ldots, k \) and \( h_i \geq f_i \) for some \( l \in \{1, 2, \ldots, k\} \), then

\[
I_k(s, t; h_1, h_2, \ldots, h_k) \geq I_k(s, t; h_1, h_2, \ldots, h_{l-1}, f_l, h_{l+1}, \ldots, h_k)
\]

for \( s > t, s, t \in \overline{T} \).
Theorem 12. Assume that \( n \) is odd and \( \iota = -1 \). If the following statements hold:

(1) there is a constant \( \lambda \) such that, for any \( t \in [t_0, \infty) \),
\[
1 < \lambda < \alpha(t); \tag{35}
\]
(2) for any \( t \in [t_0, \infty) \),
\[
\sigma(t) < \gamma(t) < \eta(t); \tag{36}
\]
(3) there is a continuous function \( \alpha : [t_0, \infty) \rightarrow [t_0, \infty) \) such that
\[
t < \alpha(t), \tag{37}
\]
(4) for each even \( l \) with \( 4 \leq l \leq n \),
\[
\limsup_{t \rightarrow \infty} KI_{l-1}(t, h^{-1}(g(t)); p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times I_{n-l+1}(t, (\ast); \frac{p_n}{a(g^{-1}(h))}, p_{n-1}, \ldots, p_l) > 1; \tag{39}
\]
(5)
\[
\limsup_{t \rightarrow \infty} KI_n(t, h^{-1}(g(t)); p_1, p_2, \ldots, p_{n-1}, \frac{p_n}{a(g^{-1}(h))}) > 1, \tag{40}
\]
then, for every nonoscillatory solution \( x \in W \) to (1),
\[
\lim_{t \rightarrow \infty} x_i(t) = 0, \forall i = 1, 2, \ldots, n.
\]

Proof. Let \( x \in W \) be a nonoscillatory solution to (1). Without loss of generality, we may assume that \( x_i(t) > 0 \) for all \( t \in [t_1, \infty) \subset [t_0, \infty) \). Because \( n \) is odd and \( \iota = -1 \), the expression (26) holds. We consider the following five cases.

(I) \( x \in N^+_3 \) on \( [t_1, \infty) \). In this case, we have
\[
x_1(t) > 0, \quad \varepsilon(t) > 0, \quad x_2(t) < 0, \quad x_3(t) > 0, \ldots, x_n(t) > 0,
\]
\[
t \in [t_1, \infty) \quad \iota.
\]

By Lemma 4, \( \lim_{t \rightarrow \infty} x_i(t) = 0 \). From (1) and (41), we have \( z(t) \) is a positive and nonincreasing function, which implies that \( \lim_{t \rightarrow \infty} |z(t)| = L_1 < \infty \). It follows, from Lemma 10, that \( \lim_{t \rightarrow \infty} x_i(t) = 0, \forall i = 1, 2, \ldots, n \).

(II) \( x \in N^+_3 \cup N^+_2 \cup \cdots \cup N^+_n \) on \( [t_1, \infty) \). In this case, we have
\[
x_1(t) > 0, \quad \varepsilon(t) > 0, \quad x_2(t) > 0, \quad x_3(t) > 0, \quad t \in [t_1, \infty) \iota.
\]

From (1) and (42), there is a positive constant \( M \) such that \( z_2(t) \geq M \) for \( t \in [t_1, \infty) \). Integrating the first equation of (1) over \( [t_1, t] \), and using the inequality above, we get
\[
z(t) \geq z(t_1) + M \int_{t_1}^t p_1(r) \Delta r, \quad t \in [t_1, \infty) \iota. \tag{43}
\]

By the condition (b), \( \lim_{t \rightarrow \infty} z(t) = \infty \). The conditions (35), (36), and (42) imply that \( x_1(t) \) is bounded on \( [t_1, \infty) \), which arrives a contradiction as \( z(t) < x_2(t) \) for \( t \in [t_1, \infty) \iota \).

(III) \( x \in N^+_2 \) on \( [t_1, \infty) \). In this case, we have
\[
x_1(t) > 0, \quad \varepsilon(t) > 0, \quad x_2(t) > 0, \quad x_3(t) > 0, \ldots, x_n(t) > 0,
\]
\[
t \in [t_1, \infty) \iota. \tag{44}
\]

Integrating the second equation of (1) from \( t \) to \( s \), we obtain that, for \( s \geq t \) and \( s, t \in [t_1, \infty) \),
\[
x_2(t) \leq -\int_s^t p_2(u_2) x_3(u_2) \Delta u_2. \tag{45}
\]

Integrating the third equation of (1) from \( u_2 \) to \( s \), we get that, for \( s \geq u_2 \geq t \) and \( s, u_2, t \in [t_1, \infty) \),
\[
-x_3(u_2) \leq \int_{u_2}^s p_3(u_3) x_4(u_3) \Delta u_3. \tag{46}
\]

Continuing this progress, we have that, for any \( s \geq u_{k-1} \geq u_{k-2} \geq \cdots \geq u_2 \geq t \) and \( s, u_k, t \in [t_1, \infty) \),\( 2 \leq k \leq n - 1 \),
\[
(-1)^k x_k(u_{k-1}) \leq (-1)^{k+1} \int_{u_{k-1}}^s p_k(u_k) x_{k+1}(u_k) \Delta u_k, \tag{47}
\]
\[
3 \leq k \leq k - 1.
\]

Combining \( n - 1 \) inequalities above, we obtain that, for \( s \geq t \) and \( s, t \in [t_1, \infty) \),
\[
x_2(t) \leq \int_t^s p_2(u_2) \int_{u_2}^s p_3(u_3) \cdots \int_{u_{n-2}}^s p_{n-1}(u_{n-1}) \int_{u_{n-1}}^s x_n(u_n) \Delta u_n \cdots \Delta u_2
\]
\[
= J_{n-1}(s, t; x_n^*, p_{n-1}, \ldots, p_2). \tag{48}
\]
From (1), (b), (d), and (44), we have that, for \( s \geq u_n \geq t \) and \( s, u_n, t \in [t_1, \infty)_T \),
\[
    x_n (u_n) = -K p_n (u_n) x_1 (h(u_n)) .
\]  

(49)

Substituting (49) to (48), we have that, for \( s \geq t \) and \( s, t \in [t_1, \infty)_T \),
\[
    x_2 (t) \leq K f_{n-1} (s, t; -p_n x_1 (h), p_{n-1}, \ldots, p_2) .
\]  

(50)

Because of \( x_1 (t) > 0 \) for \( t \in [t_1, \infty)_T \),
\[
    -x_1 (h(t)) < \frac{z (g^{-1} (h(t)))}{a (g^{-1} (h(t)))}, \quad t \in [t_1, \infty)_T .
\]  

(51)

Using the monotonicity of \( z (g^{-1} (h(t))) \), we have that, for \( s \geq u_n \geq t \) and \( s, u_n, t \in [t_1, \infty)_T \),
\[
    -x_1 (h(u_n)) \leq \frac{z (g^{-1} (h(t)))}{a (g^{-1} (h(u_n)))} .
\]  

(52)

Combining (50) and (52), we have that, for \( s \geq t \) and \( s, t \in [t_1, \infty)_T \),
\[
    x_2 (t) \leq \left[ K f_{n-1} \left( s, t; \frac{p_n}{a (g^{-1} (h))}, p_{n-1}, \ldots, p_2 \right) \right] z (g^{-1} (h(t))) .
\]  

(53)

Multiplying the inequality above by \( p_1 (t) \) and putting \( s = \alpha(t) \), we get that
\[
    z^\alpha (t) - \left[ K p_1 (t) f_{n-1} \left( \alpha(t), t; \frac{p_n}{a (g^{-1} (h))}, p_{n-1}, \ldots, p_2 \right) \right] \times z (g^{-1} (h(t))) \leq 0, \quad t \in [t_1, \infty)_T .
\]  

(54)

By Lemma 6, the inequality above has no eventually negative solution, a contradiction. Hence, \( N_2^- = 0 \).

(IV) \( x \in N_1^- \) on \([t_1, \infty)_T \), \( l = 4, 6, \ldots, n - 1 \). In this case, we have that, for \( t \in [t_1, \infty)_T \),
\[
    x_1 (t) > 0, \quad z (t) < 0, \quad x_2 (t) < 0, \ldots, x_l (t) < 0, \quad x_{l+1} (t) > 0, \quad x_{l+2} (t) < 0, \ldots, x_n (t) > 0 .
\]  

(55)

Integrating the first equation of (1) from \( s \) to \( t \), we get that, for \( s \leq t \) and \( s, t \in [t_1, \infty)_T \),
\[
    z (t) \leq \int_s^t p_1 (u_1) x_2 (u_1) \Delta u_1 .
\]  

(56)

Integrating the 2th, \( \ldots, (l - 1) \)th equation of (1), we obtain, for \( t \geq u_1 \geq u_2 \geq \ldots \geq u_{l-2} \geq s \) and \( s, u_k \ (1 \leq k \leq l - 2) \), \( t \in [t_1, \infty)_T \), that
\[
    x_k (u_{k-1}) \leq \int_s^{u_{k-1}} p_k (u_k) x_{k+1} (u_k) \Delta u_k, \quad 2 \leq k \leq l - 1 .
\]  

(57)

Integrating the \( l \)th, \( \ldots, (n - 1) \)th equation of (1), we obtain, for \( t \geq u_{n-1} \geq \ldots \geq u_2 \geq s \) and \( s, u_k \ (l - 1 \leq k \leq n - 1) \), \( t \in [t_1, \infty)_T \), that
\[
    (-1)^k x_k (u_{k-1}) \leq (-1)^{k+1} \int_{u_{k-1}}^t p_k (u_k) x_{k+1} (u_k) \Delta u_k, \quad l \leq k \leq n - 1 .
\]  

(58)

Combining \( n - 1 \) inequalities above, we obtain that, for \( t \geq s \) and \( s, t \in [t_1, \infty)_T \),
\[
    z (t) \leq - \int_{u_{n-1}}^t p_1 (u_1) \int_{u_2}^t p_2 (u_2) \ldots \int_{u_{n-2}}^{u_{n-1}} p_{n-1} (u_{n-1}) \Delta u_{n-1} \times \int_{u_{n-1}}^{u_{n-2}} p_{n-1} (u_{n-1}) x_n (u_{n-1}) \Delta u_{n-1} \ldots, \Delta u_1
\]  

(Note that (52) holds if \( x_1 (t) > 0, z (t) < 0, x_2 (t) < 0, \) and \( x_l (t) > 0 \) for \( t \in [t_1, \infty)_T \)). So we have that, for \( t \geq u_{n-1} \geq s \) and \( s, u_{n-1}, t \in [t_1, \infty)_T \),
\[
    -x_n (u_{n-1}) < -K \left[ \int_{u_{n-1}}^t \frac{p_n (u_n)}{a (g^{-1} (h(u_n)))} \Delta u_n \right] z (g^{-1} (h(s))) .
\]  

(60)

Putting \( s = h^{-1} (g(t)) \in [t_1, \infty)_T \), then
\[
    -x_n (u_{n-1}) < -K \left[ \int_{u_{n-1}}^t \frac{p_n (u_n)}{a (g^{-1} (h(u_n)))} \Delta u_n \right] z (t) .
\]  

(61)

Substituting (61) to (59), we have, on \([t_2, \infty)_T \) for some sufficiently large \( t_2 \in T \), that
\[
    z (t) \leq \left[ K f_{l-1} (t, h^{-1} (g(t)); p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times \int_{u_{n-1}}^t \frac{p_n (u_n)}{a (g^{-1} (h(u_n)))} \Delta u_n \right] z (t) .
\]  

(62)

So, for \( t \in [t_2, \infty)_T \)
\[
    K f_{l-1} (t, h^{-1} (g(t)); p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times \int_{u_{n-1}}^t \frac{p_n (u_n)}{a (g^{-1} (h(u_n)))} \Delta u_n \leq 1 .
\]  

(63)

This contradicts (39) and hence, \( N_4^- \cup N_6^- \cup \cdots \cup N_{n-1}^- = 0 \).

(V) \( x \in N_1^- \) on \([t_1, \infty)_T \). In this case, we have that, for \( t \in [t_1, \infty)_T \),
\[
    x_1 (t) > 0, \quad z (t) < 0, \quad x_2 (t) < 0, \quad x_3 (t) < 0, \ldots, x_n (t) < 0 .
\]  

(64)
Similarly as in the case (IV) of the proof of this theorem, we can get, for some sufficiently large $t_2 \in [t_1, \infty)_T$, that
\[ z(t) \leq K I_n \left( t, s; p_1, \ldots, p_{n-1}, \frac{p_n}{a(g^{-1}(h))} \right), \quad t \in [t_2, \infty)_T. \]  
(65)

Using the monotonicity of $z(g^{-1}(h(t)))$, we have, for $t \in [t_2, \infty)_T$, that
\[ z(t) \leq \left[ K I_n \left( t, s; p_1, \ldots, p_{n-1}, \frac{p_n}{a(g^{-1}(h))} \right) \right] z(t), \quad t \in [t_2, \infty)_T. \]  
(66)

So, for $t \in [t_2, \infty)_T$
\[ K I_n \left( t, s; p_1, \ldots, p_{n-1}, \frac{p_n}{a(g^{-1}(h))} \right) \leq 1, \]  
(67)

which gives a contradiction with (40). Thus $N_n^- = \emptyset$. The proof is complete.

**Theorem 13.** Assume that $n$ is even, $t = 1$ in the system (1) and conditions (35)–(39) hold. Then, for every nonoscillatory solution $x \in W$ to (1), we have $\lim_{t \to \infty} x_i(t) = 0$, $i = 1, 2, \ldots, n$.

**Proof.** Let $x \in W$ be a nonoscillatory solution to (1) and the expression (27) holds. Without loss of generality, we may assume that $x_1(t) > 0$ for all $t \in [t_1, \infty)_T < [t_0, \infty)_T$. We consider the following five cases.

(I) $x \in N_1^+$ on $[t_1, \infty)_T$. Analogically as the case (I) in the proof of Theorem 12, we can show that $\lim_{t \to \infty} x_i(t) = 0$, $i = 1, 2, \ldots, n$.

(II) $x \in N_1^+ \cup N_2^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+$ on $[t_1, \infty)_T$. Also analogically as in the proof of Theorem 12(II), we can prove that $N_2^+ \cup N_3^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ = \emptyset$.

(III) $x \in N_2^-$ on $[t_1, \infty)_T$. In this case, we have
\[ x_1(t) > 0, \quad z(t) < 0, \quad x_2(t) < 0, \quad x_3(t) > 0, \ldots, x_n(t) < 0, \]  
$t \in [t_1, \infty)_T$.  
(68)

We continue analogically as (III) in the proof of Theorem 12. We can get, for $s \geq t$ and $s, t \in [t_1, \infty)_T$, that
\[ x_2(t) \leq - \int_{u_2}^{t} p_2(u_2) \int_{u_2}^{t} p_3(u_3) \cdots \int_{u_{n-2}}^{t} p_{n-1}(u_{n-1}) \]  
\[ \times \int_{u_{n-1}}^{t} x_n^\Delta(u_n) \Delta u_n \cdots \Delta u_2 \]  
\[ = -I_{n-1} \left( s, t; x_n^\Delta, p_{n-1}, \ldots, p_2 \right). \]  
(69)

From (I), (b), (d), and (68), we have that, for $s \geq u_n \geq t$ and $s, u_n, t \in [t_1, \infty)_T$,  
\[ -x_n^\Delta(u_n) \leq -K p_n(u_n) x_1(h(u_n)). \]  
(70)

We substitute (70) with (69), we have that, for $s \geq u_n \geq t$ and $s, u_n, t \in [t_1, \infty)_T$,  
\[ x_2(t) \leq K J_{n-1} \left( s, t; -p_n x_1(h), p_{n-1}, \ldots, p_2 \right). \]  
(71)

By the same way as in the proof of Theorem 12(III), we can obtain that $N_n^- = \emptyset$.

(IV) $x \in N_n^-$ on $[t_1, \infty)_T$, $l = 4, 6, \ldots, n - 2$. In this case, we have that, for $t \in [t_1, \infty)_T$,  
\[ x_1(t) > 0, \quad z(t) < 0, \quad x_2(t) < 0, \ldots, x_l(t) < 0, \]  
\[ x_{l+1}(t) > 0, \quad x_{l+2}(t) < 0, \ldots, x_n(t) < 0. \]  
(72)

By the same way as in the proof of Theorem 12(IV), we can obtain, for $t \geq s$ and $s, t \in [t_1, \infty)_T$, that
\[ z(t) \leq I_{l-1} \left( t, s; p_1, \ldots, p_{l-2}, p_{l-1} \right) \]  
\[ \times J_{n-l} \left( t, s; x_n, p_n \right). \]  
(73)

From (I), (b), (d) and (72), we have, for $s \geq u_{n-1} \geq t$ and $s, u_{n-1}, t \in [t_1, \infty)_T$, that
\[ x_n(u_{n-1}) \leq -K \int_{u_{n-1}}^{t} p_n(x_n) x_1(1(h(x_n))) \Delta u_n. \]  
(74)

Therefore, it holds that, for $t \geq s$ and $t \in [t_1, \infty)_T$,  
\[ z(t) \leq -K I_{n-2} \left( t, s; p_1, \ldots, p_{n-2}, p_{n-1} \right) \]  
\[ \times J_{n-2} \left( t, s; x_n, p_n \right). \]  
(75)

From (I), (b), (d), and (72), we have that, for $s \geq u_{n-1} \geq t$ and $s, u_{n-1}, t \in [t_1, \infty)_T$,  
\[ -x_n^\Delta(u_{n-1}) \leq -K p_n(u_{n-1}) x_1(1(h(u_{n-1}))). \]  
(76)

And at the end to contradict to (39) for $l = n$. Thus $N_n^- = \emptyset$. This completes the proof.

**Theorem 14.** Let $n$ be odd, $t = 1$ in the system (1). If (37) the following statements hold:

1. there is a constant $\lambda$ such that, for any $t \in [t_0, \infty)_T$,  
\[ 0 < a(t) \leq \lambda < 1; \]  
(77)

2. for any $t \in [t_0, \infty)_T$,  
\[ g(t) < t < h(t); \]  
(78)

3. the function $\alpha(t)$ in (37) satisfies that
\[ \lim_{t \to \infty} \int_{\sigma(t)}^{h(t)} K p_1(v) I_{l-1}(\alpha(v), v, p_n, p_{n-1}, \ldots, p_2) \Delta v > \frac{1}{e}; \]  
(79)
(4) for each even \( l \) with \( 4 \leq l \leq n \),
\[
\lim_{t \to \infty} \sup K_{I_{l-1}}(t, h^{-1}(t); p_1, \ldots, p_{l-1} ; *) \times J_{n-l+1}(t,*; p_n, \ldots, p_l) > 1;
\]
(80)

(5)
\[
\lim_{t \to \infty} \sup K_{I_{n}}(t, h^{-1}(t); p_1, p_2, \ldots, p_n) > 1.
\]
(81)

Then, for every nonoscillatory solution \( x \in W \) to (1),
\[
\lim_{t \to \infty} x_i(t) = 0, \quad i = 1, 2, \ldots, n.
\]

Proof. Let \( x \in W \) be a nonoscillatory solution to (1). Without loss of generality, we may assume that \( x_i(t) > 0 \) for all \( t \in [t_1, \infty)_T \). By the expression (25) and Remark 9, we have \( N = N_2^+ \cup N_2^- \cup \cdots \cup N_{n-1}^+ \cup N_{n-1}^- \cup N_1^- \). We consider the following four cases.

(I) \( x \in N_2^+ \) on \( [t_1, \infty)_T \). In this case, we have that
\[
x_1(t) > 0, \quad z(t) > 0, \quad x_2(t) > 0, \quad x_3(t) < 0, \ldots, x_n(t) < 0, \quad t \in [t_1, \infty)_T.
\]
(82)

Integrating the second equation of (1) from \( t \) to \( s \), we obtain that, for \( s \geq t \) and \( s, t \in [t_1, \infty)_T \),
\[
x_2(s) \geq -\int_t^s p_2(u_2) x_3(u_2) \Delta u_2.
\]
(83)

Integrating the \( k \)th equation of (1) from \( u_{k-1} \) to \( s \), we get, for \( s \geq u_{k-1} \geq t \) and \( s, u_{k-1}, t \in [t_1, \infty)_T \), that
\[
(-1)^k x_k(u_{k-1}) \geq (-1)^{k+1} \int_{u_{k-1}}^s p_k(u_k) x_{k+1}(u_k) \Delta u_k, \quad 3 \leq k \leq k-1
\]
(84)

\[-x_n(u_{n-1}) \geq \int_{u_{n-1}}^s x_n^\Delta(u_n) \Delta u_n.
\]

Combining \( n-1 \) inequalities above, we obtain that, for \( s \geq t \) and \( s, t \in [t_1, \infty)_T \),
\[
x_2(s) \geq -\int_t^s p_2(u_2) \int_{u_2}^s p_3(u_3) \cdots \int_{u_{n-1}}^s p_{n-1}(u_{n-1}) \times \int_{u_{n-1}}^s x_n^\Delta(u_n) \Delta u_n \quad \Delta u_2
\]
(85)

\[= J_{n-1}(s, t; x_n^\Delta, p_n, \ldots, p_2).
\]

From (1), (b), (d), and (44), we have that, for \( s \geq u_{n-1} \geq t \) and \( s, u_{n-1}, t \in [t_1, \infty)_T \),
\[
x_n^\Delta(u_n) \geq K p_n(u_n) x_1(h(u_n)).
\]
(86)

Substitute (86) with (71), and notice that the inequality \( x_1(h(t)) \geq z(h(t)) \) holds for any \( t \in [t_1, \infty)_T \). Then we have that, for \( s \geq t \) and \( s, t \in [t_1, \infty)_T \),
\[
x_2(s) \geq K J_{n-1}(s, t; p_n, p_{n-1}, \ldots, p_2) \geq K J_{n-1}(s, t; p_n, p_{n-1}, \ldots, p_2) z(h(t)).
\]
(87)

Multiplying the inequality above by \( p_1(t) \) and putting \( s = a(t) \), we get that
\[
z_\Delta(t) - [K p_1(t) J_{n-1}(a(t), t; p_n, p_{n-1}, \ldots, p_2)] z(h(t)) \geq 0,
\]
(88)

By Lemma 6, the inequality above has no eventually positive solution, a contradiction. Hence, \( N_2^+ = \emptyset \).

(II) \( x \in N_1^- \) on \( [t_1, \infty)_T \), \( l = 4, 6, \ldots, n - 1 \). In this case, we have that, for any \( t \in [t_1, \infty)_T \),
\[
x_1(t) > 0, \quad z(t) > 0, \quad x_2(t) > 0, \ldots, x_l(t) > 0, \quad x_{l+1}(t) < 0, \ldots, x_n(t) < 0.
\]
(89)

Integrating the first equation of (1) over \([s, t]_T \), we get that
\[
z(t) \geq \int_s^t p_1(u_1) \Delta u_1, \quad s \leq t, \quad s, t \in [t_1, \infty)_T.
\]
(90)

Similarly, by integrating the 2th, \( \ldots, (l-1) \)th equations of (1) and the obtained inequalities institute to (90), we get, for \( s \geq t \) and \( s, t \in [t_1, \infty)_T \), that
\[
z(t) \geq \int_s^t p_1(u_1) \int_{u_1}^{u_2} p_2(u_2) \cdots \int_{u_{l-1}}^{u_l} p_l(u_l) \Delta u_l \cdots \Delta u_1
\]
(91)

\[= I_{l-1}(s, t; p_1, \ldots, p_{l-1}, x_{l+1}, \ldots, x_n).
\]

Integrating the \( l \)th, \( \ldots, n \)th equation of (1), we obtain, for \( t \geq u_{n-1} \geq \cdots \geq u_{l+1} \geq s \geq t_1 \) and \( s, u_k \) \( (l-1) \leq k \leq n-1 \), \( t \in [t_1, \infty)_T \), that
\[
(-1)^k x_k(u_{k-1}) \geq (-1)^{k+1} \int_{u_{k-1}}^t p_k(u_k) x_{k+1}(u_k) \Delta u_k, \quad l \leq k \leq n-1.
\]
(92)

\[-x_n(u_{n-1}) \geq \int_{u_{n-1}}^t x_n^\Delta(u_n) \Delta u_n.
\]
Combining $n - l + 1$ inequalities above and (91), we obtain that, for $t \geq s$ and $s, t \in [t_1, \infty)$,

$$z(t) \geq \int_{t_1}^{t} p_1(u_1) \int_{u_1}^{t_1} p_2(u_2) \cdots \int_{u_{n-1}}^{t_{n-1}} p_{n-1}(u_{n-1}) \frac{x_n(u_n)}{u_n} \Delta u_n \cdots \Delta u_1 \tag{93}$$

$$= I_{l-1}(t, s; p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times I_{n-l+1}(t, s; p_n z(h), p_{n-1}, \ldots, p_1)).$$

Note that (86) and the inequality $x_i(h(t)) \geq z(h(t))$ hold. So we have that, for $t \geq s$ and $s, t \in [t_1, \infty)$,

$$z(t) \geq K I_{l-1}(t, s; p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times I_{n-l+1}(t, s; p_n z(h), p_{n-1}, \ldots, p_1)).$$

Putting $s = h^{-1}(t) \in [t_1, \infty)$, using the monotonicity of $z(h(t))$, then we have, for some sufficiently large $t_2 \in [t_1, \infty)$, that

$$z(t) \geq K I_{l-1}(t, h^{-1}(t); p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times I_{n-l+1}(t, h^{-1}(t); p_n z(h), p_{n-1}, \ldots, p_1).$$

So, for $t \in [t_2, \infty)$,

$$K I_{l-1}(t; s; p_1, \ldots, p_{l-2}, p_{l-1} (\ast)) \times I_{n-l+1}(t; s; p_n z(h), p_{n-1}, \ldots, p_1) \leq 1. \tag{96}$$

This contradicts (80) and hence, $N^+_1 \cup N^+_2 \cdots \cup N^+_n = \emptyset$.

(III) $x \in N^+_n$ on $[t_1, \infty)$. In this case, we have that, for $t \in [t_1, \infty)$,

$$x(t) > 0, \ z(t) > 0, \ x_2(t) > 0, \ x_3(t) > 0, \ldots, x_n(t) > 0. \tag{97}$$

Analogously as derived at part (II) of the proof of this theorem, we have that

$$z(t) \geq I_n(t, s; p_1, \ldots, p_{n-1}, x^\Delta_n), \quad t \geq s, \ s, t \in [t_1, \infty). \tag{98}$$

And putting $s = h^{-1}(t)$, we can obtain that

$$K I_n(t, h^{-1}(t); p_1, \ldots, p_n) \leq 1, \quad t \in [t_1, \infty). \tag{99}$$

which give a contradiction with (81), and so $N^+_n = \emptyset$.

(IV) $x \in N^+_n$ on $[t_1, \infty)$. In this case, we have that

$$x_1(t) > 0, \ z(t) < 0, \ x_2(t) > 0, \ x_3(t) < 0, \ldots, x_n(t) < 0, \quad t \in [t_1, \infty). \tag{100}$$

By Lemma 5, $\lim_{t \to \infty} x_i(t) = 0$. From (100), we have $\lim_{t \to \infty} z(t) = L_1 < +\infty$. Then it follows, by Lemma 10, that $\lim_{t \to \infty} x_i(t) = 0, i = 2, 3, \ldots, n$. The proof is complete. \qed

Similarly to Theorem 14, we can prove the following theorem.

**Theorem 15.** Assume that $n$ is even, $i = -1$ in the system (1) and conditions (77)–(80) hold. Then, for every nonoscillatory solution $x \in W$ to (1), we have $\lim_{t \to \infty} x_i(t) = 0, i = 1, 2, \ldots, n$.

**Example 16.** We consider a system on the time scale $T = \mathbb{Q}$.

In order to simplify calculations, we may suppose that $q = 2$.

Let $a(t) = 8, g(t) = 16t, h(t) = 512t, p_1(t) = (1/4)t, p_2(t) = (7/8)t, p_3(t) = (31/32)t, p_4(t) = (127/128)t, p_5(t) = 511/t^5, f(y) = y, K = 1, a(t) = 2t, i = -1, and $n = 5$; that is to say, the system has the following form:

$$[x_1(t) - 8x_1(16t)]^3 = \frac{1}{4} x_2(t),$$

$$x_2(t) = \frac{7}{8} x_3(t),$$

$$x_3(t) = \frac{31}{32} x_4(t),$$

$$x_4(t) = \frac{127}{128} x_5(t),$$

$$x_5(t) = -\frac{511}{128} x_4(512t), \quad t \in T.$$

We calculate the conditions of Theorem 12 as follows:

$$\int_{2t_2}^{2t} \int_{s_2}^{x_2(t)} \int_{s_2}^{x_3(t)} \int_{s_2}^{x_4(t)} \int_{s_2}^{x_5(t)} 2^7 31 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 = 234.05 > 1, \tag{102}$$

$$\int_{2t_2}^{t} \int_{s_2}^{x_2(t)} \int_{s_2}^{x_3(t)} \int_{s_2}^{x_4(t)} \int_{s_2}^{x_5(t)} 2^7 31 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 = 332233.05 > 1,$$

$$\int_{2t_2}^{t} \int_{s_2}^{x_2(t)} \int_{s_2}^{x_3(t)} \int_{s_2}^{x_4(t)} \int_{s_2}^{x_5(t)} 2^7 31 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 = 3431.69 > 1.$$

It follows, from Theorem 12, that for every nonoscillatory solution $x \in W$ to (110), $\lim_{t \to \infty} x_i(t) = 0, i = 1, 2, \ldots, 5$. In fact, functions $x_1(t) = 1/t, x_2(t) = -1/t^3, x_3(t) = 1/t^5, x_4(t) = -1/t^7, and x_5(t) = 1/t^9$ are particular components of such a kind of solutions.
Example 17. We also consider a system on the time scale $T = \mathbb{q}^7$ and let $q = 2$. Let $a(t) = 4, g(t) = 128t, p_1(t) = (1/8)t, p_2(t) = (1/4)t, p_3(t) = (7/32)t, p_4(t) = 31/t^7, f(y) = 127y, K = 127, \alpha(t) = 16t, i = 1,$ and $n = 4$; that is to say, the system has the following form:

$$[x_1(t) - 2x_1(4t)]^\Delta = \frac{1}{8}tx_2(t),$$

$$x_2^\Delta(t) = \frac{1}{4}tx_3(t),$$

$$x_3^\Delta(t) = \frac{7}{32}tx_4(t),$$

$$x_4^\Delta(t) = \frac{31}{t^7} \cdot 127 \cdot x_1(128t), \quad t \in T.$$

We calculate the conditions of Theorem 13 as follows:

$$\int_{2^3}^{32} \frac{127}{2^3} \int_{x_1}^{x_2} \frac{1}{4} dx_1 \int_{x_1}^{x_2} \frac{7}{32} dx_2$$

$$\times \left( \int_{x_2}^{x_3} \frac{31}{2^3} \Delta x_3 \Delta x_1 \Delta x_1 \Delta v \approx 77.39 > \frac{1}{e} \right),$$

$$\int_{1/2}^{t} \frac{1}{8} x_1 \int_{x_1}^{x_2} \frac{1}{4} dx_1 \int_{x_1}^{x_2} \frac{7}{32} dx_2$$

$$\times \left( \int_{x_2}^{x_3} \frac{31}{2^3} \Delta x_3 \Delta x_1 \Delta x_1 \Delta v \approx 331582.88 > 1. \right)$$

It follows, from Theorem 13, that for every nonoscillatory solution $x \in W$ to (103), $\lim_{t \to \infty} x_i(t) = 0, i = 1, 2, \ldots, 4$. In fact, functions $x_i(t) = 1/t$, $x_i(t) = -2/t^3$, $x_i(t) = 7/t^5$, and $x_i(t) = -31/t^7$ are particular components of such a kind of solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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